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ON COMPLEX APPROXIMATION

LAWRENCE CARL EGGAN AND EUGENE A. MAIER

ON COMPLEX APPROXIMATION

L. C. EGGAN and E. A. MAIER

1. Let C denote the set of complex numbers and G the set of Gaussian integers. In this note we prove the following theorem which is a two-dimensional analogue of Theorem 2 in [3].

THEOREM 1. *If $\beta, \gamma \in C$, then there exists $u \in G$ such that $|\beta - u| < 2$ and*

$$|\beta - u| |\gamma - u| < \begin{cases} 27/32 & \text{if } |\beta - \gamma| < \sqrt{11/8} \\ \sqrt{2} |\beta - \gamma|/2 & \text{if } |\beta - \gamma| \geq \sqrt{11/8}. \end{cases}$$

As an illustration of the application of Theorem 1 to complex approximation, we use it to prove the following result.

THEOREM 2. *If $\theta \in C$ is irrational and $a \in C$, $a \neq m\theta + n$ where $m, n \in G$, then there exist infinitely many pairs of relatively prime integers $x, y \in G$ such that*

$$|x(x\theta - y - a)| < 1/2.$$

The method of proof of Theorem 2 is due to Niven [6]. Also in [7], Niven uses Theorem 1 to obtain a more general result concerning complex approximation by nonhomogeneous linear forms.

Alternatively, Theorem 2 may be obtained as a consequence of a theorem of Hlawka [5]. This was done by Eggen [2] using Chalk's statement [1] of Hlawka's Theorem.

2. Theorem 1 may be restated in an equivalent form. For $u, b, c \in C$, define

$$g(u, b, c) = |u - (b + c)| |u - (b - c)|.$$

Then Theorem 1 may be stated as follows.

THEOREM 1'. *If $b, c \in C$, then there exist $u_1, u_2 \in G$ such that*

$$(i) \quad |u_1 - (b + c)| < 2, |u_2 - (b - c)| < 2$$

and for $i = 1, 2$,

$$(ii) \quad g(u_i, b, c) < \begin{cases} 27/32 & \text{if } |c| < \sqrt{11/32} \\ \sqrt{2} |c| & \text{if } |c| \geq \sqrt{11/32}. \end{cases}$$

It is clear that Theorem 1' implies Theorem 1 by taking

$$b = (\beta + \gamma)/2, \quad c = (\beta - \gamma)/2.$$

To see that Theorem 1 implies Theorem 1', first apply Theorem 1 with $\beta = b + c$, $\gamma = b - c$ and then apply Theorem 1 with $\beta = b - c$, $\gamma = b + c$.

3. We precede the proof of Theorem 1' with a few remarks concerning the nature of the proof.

Given $b, c \in C$, introduce a rectangular coordinate system for the complex plane such that b has coordinates $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y)

$$\begin{aligned} g^2(u, b, c) &= |u - b - c|^2 |u - b + c|^2 \\ &= ((x - k)^2 + y^2)((x + k)^2 + y^2) \\ &= (x^2 + y^2 + k^2)^2 - 4k^2x^2. \end{aligned}$$

Now for k a positive real number let $R(k)$ be the set of all points (x, y) such that

$$(x^2 + y^2 + k^2)^2 - 4k^2x^2 < \begin{cases} (27/32)^2 & \text{if } k < \sqrt{11/32} \\ 2k^2 & \text{if } k \geq \sqrt{11/32}. \end{cases}$$

Theorem 1' depends upon showing that $R(k)$ under any rigid motion always contains two lattice points, not necessarily distinct. These lattice points correspond to the integers u_1 and u_2 of the theorem.

For $k > 1/\sqrt{2}$, $R(k)$ contains two circles with centers at

$$(\pm\sqrt{k^2 - 1/2}, 0)$$

and each of radius $1/\sqrt{2}$. Each of these circles contains a lattice point no matter how $R(k)$ is displaced in the plane. In this case, u_1 and u_2 correspond to these lattice points.

For $k < \sqrt{11/32}$, $R(k)$ contains the circle with center at $(0, 0)$ and radius $1/\sqrt{2}$. In this case, $u_1 = u_2$ corresponds to a lattice point in this circle. Finally if $\sqrt{11/32} \leq k \leq 1/\sqrt{2}$ $R(k)$ contains a region described by Sawyer [8] which always contains a lattice point no matter how it is displaced and $u_1 = u_2$ corresponds to a lattice point in this region.

4. We turn now to the proof of Theorem 1'. As above, for given $b, c \in C$, introduce a coordinate system so that b has coordinate $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y) ,

$$(1) \quad g^2(u, b, c) = (x^2 + y^2 + k^2)^2 - 4k^2x^2.$$

Suppose that $|c| = k > 1/\sqrt{2}$. For $i = 1, 2$ let

$$d_i = (\delta_i \sqrt{k^2 - 1/2}, 0)$$

where $\delta_i = (-1)^{i+1}$ and let $u_i \in G$ be a closest Gaussian integer to d_i (i. e. $|d_i - u_i| \leq |d_i - t|$, $t \in G$). Then, omitting the subscripts,

$$|d - (b + \delta c)| = |\delta \sqrt{k^2 - 1/2} - \delta k| = k - \sqrt{k^2 - 1/2} < 1/\sqrt{2}.$$

Hence

$$|u - (b + \delta c)| \leq |u - d| + |d - (b + \delta c)| < 2(1/\sqrt{2}) < 2$$

and condition (i) is satisfied.

Now let u_i have coordinates (x_i, y_i) . Then, again omitting subscripts, since $|d - u| \leq 1/\sqrt{2}$, we have

$$(2) \quad (x - \delta \sqrt{k^2 - 1/2})^2 + y^2 \leq 1/2,$$

equality holding if and only if d is the center of a unit square with Gaussian integers as vertices. Also, since for any two real numbers a and b , $2ab \leq a^2 + b^2$, equality holding if and only if $a = b$, we have

$$(3) \quad 2\delta x \sqrt{k^2 - 1/2} \leq x^2 + k^2 - 1/2,$$

equality holding if and only if $x = \sqrt{k^2 - 1/2}/\delta$. Thus

$$\begin{aligned} (1 + 2\delta x \sqrt{k^2 - 1/2})^2 &= 4\delta x \sqrt{k^2 - 1/2} + 4x^2(k^2 - 1/2) + 1 \\ &\leq 2x^2 + 2k^2 - 1 + 4x^2(k^2 - 1/2) + 1 \\ &= k^2(2 + 4x^2) \end{aligned}$$

and since k and $k^2(2 + 4x^2)$ are positive,

$$1 + 2\delta x \sqrt{k^2 - 1/2} \leq k \sqrt{2 + 4x^2}.$$

Hence

$$(4) \quad 1/2 - (x - \delta \sqrt{k^2 - 1/2})^2 = 1 + 2\delta x \sqrt{k^2 - 1/2} - x^2 - k^2 \leq k \sqrt{2 + 4x^2} - x^2 - k^2.$$

Using (4) and (2), we have

$$\begin{aligned} x^2 + k^2 + y^2 &\leq k \sqrt{2 + 4x^2} + (x - \delta \sqrt{k^2 - 1/2})^2 - 1/2 + y^2 \\ &\leq k \sqrt{2 + 4x^2}, \end{aligned}$$

$$(5) \quad (x^2 + k^2 + y^2)^2 \leq 2k^2 + 4k^2x^2.$$

Thus, from (1) and (5), $g^2(u, b, c) \leq 2k^2$, the equality holding if and only if equality holds in both (2) and (3). If equality holds in (2), then there exist four possible choices for u , at least two of these

choices having unequal first coordinates. Now equality holds in (3) if and only if, for fixed k , x is unique. Thus if equality holds in (2), u may be chosen so that equality does not hold in (3). For this choice of u , $g^2(u, b, c) < 2k^2$ which establishes condition (ii).

Next suppose $|c| = k < \sqrt{11/32}$. Now there exists $u \in G$ such that $|u - b| \leq 1/\sqrt{2}$. Thus

$$|u - (b \pm c)| \leq |u - b| + |c| < 2(1/\sqrt{2}) < 2.$$

Also, if u has coordinates (x, y) , $x^2 + y^2 \leq 1/2$ and thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &< \frac{1}{4} + 2\left(\frac{11}{32}\right)\frac{1}{2} + \left(\frac{11}{32}\right)^2 = \left(\frac{27}{32}\right)^2 \end{aligned}$$

which establishes the theorem for $|c| < \sqrt{11/32}$.

Finally, for $\sqrt{11/32} \leq |c| = k \leq 1/\sqrt{2}$, we use a result due to Sawyer [8] which states that the region defined by $|x| \leq 3/4 - y^2$, $|y| \leq 1/2$ always contains a lattice point no matter how it is displaced in the plane. Thus there exists $u \in G$ with coordinates (x, y) such that $|x| \leq 3/4 - y^2$, $|y| \leq 1/2$.

If $|x| < 1/2$, then

$$|u - (b \pm c)| \leq |u - b| + |c| = \sqrt{x^2 + y^2} + |c| \leq \sqrt{2}.$$

Also since $|x^2 - k^2| \leq 1/2$,

$$\begin{aligned} g^2(u, b, c) &= (x^2 - k^2)^2 + 2y^2(x^2 + k^2) + y^4 \\ &< \frac{1}{4} + 2\frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{16} = \frac{11}{16} \leq 2|c|^2. \end{aligned}$$

If $1/2 \leq |x| \leq 3/4 - y^2$, then

$$x^2 + y^2 \leq \frac{9}{16} - \frac{1}{2}y^2 + y^2 = \frac{1}{2} + \left(y^2 - \frac{1}{4}\right)^2 \leq \frac{9}{16}.$$

Hence

$$|u - (b + c)| \leq \sqrt{x^2 + y^2} + |c| \leq \frac{3}{4} + \frac{1}{\sqrt{2}} < 2.$$

Also $-x^2 \leq -1/4$ so $y^2 - x^2 \leq 0$. Thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &\leq \left(\frac{9}{16}\right)^2 + 0 + \frac{1}{4} < \frac{11}{16} \leq 2|c|^2. \end{aligned}$$

This completes the proof of Theorem 1'.

5. To prove Theorem 2, we require a well-known result of Ford [4] which states that for any irrational $\theta \in C$, there exist infinitely many pairs of relatively prime $h, k \in G$ such that

$$(6) \quad |k(k\theta - h)| < 1/\sqrt{3} :$$

For θ and a is in the statement of Theorem 2, choose h, k satisfying (6) and let $t \in G$ be such that $|t - ka| \leq 1/\sqrt{2}$. Since h and k are relatively prime, there exist $r, s \in G$ such that $rh - sk = t$ and hence

$$(7) \quad |rh - sk - ka| \leq 1/\sqrt{2} .$$

Now, in Theorem 1, let

$$\beta = \frac{r\theta - s - a}{k\theta - h}, \quad \gamma = \frac{r}{k}$$

and set

$$x = r - ku, \quad y = s - hu$$

where u is the Gaussian integer whose existence is guaranteed by the theorem. Then $x, y \in G$ and

$$|x\theta - y - a||x| = |\beta - u||\gamma - u||k||k\theta - h| .$$

Hence if $|\beta - \gamma| < \sqrt{11/8}$ we have, using Theorem 1 and (6),

$$|x\theta - y - a||x| < \frac{27}{32}|k(k\theta - h)| < \frac{27}{32} \quad \frac{1}{\sqrt{3}} < \frac{1}{2} .$$

If $|\beta - \gamma| \geq \sqrt{11/8}$, using Theorem 1 and (7), we have

$$\begin{aligned} |x\theta - y - a||x| &< \frac{1}{2}\sqrt{2}|\gamma - \beta||k(k\theta - h)| \\ &= \frac{1}{2}\sqrt{2}\left|\frac{hr - ks - ka}{k(k\theta - h)}\right||k(k\theta - h)| \leq \frac{1}{2} . \end{aligned}$$

Thus for each pair h, k satisfying (6) we have a solution in G of

$$(8) \quad |x(x\theta - y - a)| < 1/2 .$$

To show that there are infinitely many solutions to (8), we note that since $|\beta - u| < 2$ and $a \neq m\theta + n, m, n \in G$, we have with the use of (6).

$$(9) \quad 0 < |x\theta - y - a| = |\beta - u||k\theta - h| < 2/(\sqrt{3}|k|) .$$

If there are only a finite number of solutions of (8), let M be the minimum of $|x\theta - y - a|$ for these solutions. Then from (9), for every h, k satisfying (6) we have $|k| < 2/(\sqrt{3}M)$ and

$$|h| \leq |h - k\theta| + |k\theta| < 1/(\sqrt{3}|k|) + |k||\theta| < N,$$

say. But this is impossible since there are infinitely many pairs $h, k \in G$ which satisfy (6).

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