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**WEAK CONTAINMENT AND KRONECKER PRODUCTS OF  
GROUP REPRESENTATIONS**

JAMES MICHAEL GARDNER FELL

# WEAK CONTAINMENT AND KRONECKER PRODUCTS OF GROUP REPRESENTATIONS

J. M. G. FELL

**Introduction.** Throughout this paper  $G$  is a fixed locally compact group. Let us recall some concepts bearing on the representation theory of  $G$ . The family of all unitary equivalence classes of unitary representations of  $G$  will be called  $\mathcal{T}(G)$ . A function  $\varphi$  of positive type on  $G$  is *associated* with a subset  $\mathcal{S}$  of  $\mathcal{T}(G)$  if there is an  $S$  in  $\mathcal{S}$ , and a vector  $\xi$  in the space  $H(S)$  of  $S$ , such that  $\varphi(x) = (S_x \xi, \xi)$  for all  $x$  in  $G$ . An element  $T$  of  $\mathcal{T}(G)$  is *weakly contained* in a subset  $\mathcal{S}$  of  $\mathcal{T}(G)$  if every function of positive type on  $G$  associated with  $T$  can be approximated uniformly on compact sets by sums of functions of positive type associated with  $\mathcal{S}$ . The notion of weak containment leads to that of the *inner hull-kernel topology* of  $\mathcal{T}(G)$ : A net  $\{T^i\}$  of elements of  $\mathcal{T}(G)$  converges to  $T$  in this topology if and only if every subnet of  $\{T^i\}$  weakly contains  $T$ . Relativized to the subset  $\hat{G}$  of  $\mathcal{T}(G)$  consisting of the irreducible representations of  $G$ , this topology becomes the ordinary *hull-kernel topology* of  $\hat{G}$ . (For these notions and facts see [1] and [2]).

If  $H$  is a Hilbert space, the *adjoint space*  $\bar{H}$  of  $H$  can be defined as the Hilbert space whose underlying set is the same as that of  $H$ , and which is conjugate-isomorphic with  $H$  under the identity map. If  $T$  is a unitary representation of  $G$ , the *adjoint representation*  $\bar{T}$  is defined by the requirements:  $H(\bar{T}) = H(T)^-$ ,  $\bar{T}_x = T_x^*(x \in G)$ . The *Kronecker product*  $S \otimes T$  of two unitary representations  $S$  and  $T$  of  $G$  is that representation whose space is  $H(S) \otimes H(T)$ , and for which  $(S \otimes T)_x(\xi \otimes \eta) = (S_x \xi) \otimes (T_x \eta)$ . We can also describe the Kronecker product  $S \otimes \bar{T}$  as follows:  $H(S \otimes \bar{T})$  is the Hilbert space of all Hilbert-Schmidt operators on  $H(T)$  to  $H(S)$ , and  $(S \otimes \bar{T})_x(A) = S_x A T_x^{-1}$ .

If  $\mathcal{S} \subset \mathcal{T}(G)$  and  $\mathcal{T} \subset \mathcal{T}(G)$ , let  $\mathcal{S} \otimes \mathcal{T}$  denote  $\{S \otimes T \mid S \in \mathcal{S}, T \in \mathcal{T}\}$ .

Throughout this paper  $I$  will be the one-dimensional identity representation of  $G$ . It is well known and easily verified that if  $S$  and  $T$  are finite-dimensional unitary representations of  $G$  and  $T$  is irreducible,  $S \otimes \bar{T}$  contains  $I$  if and only if  $S$  contains  $T$ . Can this be generalized to the case where  $S$  and  $T$  are infinite-dimensional and 'containment' is replaced by 'weak containment'? The main object of this note is to answer this question affirmatively for the case that  $S$  is infinite-dimensional but  $T$  is still finite-dimensional (Theorem 4). In preparation for this we shall show (Theorem 2) that the Kronecker product oper-

ation is continuous with respect to the inner hull-kernel topology of  $\mathcal{T}(G)$ .

Another by-product of the main result is the following strengthening (Theorem 3) of a remark of Godement ([4], p. 77): If the regular representation  $R$  of  $G$  weakly contains some finite-dimensional irreducible unitary representation of  $G$ , then  $R$  weakly contains all unitary representations of  $G$ .

### 1. The continuity of the Kronecker product.

LEMMA 1. *Suppose that  $\mathcal{S} \subset \mathcal{T}(G)$  and  $T \in \mathcal{T}(G)$ ; and let  $K$  be the set of all those  $\xi$  in  $H(T)$  such that the function  $\varphi$  defined by  $\varphi(x) = (T_x \xi, \xi)(x \in G)$  can be approximated, uniformly on compact sets, by sums of functions of positive type associated with  $\mathcal{S}$ . Then  $K$  is a closed  $T$ -invariant linear subspace of  $H(T)$ .*

*Proof.* Obviously  $K$  is closed in the norm and under scalar multiplication. By the easy argument of [1], p. 368, (ii'),  $\sum_{i=1}^n a_i T_{x_i} \xi$  is in  $K$  whenever  $\xi \in K$ , the  $x_i$  are in  $G$ , and the  $a_i$  are complex; in particular  $K$  is  $T$ -invariant. It remains only to show  $K$  closed under addition.

Let  $\xi$  and  $\eta$  be elements of  $K$ ; let  $L$  and  $M$  be the closed invariant subspaces of  $H(T)$  generated by  $\xi$  and  $\eta$  respectively; and let  $Q$  be the closure of  $L + M$ . By the preceding paragraph

$$(1) \quad L \subset K \quad \text{and} \quad M \subset K.$$

If  $A$  is projection onto  $L^\perp$ ,  $A(M)$  is a dense subspace of  $Q \cap L^\perp$ . So by Mackey's form of Schur's Lemma ([7], Theorem 1.2), the restriction of  $T$  to the invariant subspace  $Q \cap L^\perp$  is equivalent to a subrepresentation of the restriction of  $T$  to  $M$ . This and (1) show that

$$(2) \quad Q \cap L^\perp \subset K.$$

Putting  $\zeta = \xi + \eta$ , we have  $\zeta = \xi' + \eta'$ , where  $\xi' \in L$  and  $\eta' \in Q \cap L^\perp$ . Since  $L$  and  $Q \cap L^\perp$  are orthogonal and  $T$ -invariant,

$$(3) \quad (T_x \zeta, \zeta) = (T_x \xi', \xi') + (T_x \eta', \eta')$$

( $x \in G$ ). By (1) and (2)  $\xi'$  and  $\eta'$  are in  $K$ ; so by (3)  $\zeta \in K$ , and  $K$  is closed under addition.

REMARK 1. If  $A$  is a  $C^*$ -algebra,  $\mathcal{T}(A)$  is defined as the set of all equivalence classes of  $*$ -representations of  $A$ . Exactly the same proof shows that Lemma 1 is valid for  $C^*$ -algebras, provided that we replace functions of positive type by positive functionals, and uniform approximation on compact sets by weak\* approximation.

REMARK 2. According to Lemma 1,  $T$  will be weakly contained in  $\mathcal{S}$  provided  $H(T)$  is generated (under  $T$ ) by those  $\xi$  in  $H(T)$  whose associated functions of positive type are approximated by sums of functions of positive type associated with  $\mathcal{S}$ . For example, we have immediately:

THEOREM 1. Suppose that  $\mathcal{S}_k \subset \mathcal{T}(G)$  and  $\mathcal{S}_k$  weakly contains  $T_k$  ( $k = 1, 2$ ). Then  $\mathcal{S}_1 \otimes \mathcal{S}_2$  weakly contains  $T_1 \otimes T_2$ .

THEOREM 2. The map  $\langle S, T \rangle \rightarrow S \otimes T$  (of  $\mathcal{T}(G) \times \mathcal{T}(G)$  into  $\mathcal{T}(G)$ ) is continuous with respect to the inner hull-kernel topology of  $\mathcal{T}(G)$ .

*Proof.* Let  $S^i \rightarrow S$  and  $T^i \rightarrow T$  in  $\mathcal{T}(G)$ . By the definition of the topology of  $\mathcal{T}(G)$ , we have only to show that the net  $\{S^i \otimes T^i\}$  (and hence by the same argument every subnet of it) weakly contains  $S \otimes T$ . But Theorem 2.2 of [2] clearly shows that the function of positive type associated with each product vector  $\xi \otimes \eta$  in  $H(S) \otimes H(T)$  can be approximated by functions of positive type associated with the  $S^i \otimes T^i$ . Hence by Lemma 1  $S \otimes T$  is weakly contained in  $\{S^i \otimes T^i\}$ .

It should be mentioned that the "easy verification" of the proposition used in the proof of [2], p. 260, Corollary 1, actually requires the above Theorem 1.

2. When does  $S \otimes \bar{T}$  weakly contain  $I$ ? In this section  $G$  is assumed to satisfy the second axiom of countability; and we shall consider only unitary representations acting in a separable space.

Suppose that  $T \in \hat{G}$  and  $S \in \mathcal{T}(G)$ . Is it true that  $S \otimes \bar{T}$  weakly contains  $I$  if and only if  $S$  weakly contains  $T$ ? In general, as we next show, the implication is false in both directions, even if  $S$  is assumed irreducible.

Let  $R$  be the regular representation of  $G$ , and  $T$  some irreducible representation weakly contained in  $R$ . Clearly  $R \cong \bar{R}$ . By [6], Theorem 12.2,  $R \otimes R$  is a multiple of  $R$ . So  $R \otimes \bar{R}$  weakly contains  $I$  if and only if  $R$  does. Choose  $G$  so that  $R$  does not weakly contain  $I$ ; for example  $G$  might be the free group on two generators, or a non-compact connected semisimple Lie group (see [8]). Then  $R \otimes \bar{R}$  does not weakly contain  $I$ , and hence, by Theorem 1, nor does  $T \otimes \bar{T}$ .

For an easy counter-example in the other direction take  $G$  to be the " $ax + b$ " group, and  $T$  to be one of the two infinite-dimensional irreducible representations of  $G$ . Then  $\bar{T} = I \otimes \bar{T}$  weakly contains  $I$  (see [2], Theorem 5.1), but  $I$  does not weakly contain  $T$ . A "better" example, in which  $S \otimes \bar{T}$  weakly contains  $I$  but neither  $S$  nor  $T$  weakly contains the other, will be given in § 3.

However, if  $T$  is finite-dimensional, the answer to the question posed above is affirmative (Theorem 4).

**LEMMA 2.** *If  $\mathcal{S} \subset \mathcal{F}(G)$  and  $\mathcal{S}$  weakly contains a finite-dimensional irreducible unitary representation  $T$  of  $G$ , then  $\mathcal{S} \otimes \bar{T}$  weakly contains  $I$ .*

*Proof.*  $\mathcal{S} \otimes \bar{T}$  weakly contains  $T \otimes \bar{T}$  by Theorem 1. Since  $T$  is finite-dimensional,  $T \otimes \bar{T}$  contains  $I$ .

Here is an interesting consequence of Lemma 2:

**THEOREM 3.** *If the regular representation  $R$  of  $G$  weakly contains some finite-dimensional irreducible representation  $T$  of  $G$ , it weakly contains all unitary representations of  $G$ .*

*Proof.* By Lemma 2  $R \otimes \bar{T}$  weakly contains  $I$ . But by [2], Lemma 4.2,  $R \otimes \bar{T}$  is a multiple of  $R$ . Hence  $R$  weakly contains  $I$ , and the conclusion follows from Godement's remark ([4], p. 77, or [2], p. 260).

**LEMMA 3.** *Let  $T$  be an irreducible finite-dimensional unitary representation of  $G$ . To each  $\delta > 0$ , there is a finite subset  $F$  of  $G$  and an  $\varepsilon > 0$  such that, whenever  $A$  is a positive linear operator on  $H(T)$  satisfying (i)  $\|A\| = 1$  and (ii)  $\|AT_x - T_xA\| < \varepsilon$  for all  $x$  in  $F$ , then  $\|A - E\| < \delta$  ( $E$  being the identity operator on  $H(T)$ ).*

*Proof.* Assume the lemma false. Then there is a  $\delta > 0$  and a net  $\{A_i\}$  of positive operators in  $Q$  such that  $A_iT_x - T_xA_i \xrightarrow{i} 0$  for all  $x$  in  $G$ ; here  $Q$  is the compact set of those positive operators  $A$  on  $H(T)$  for which  $\|A\| = 1$  and  $\|A - E\| \geq \delta$ . Replacing  $\{A_i\}$  by a subnet, we may assume that  $A_i \rightarrow A$  in  $Q$ . Passing to the limit, we deduce that  $AT_x = T_xA$  for all  $x$ , whence  $A = \lambda E$ . Since  $A$  is positive and of norm 1, we must have  $\lambda = 1$ ; but this contradicts  $\|A - E\| \geq \delta$ .

**LEMMA 4.** *Suppose that  $\mathcal{S} \subset \mathcal{F}(G)$ , and  $T$  is a finite-dimensional irreducible unitary representation of  $G$  such that  $\mathcal{S} \otimes \bar{T}$  weakly contains  $I$ . Then  $\mathcal{S}$  weakly contains  $T$ .*

*Proof.* The family of all finite direct sums of elements of  $\mathcal{S}$  weakly contains  $T$  if and only if  $\mathcal{S}$  does; hence we may assume without loss of generality that  $\mathcal{S}$  is closed under finite direct sums. But then  $I$  belongs to the quotient closure of  $\mathcal{S} \otimes \bar{T}$  ([2], Theorem 1.1).

Let  $C$  be a compact subset of  $G$ . For fixed  $\delta > 0$ , choose  $F$  and  $\varepsilon$  as in Lemma 3. Let  $r$  be the dimension of  $H(T)$ ; and put  $C' = (C \cup F) \cup (C \cup F)^{-1}$ .

By [2], Lemma 1.1, there is an  $S$  in  $\mathcal{S}$  and a unit vector  $\zeta$  in  $H(S \otimes \bar{T})$  such that

$$(4) \quad \|(S \otimes \bar{T})_x \zeta - \zeta\| < \frac{\varepsilon}{2r^4}$$

for all  $x$  in  $C'$ . Fixing an orthonormal basis  $\xi_1, \dots, \xi_r$  of  $H(T)$ , let us write  $\zeta = \sum_{i=1}^r \eta_i \otimes \xi_i$  ( $\eta_i \in H(S)$ ), where

$$(5) \quad 1 = \|\zeta\|^2 = \sum_{i=1}^r \|\eta_i\|^2.$$

If the matrix of  $T_x$  in the basis  $\{\xi_i\}$  is  $\{\tau_{ij}(x)\}$ , we have  $\bar{T}_x \xi_i = \sum_{j=1}^r \overline{\tau_{ji}(x)} \xi_j$ . So  $(S \otimes \bar{T})_x \zeta = \sum_j (\sum_i \overline{\tau_{ji}(x)} S_x \eta_i) \otimes \xi_j$ , whence

$$(6) \quad \|(S \otimes \bar{T})_x \zeta - \zeta\|^2 = \sum_j \left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\|^2.$$

By (4) and (6),

$$(7) \quad \left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\| < \frac{\varepsilon}{2r^4}$$

( $x \in C', j = 1, \dots, r$ ). From (7) and the unitariness of  $\tau(x)$ ,

$$(8) \quad \begin{aligned} & \left\| S_x \eta_k - \sum_j \tau_{jk}(x) \eta_j \right\| \\ & \leq \sum_j |\tau_{jk}(x)| \left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\| \\ & < \frac{\varepsilon}{2r^3}. \end{aligned}$$

Let  $A$  be the linear map of  $H(T)$  into  $H(S)$  sending  $\xi_i$  into  $\eta_i$  ( $i = 1, \dots, r$ ). Then (8) gives

$$(9) \quad \|S_x A - A T_x\| < \frac{\varepsilon}{2r^2} \quad (x \in C').$$

From this and the symmetry of  $C'$ ,

$$(10) \quad \|A^* S_x - T_x A^*\| < \frac{\varepsilon}{2r^2} \quad (x \in C').$$

By (5),  $\|A\| = \|A^*\| \leq r$  and also

$$(11) \quad \|A^* A\| \geq \frac{1}{r}.$$

Hence, denoting  $A^* A / \|A^* A\|$  by  $B$ , we obtain from (9) and (10)  $\|B T_x - T_x B\| < \varepsilon$  ( $x \in C'$ ). Since  $B$  is positive,  $\|B\| = 1$ , and  $F \subset C'$ ,

Lemma 3 asserts that  $\|B - E\| < \delta$ . From this, setting  $\eta'_i = \eta_i/\|A\|$ , we get

$$(12) \quad |(\eta'_i, \eta'_j) - \delta_{ij}| < \delta$$

for all  $i, j$ . Let  $\varphi(x) = (S_x \eta'_i, \eta'_i)(x \in G)$ . By (8) and (11)  $\|S_x \eta'_i - \sum_j \tau_{j1}(x) \eta'_j\| < \varepsilon/2r^2$ . Combining this with (12) we have for  $x$  in  $C$

$$\begin{aligned} |\varphi(x) - \tau_{11}(x)| &\leq \left| \left( \left( S_x \eta'_i - \sum_j \tau_{j1}(x) \eta'_j \right), \eta'_i \right) \right| \\ &\quad + \left| \left( \sum_j \tau_{j1}(x) \eta'_j, \eta'_i \right) - \tau_{11}(x) \right| \\ &\leq \sum_j |\tau_{j1}(x)| |(\eta'_j, \eta'_i) - \delta_{j1}| + \frac{\varepsilon}{2r^2} \|\eta'_i\| \\ &\leq r\delta + \frac{\varepsilon}{2r}, \end{aligned}$$

which is as small as we wish. Thus we have an  $S$  in  $\mathcal{S}$  and a function  $\varphi$  of positive type associated with  $S$  which differs from  $\tau_{11}$  on  $C$  by an arbitrarily small quantity. So  $\mathcal{S}$  weakly contains  $T$ .

Combining Lemmas 2 and 4 we get:

**THEOREM 4.** *Let  $\mathcal{S}$  be a family of unitary representations of  $G$  and  $T$  a finite-dimensional irreducible unitary representation of  $G$ . Then  $\mathcal{S}$  weakly contains  $T$  if and only if  $\mathcal{S} \otimes \bar{T}$  weakly contains  $I$ .*

As a corollary we mention the following weak "Frobenius-like" proposition. As usual,  $U^S$  denotes the representation of  $G$  induced from the representation  $S$  of a subgroup.

**COROLLARY.** *Let  $K$  be a closed subgroup of  $G$ , and  $J$  and  $I$  the identity representations of  $K$  and  $G$  respectively. We assume that  $U^J$  weakly contains  $I$ . If  $\mathcal{S} \subset \mathcal{S}(K)$ ,  $T$  is a finite-dimensional irreducible unitary representation of  $G$ , and  $\mathcal{S}$  weakly contains some irreducible component of  $T|_K$ , then  $\{U^S | S \in \mathcal{S}\}$  weakly contains  $T$ .*

*Proof.* By Theorem 4  $\mathcal{S} \otimes \bar{T}|_K$  weakly contains  $J$ . Hence by [2], Theorem 4.2,  $\{U^{S \otimes \bar{T}|_K} | S \in \mathcal{S}\}$  weakly contains  $U^J$ . By hypothesis the latter weakly contains  $I$ ; so  $\{U^{S \otimes \bar{T}|_K} | S \in \mathcal{S}\}$  weakly contains  $I$ . But by [2], Lemma 4.2,  $U^{S \otimes \bar{T}|_K} \cong U^S \otimes \bar{T}$ . Hence another application of Theorem 4 gives the required conclusion.

**3. A counter-example.** Let  $G$  be the proper Euclidean group in three-dimensional real space  $R^3$ . We observe that the hull-kernel

topology of  $\hat{G}$  is  $T_1$  (i.e. points are closed). Indeed, the results of [5] show that  $T_f$  is completely continuous whenever  $T \in \hat{G}$  and  $f \in L_1(G)$ . So, by [1], Lemma 1.11,  $\hat{G}$  is  $T_1$ . Thus, if  $S$  and  $T$  are inequivalent elements of  $\hat{G}$ , neither weakly contains the other. We shall now construct two inequivalent elements  $S$  and  $T$  of  $\hat{G}$  such that  $S \otimes \bar{T}$  weakly contains  $I$  (see the beginning of § 2).

Let  $N$  and  $K$  be the translation and rotation subgroups of  $G$  respectively;  $\tau_u$  will denote translation by  $u$ :  $\tau_u(v) = u + v$  ( $u, v \in \mathbb{R}^3$ ). Let  $\chi$  be the fixed character of  $N$  defined by  $\chi(\tau_u) = e^{iu_1}$ . The "stationary subgroup" for  $\chi$  (consisting of those  $\sigma$  in  $G$  such that  $\chi(\sigma\tau_u\sigma^{-1}) = \chi(\tau_u)$  for all  $u$ ) is  $Z = HN$ , where  $H = \{\rho \in K \mid \rho(1, 0, 0) = (1, 0, 0)\}$ . Thus, by [6], Theorem 14.1, to each character  $\varphi$  of the Abelian group  $H$  we get an irreducible representation  $T^\varphi$  of  $G$ , namely, that induced from the character  $\psi$  of  $Z$ , where

$$(13) \quad \psi(\rho\tau_u) = \varphi(\rho)\chi(\tau_u) \quad (\rho \in H, u \in \mathbb{R}^3).$$

Further, if  $\varphi$  and  $\varphi'$  are distinct characters of  $H$ ,  $T^\varphi$  and  $T^{\varphi'}$  are inequivalent.

Now let  $\varphi$  and  $\varphi'$  be distinct characters of  $H$ . Let  $0 < \theta < \pi/2$  and let  $\rho$  be the element of  $K$  consisting of rotation through an angle  $\theta$  about the third axis. We verify easily that  $Z \cap \rho Z \rho^{-1} = N$ . Hence by [3], Theorem 5.4 (the 'weak containment' version of Mackey's Kronecker Product Theorem),  $T^\varphi \otimes (T^{\varphi'})^-$  weakly contains the representation of  $G$  induced from the character  $\chi_\theta$  of  $N$  given by  $\chi_\theta(\tau_u) = \chi(\tau_{\rho(u)})\chi(\tau_u)$ . (Here  $(T^{\varphi'})^-$  is the adjoint of  $T^{\varphi'}$ ). Since this is true whenever  $0 < \theta < \pi/2$ , we can use [2], Theorem 4.2, to pass to the limit as  $\theta \rightarrow 0$ ; we then conclude that  $T^\varphi \otimes (T^{\varphi'})^-$  weakly contains  $U^{\chi_0}$ , where  $\chi_0$  is the identity character of  $N$ . But  $U^{\chi_0}$  is obtained by lifting to  $G$  the regular representation of the compact group  $K$ ; hence it contains  $I$  as a direct summand. Thus we conclude that  $T^\varphi \otimes (T^{\varphi'})^-$  weakly contains  $I$ . This is the desired example, since we have already observed that  $T^\varphi$  and  $T^{\varphi'}$  are inequivalent irreducible representations of  $G$ .

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