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# WEAK CONTAINMENT AND KRONECKER PRODUCTS OF GROUP REPRESENTATIONS

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**Introduction.** Throughout this paper $G$ is a fixed locally compact group. Let us recall some concepts bearing on the representation theory of $G$. The family of all unitary equivalence classes of unitary representations of $G$ will be called $\mathscr{T}(G)$. A function $\varphi$ of positive type on $G$ is *associated* with a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if there is an $S$ in $\mathscr{S}$, and a vector $\xi$ in the space $H(S)$ of $S$, such that $\varphi(x) = (S_x\xi, \xi)$ for all $x$ in $G$. An element $T$ of $\mathscr{T}(G)$ is *weakly contained* in a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if every function of positive type on $G$ associated with $T$ can be approximated uniformly on compact sets by sums of functions of positive type associated with $\mathscr{S}$. The notion of weak containment leads to that of the *inner hull-kernel topology* of $\mathscr{T}(G)$: A net $\{T^i\}$ of elements of $\mathscr{T}(G)$ converges to $T$ in this topology if and only if every subnet of $\{T^i\}$ weakly contains $T$. Relativized to the subset $\hat{G}$ of $\mathscr{T}(G)$ consisting of the irreducible representations of $G$, this topology becomes the ordinary *hull-kernel topology* of $\hat{G}$. (For these notions and facts see [1] and [2]).

If $H$ is a Hilbert space, the *adjoint space* $\bar{H}$ of $H$ can be defined as the Hilbert space whose underlying set is the same as that of $H$, and which is conjugate-isomorphic with $H$ under the identity map. If $T$ is a unitary representation of $G$, the *adjoint representation* $\bar{T}$ is defined by the requirements: $H(\bar{T}) = H(T)^-$, $\bar{T}_x = T_x (x \in G)$. The *Kronecker product* $S \otimes T$ of two unitary representations $S$ and $T$ of $G$ is that representation whose space is $H(S) \otimes H(T)$, and for which $(S \otimes T)_x(\xi \otimes \eta) = (S_x\xi) \otimes (T_x\eta)$. We can also describe the Kronecker product $S \otimes \bar{T}$ as follows: $H(S \otimes \bar{T})$ is the Hilbert space of all Hilbert-Schmidt operators on $H(T)$ to $H(S)$, and $(S \otimes \bar{T})_x(A) = S_x A T_x^{-1}$.

If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{T} \subset \mathscr{T}(G)$, let $\mathscr{S} \otimes \mathscr{T}$ denote $\{S \otimes T \mid S \in \mathscr{S}, T \in \mathscr{T}\}$.

Throughout this paper $I$ will be the one-dimensional identity representation of $G$. It is well known and easily verified that if $S$ and $T$ are finite-dimensional unitary representations of $G$ and $T$ is irreducible, $S \otimes \bar{T}$ contains $I$ if and only if $S$ contains $T$. Can this be generalized to the case where $S$ and $T$ are infinite-dimensional and 'containment' is replaced by 'weak containment'? The main object of this note is to answer this question affirmatively for the case that $S$ is infinite-dimensional but $T$ is still finite-dimensional (Theorem 4). In preparation for this we shall show (Theorem 2) that the Kronecker product oper-

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ation is continuous with respect to the inner hull-kernel topology of $\mathscr{T}(G)$.

Another by-product of the main result is the following strenthening (Theorem 3) of a remark of Godement ([4], p. 77): If the regular representation $R$ of $G$ weakly contains some finite-dimensional irreducible unitary representation of $G$, then $R$ weakly contains all unitary representations of $G$.

## 1. The continuity of the Kronecker product.

LEMMA 1. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$ and $T \in \mathscr{T}(G)$; and let $K$ be the set of all those $\xi$ in $H(T)$ such that the function $\varphi$ defined by $\varphi(x) = (T_x\xi, \xi)(x \in G)$ can be approximated, uniformly on compact sets, by sums of functions of positive type associated with $\mathscr{S}$. Then $K$ is a closed $T$-invariant linear subspace of $H(T)$.*

*Proof.* Obviously $K$ is closed in the norm and under scalar multiplication. By the easy argument of [1], p. 368, (ii′), $\sum_{i=1}^{n} a_i T_{x_i}\xi$ is in $K$ whenever $\xi \in K$, the $x_i$ are in $G$, and the $a_i$ are complex; in particular $K$ is $T$-invariant. It remains only to show $K$ closed under addition.

Let $\xi$ and $\eta$ be elements of $K$; let $L$ and $M$ be the closed invariant subspaces of $H(T)$ generated by $\xi$ and $\eta$ respectively; and let $Q$ be the closure of $L + M$. By the preceding paragraph

$$L \subset K \quad \text{and} \quad M \subset K . \tag{1}$$

If $A$ is projection onto $L^{\perp}$, $A(M)$ is a dense subspace of $Q \cap L^{\perp}$. So by Mackey's form of Schur's Lemma ([7], Theorem 1.2), the restriction of $T$ to the invariant subspace $Q \cap L^{\perp}$ is equivalent to a subrepresentation of the restriction of $T$ to $M$. This and (1) show that

$$Q \cap L^{\perp} \subset K . \tag{2}$$

Putting $\zeta = \xi + \eta$, we have $\zeta = \xi' + \eta'$, where $\xi' \in L$ and $\eta' \in Q \cap L^{\perp}$. Since $L$ and $Q \cap L^{\perp}$ are orthogonal and $T$-invariant,

$$(T_x\zeta, \zeta) = (T_x\xi', \xi') + (T_x\eta', \eta') \tag{3}$$

$(x \in G)$. By (1) and (2) $\xi'$ and $\eta'$ are in $K$; so by (3) $\zeta \in K$, and $K$ is closed under addition.

REMARK 1. If $A$ is a $C^*$-algebra, $\mathscr{T}(A)$ is defined as the set of all equivalence classes of *-representations of $A$. Exactly the same proof shows that Lemma 1 is valid for $C^*$-algebras, provided that we replace functions of positive type by positive functionals, and uniform approximation on compact sets by weak* approximation.

REMARK 2. According to Lemma 1, $T$ will be weakly contained in $\mathscr{S}$ provided $H(T)$ is generated (under $T$) by those $\xi$ in $H(T)$ whose associated functions of positive type are approximated by sums of functions of positive type associated with $\mathscr{S}$. For example, we have immediately:

THEOREM 1. *Suppose that* $\mathscr{S}_k \subset \mathscr{T}(G)$ *and* $\mathscr{S}_k$ *weakly contains* $T_k\,(k = 1, 2)$. *Then* $\mathscr{S}_1 \otimes \mathscr{S}_2$ *weakly contains* $T_1 \otimes T_2$.

THEOREM 2. *The map* $\langle S, T\rangle \rightarrow S \otimes T$ *(of* $\mathscr{T}(G) \times \mathscr{T}(G)$ *into* $\mathscr{T}(G)$*) is continuous with respect to the inner hull-kernel topology of* $\mathscr{T}(G)$.

*Proof.* Let $S^i \rightarrow S$ and $T^i \rightarrow T$ in $\mathscr{T}(G)$. By the definition of the topology of $\mathscr{T}(G)$, we have only to show that the net $\{S^i \otimes T^i\}$ (and hence by the same argument every subnet of it) weakly contains $S \otimes T$. But Theorem 2.2 of [2] clearly shows that the function of positive type associated with each product vector $\xi \otimes \eta$ in $H(S) \otimes H(T)$ can be approximated by functions of positive type associated with the $S^i \otimes T^i$. Hence by Lemma 1 $S \otimes T$ is weakly contained in $\{S^i \otimes T^i\}$.

It should be mentioned that the "easy verification" of the proposition used in the proof of [2], p. 260, Corollary 1, actually requires the above Theorem 1.

**2. When does $S \otimes \bar{T}$ weakly contain $I$?** In this section $G$ is assumed to satisfy the second axiom of countability; and we shall consider only unitary representations acting in a separable space.

Suppose that $T \in \hat{G}$ and $S \in \mathscr{T}(G)$. Is it true that $S \otimes \bar{T}$ weakly contains $I$ if and only if $S$ weakly contains $T$? In general, as we next show, the implication is false in both directions, even if $S$ is assumed irreducible.

Let $R$ be the regular representation of $G$, and $T$ some irreducible representation weakly contained in $R$. Clearly $R \cong \bar{R}$. By [6], Theorem 12.2, $R \otimes R$ is a multiple of $R$. So $R \otimes \bar{R}$ weakly contains $I$ if and only if $R$ does. Choose $G$ so that $R$ does not weakly contain $I$; for example $G$ might be the free group on two generators, or a non-compact connected semisimple Lie group (see [8]). Then $R \otimes \bar{R}$ does not weakly contain $I$, and hence, by Theorem 1, nor does $T \otimes \bar{T}$.

For an easy counter-example in the other direction take $G$ to be the "$ax + b$" group, and $T$ to be one of the two infinite-dimensional irreducible representations of $G$. Then $\bar{T} = I \otimes \bar{T}$ weakly contains $I$ (see [2], Theorem 5.1), but $I$ does not weakly contain $T$. A "better" example, in which $S \otimes \bar{T}$ weakly contains $I$ but neither $S$ nor $T$ weakly contains the other, will be given in § 3.

However, if $T$ is finite-dimensional, the answer to the question posed above is affirmative (Theorem 4).

LEMMA 2. *If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{S}$ weakly contains a finite-dimensional irreducible unitary representation $T$ of $G$, then $\mathscr{S} \otimes \bar{T}$ weakly contains $I$.*

*Proof.* $\mathscr{S} \otimes \bar{T}$ weakly contains $T \otimes \bar{T}$ by Theorem 1. Since $T$ is finite-dimensional, $T \otimes \bar{T}$ contains $I$.

Here is an interesting consequence of Lemma 2:

THEOREM 3. *If the regular representation $R$ of $G$ weakly contains some finite-dimensional irreducible representation $T$ of $G$, it weakly contains all unitary representations of $G$.*

*Proof.* By Lemma 2 $R \otimes \bar{T}$ weakly contains $I$. But by [2], Lemma 4.2, $R \otimes \bar{T}$ is a multiple of $R$. Hence $R$ weakly contains $I$, and the conclusion follows from Godement's remark ([4], p. 77, or [2], p. 260).

LEMMA 3. *Let $T$ be an irreducible finite-dimensional unitary representation of $G$. To each $\delta > 0$, there is a finite subset $F$ of $G$ and an $\varepsilon > 0$ such that, whenever $A$ is a positive linear operator on $H(T)$ satisfying* (i) $\|A\| = 1$ *and* (ii) $\|AT_x - T_xA\| < \varepsilon$ *for all $x$ in $F$, then $\|A - E\| < \delta$ ($E$ being the identity operator on $H(T)$).*

*Proof.* Assume the lemma false. Then there is a $\delta > 0$ and a net $\{A_i\}$ of positive operators in $Q$ such that $A_iT_x - T_xA_i \xrightarrow[i]{} 0$ for all $x$ in $G$; here $Q$ is the compact set of those positive operators $A$ on $H(T)$ for which $\|A\| = 1$ and $\|A - E\| \geqq \delta$. Replacing $\{A_i\}$ by a subnet, we may assume that $A_i \rightarrow A$ in $Q$. Passing to the limit, we deduce that $AT_x = T_xA$ for all $x$, whence $A = \lambda E$. Since $A$ is positive and of norm 1, we must have $\lambda = 1$; but this contradicts $\|A - E\| \geqq \delta$.

LEMMA 4. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$, and $T$ is a finite-dimensional irreducible unitary representation of $G$ such that $\mathscr{S} \otimes \bar{T}$ weakly contains $I$. Then $\mathscr{S}$ weakly contains $T$.*

*Proof.* The family of all finite direct sums of elements of $\mathscr{S}$ weakly contains $T$ if and only if $\mathscr{S}$ does; hence we may assume without loss of generality that $\mathscr{S}$ is closed under finite direct sums. But then $I$ belongs to the quotient closure of $\mathscr{S} \otimes \bar{T}$ ([2], Theorem 1.1).

Let $C$ be a compact subset of $G$. For fixed $\delta > 0$, choose $F$ and $\varepsilon$ as in Lemma 3. Let $r$ be the dimension of $H(T)$; and put $C' = (C \cup F) \cup (C \cup F)^{-1}$.

By [2], Lemma 1.1, there is an $S$ in $\mathscr{S}$ and a unit vector $\zeta$ in $H(S \otimes \bar{T})$ such that

$$\|(S \otimes \bar{T})_x \zeta - \zeta\| < \frac{\varepsilon}{2r^4} \tag{4}$$

for all $x$ in $C'$. Fixing an orthonormal basis $\xi_1, \cdots, \xi_r$ of $H(T)$, let us write $\zeta = \sum_{i=1}^r \eta_i \otimes \xi_i (\eta_i \in H(S))$, where

$$1 = \|\zeta\|^2 = \sum_{i=1}^r \|\eta_i\|^2 . \tag{5}$$

If the matrix of $T_x$ in the basis $\{\xi_i\}$ is $\{\tau_{ij}(x)\}$, we have $\bar{T}_x \xi_i = \sum_{j=1}^r \overline{\tau_{ji}(x)} \xi_j$. So $(S \otimes \bar{T})_x \zeta = \sum_j (\sum_i \overline{\tau_{ji}(x)} S_x \eta_i) \otimes \xi_j$, whence

$$\|(S \otimes \bar{T})_x \zeta - \zeta\|^2 = \sum_j \left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\|^2 . \tag{6}$$

By (4) and (6),

$$\left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\| < \frac{\varepsilon}{2r^4} \tag{7}$$

$(x \in C', j = 1, \cdots, r)$. From (7) and the unitariness of $\tau(x)$,

$$\begin{aligned} &\left\| S_x \eta_k - \sum_j \tau_{jk}(x) \eta_j \right\| \\ &\leqq \sum_j |\tau_{jk}(x)| \left\| \left( \sum_i \overline{\tau_{ji}(x)} S_x \eta_i \right) - \eta_j \right\| \\ &< \frac{\varepsilon}{2r^3} . \end{aligned} \tag{8}$$

Let $A$ be the linear map of $H(T)$ into $H(S)$ sending $\xi_i$ into $\eta_i (i = 1, \cdots, r)$. Then (8) gives

$$\|S_x A - A T_x\| < \frac{\varepsilon}{2r^2} \qquad (x \in C') . \tag{9}$$

From this and the symmetry of $C'$,

$$\|A^* S_x - T_x A^*\| < \frac{\varepsilon}{2r^2} \qquad (x \in C') . \tag{10}$$

By (5), $\|A\| = \|A^*\| \leqq r$ and also

$$\|A^* A\| \geqq \frac{1}{r} . \tag{11}$$

Hence, denoting $A^*A/\|A^*A\|$ by $B$, we obtain from (9) and (10) $\|BT_x - T_x B\| < \varepsilon$ $(x \in C')$. Since $B$ is positive, $\|B\| = 1$, and $F \subset C'$, Lemma 3 asserts that $\|B - E\| < \delta$. From this, setting $\eta'_i = \eta_i/\|A\|$, we get

$$(12) \qquad |(\eta'_i, \eta'_j) - \delta_{ij}| < \delta$$

for all $i$, $j$. Let $\varphi(x) = (S_x\eta'_1, \eta'_1)(x \in G)$. By (8) and (11) $\|S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\| < \varepsilon/2r^2$. Combining this with (12) we have for $x$ in $C$

$$\begin{aligned}|\varphi(x) - \tau_{11}(x)| &\leqq \left|\left(\left(S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\right), \eta'_1\right)\right| \\ &\quad + \left|\left(\sum_j \tau_{j1}(x)\eta'_j, \eta'_1\right) - \tau_{11}(x)\right| \\ &\leqq \sum_j |\tau_{j1}(x)|\,|(\eta'_j, \eta'_1) - \delta_{j1}| + \frac{\varepsilon}{2r^2}\|\eta'_1\| \\ &\leqq r\delta + \frac{\varepsilon}{2r},\end{aligned}$$

which is as small as we wish. Thus we have an $S$ in $\mathcal{S}$ and a function $\varphi$ of positive type associated with $S$ which differs from $\tau_{11}$ on $C$ by an arbitrarily small quantity. So $\mathcal{S}$ weakly contains $T$.

Combining Lemmas 2 and 4 we get:

THEOREM 4. *Let $\mathcal{S}$ be a family of unitary representations of $G$ and $T$ a finite-dimensional irreducible unitary representation of $G$. Then $\mathcal{S}$ weakly contains $T$ if and only if $\mathcal{S} \otimes \bar{T}$ weakly contains $I$.*

As a corollary we mention the following weak "Frobenius-like" proposition. As usual, $U^S$ denotes the representation of $G$ induced from the representation $S$ of a subgroup.

COROLLARY. *Let $K$ be a closed subgroup of $G$, and $J$ and $I$ the identity representations of $K$ and $G$ respectively. We assume that $U^J$ weakly contains $I$. If $\mathcal{S} \subset \mathcal{T}(K)$, $T$ is a finite-dimensional irreducible unitary representation of $G$, and $\mathcal{S}$ weakly contains some irreducible component of $T|K$, then $\{U^S \mid S \in \mathcal{S}\}$ weakly contains $T$.*

*Proof.* By Theorem 4 $\mathcal{S} \otimes \bar{T}|K$ weakly contains $J$. Hence by [2], Theorem 4.2, $\{U^{S \otimes \bar{T}|K} \mid S \in \mathcal{S}\}$ weakly contains $U^J$. By hypothesis the latter weakly contains $I$; so $\{U^{S \otimes \bar{T}|K} \mid S \in \mathcal{S}\}$ weakly contains $I$. But by [2], Lemma 4.2, $U^{S \otimes \bar{T}|K} \cong U^S \otimes \bar{T}$. Hence another application of Theorem 4 gives the required conclusion.

**3. A counter-example.** Let $G$ be the proper Euclidean group in three-dimensional real space $R^3$. We observe that the hull-kernel topology of $\hat{G}$ is $T_1$ (i.e. points are closed). Indeed, the results of [5] show that $T_f$ is completely continuous whenever $T \in \hat{G}$ and $f \in L_1(G)$. So, by [1], Lemma 1.11, $\hat{G}$ is $T_1$. Thus, if $S$ and $T$ are inequivalent elements of $\hat{G}$, neither weakly contains the other. We shall now construct two inequivalent elements $S$ and $T$ of $\hat{G}$ such that $S \otimes \bar{T}$ weakly contains $I$ (see the beginning of § 2).

Let $N$ and $K$ be the translation and rotation subgroups of $G$ respectively; $\tau_u$ will denote translation by $u$: $\tau_u(v) = u + v(u, v \in R^3)$. Let $\chi$ be the fixed character of $N$ defined by $\chi(\tau_u) = e^{iu_1}$. The "stationary subgroup" for $\chi$ (consisting of those $\sigma$ in $G$ such that $\chi(\sigma\tau_u\sigma^{-1}) = \chi(\tau_u)$ for all $u$) is $Z = HN$, where $H = \{\rho \in K \mid \rho(1, 0, 0) = (1, 0, 0)\}$. Thus, by [6], Theorem 14.1, to each character $\varphi$ of the Abelian group $H$ we get an irreducible representation $T^\varphi$ of $G$, namely, that induced from the character $\psi$ of $Z$, where

$$\psi(\rho\tau_u) = \varphi(\rho)\chi(\tau_u) \qquad (\rho \in H, u \in R^3) . \tag{13}$$

Further, if $\varphi$ and $\varphi'$ are distinct characters of $H$, $T^\varphi$ and $T^{\varphi'}$ are inequivalent.

Now let $\varphi$ and $\varphi'$ be distinct characters of $H$. Let $0 < \theta < \pi/2$ and let $\rho$ be the element of $K$ consisting of rotation through an angle $\theta$ about the third axis. We verify easily that $Z \cap \rho Z \rho^{-1} = N$. Hence by [3], Theorem 5.4 (the 'weak containment' version of Mackey's Kronecker Product Theorem), $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains the representation of $G$ induced from the character $\chi_\theta$ of $N$ given by $\chi_\theta(\tau_u) = \chi(\tau_{\rho(u)})\chi(\tau_u)$. (Here $(T^{\varphi'})^-$ is the adjoint of $T^{\varphi'}$). Since this is true whenever $0 < \theta < \pi/2$, we can use [2], Theorem 4.2, to pass to the limit as $\theta \to 0$; we then conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains $U^{\chi_0}$, where $\chi_0$ is the identity character of $N$. But $U^{\chi_0}$ is obtained by lifting to $G$ the regular representation of the compact group $K$; hence it contains $I$ as a direct summand. Thus we conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains $I$. This is the desired example, since we have already observed that $T^\varphi$ and $T^{\varphi'}$ are inequivalent irreducible representations of $G$.

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