MAXIMUM MODULUS ALGEBRAS AND LOCAL APPROXIMATION IN $C^n$

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1. In [4] W. Rudin established an important result concerning maximum modulus algebras $A$ of continuous complex-valued functions defined on the closure $K$ of a Jordan domain in the complex plane (see also [5]). Rudin’s result states, under the assumptions (a) $A$ contains a function $\psi$ which is schlicht on $K$, and (b) $A$ contains a non-constant function $\phi$ which is analytic in the interior, int $K$, of $K$, that every function in $A$ is analytic in int $K$. In this note we will establish conditions under which assumption (b) alone yields the desired conclusion in a slightly more general setting. We assume that $K$ is a compact set, with interior, of a Riemann surface, but also assume that int $K$ is essentially open in the maximal ideal space $\Sigma_A$ of $A$ ($A$ being regarded as a Banach algebra with the sup norm $||f|| = \sup_{p \in K} |f(p)|$; see [2]). This means that each point of int $K$, excepting a set of points having no limit point in int $K$, has a neighborhood in int $K$ which is open in $\Sigma_A$ under the natural mapping of $K$ into $\Sigma_A$. Under these assumptions it is easy to show, using the Local Maximum Modulus Principle of H. Rossi [3; Theorem 6.1] and Rudin’s results, that (b) is sufficient to guarantee that $A$ consists only of analytic functions. Our main purpose, however, is to establish the result by a geometric method, independent of Rudin’s work, which is based on an appropriate local approximation in $C^n$. Unfortunately the geometric approach being used here only allows us to make the desired conclusion for twice continuously differentiable functions in $A$ whereas the use of Rubin’s results would give a proof valid for any function in $A$. However it is hoped that our method will be of some interest in itself.

The basic idea of the proof is as follows. For simplicity let $K$ be the unit circle $\{z \in \mathbb{C}: |z| \leq 1\}$ in the complex plane, and let $f$ and $g$ be nonconstant functions in the maximum modulus algebra $A$. Suppose that $\Sigma_A = K$. Use $f$ and $g$ to map $K$ into $C^2$ (the space of 2 complex variables) in the obvious way. If $f$ and $g$ are twice continuously differentiable in the neighborhood of a given point in int $K$ then the image of this neighborhood in $C^2$ will be a two (real) dimensional surface possessing a tangent plane at the image $p$ of the point. Let $\pi$ be the two (real) dimensional tangent plane to this surface at $p$. If this plane is nonanalytic (Definition 1) then we can find a polynomial in the coordinates $w_1$ and $w_2$ of $C^2$ which locally peaks [3] at $p$ when...
restricted to the surface. The results of Rossi and the Arens-Calderon Theorem [1] then show that this will contradict the maximum modulus property. Thus \( \pi \) cannot be nonanalytic. This gives a relation between the complex derivatives of \( f \) and \( g \) which, in particular, implies that both functions are analytic at the pre-image of \( p \) if one of them is analytic there. In § 2 the essential geometric lemma is established and in § 3 it is used to prove the main result.

2. Let \( F: M \to \mathbb{R}^n \) be an immersion (a regular map in the sense of Whitney [6]) of a two (real) dimensional twice continuously differentiable manifold \( M \) into real Euclidean \( n \)-space \( \mathbb{R}^n \). Let \( p \in M \) and let \((U, h)\) give local coordinates about \( p \), where \( U \) is an open set in \( M \), and \( h \) is a homeomorphism from \( U \) onto \( D = \{(u, v) \in \mathbb{R}^2; \ u^2 + v^2 < 1\} \), with \( h(p) = (0, 0) \). If \( x_j (j = 1, \cdots, n) \) is a coordinate function in \( \mathbb{R}^n \) then the functions \( \phi_j(u, v) = x_j \circ F \circ h^{-1}(u, v) \) \((j = 1, \cdots, n)\) are differentiable and give a map \( \Phi: D \to \mathbb{R}^n \) defined by \( \Phi(u, v) = (\phi_1(u, v), \cdots, \phi_n(u, v)) \). Since \( F \) is an immersion, the 2 by \( n \) matrix

\[
\begin{pmatrix}
\frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\
\vdots & \vdots \\
\frac{\partial \phi_n}{\partial u} & \frac{\partial \phi_n}{\partial v}
\end{pmatrix}
\]

has rank 2 and the mapping \( \Phi \) is one-to-one in some disc \( V = \{(u, v) \in \mathbb{R}^2; \ u^2 + v^2 < r^2 < 1\} \). Further, the set \( \Phi(V) \) is a surface element having a tangent plane at \( \Phi(0, 0) \).

We can suppose for our purposes that \( \Phi(0, 0) \) is the origin \( 0 \) in \( \mathbb{R}^n \). The tangent plane \( \pi \) to \( \Phi(V) \) at \( 0 \) is then given parametrically by

\[
x_j = \frac{\partial \phi_j}{\partial u} u + \frac{\partial \phi_j}{\partial v} v \quad (j = 1, \cdots, n),
\]

where the derivatives are evaluated at \( u = v = 0 \). A change of local parameters from \( u \) and \( v \) to \( u' = u'(u, v) \) and \( v' = v'(u, v) \) with \( \partial(u', v')/\partial(u, v) \neq 0 \) (the inverse transformation being given by \( u = u(u', v') \) and \( v = v(u', v') \) in some neighborhood of \( u = v = 0 \)) yield new functions \( \phi_j(u', v') = \phi_j(u(u', v'), v(u', v')) \) and a new parametrization of the tangent plane, namely,

\[
x_j = \frac{\partial \phi_j}{\partial u'} u' + \frac{\partial \phi_j}{\partial v'} v' = \left( \frac{\partial \phi_j}{\partial u} \frac{\partial u}{\partial u'} + \frac{\partial \phi_j}{\partial v} \frac{\partial u}{\partial u'} \right) u' + \left( \frac{\partial \phi_j}{\partial u} \frac{\partial v}{\partial v'} + \frac{\partial \phi_j}{\partial v} \frac{\partial v}{\partial v'} \right) v' \quad (j = 1, \cdots, n).
\]
Note that the rank of the matrix \((\partial \phi_j/\partial u, \partial \phi_j/\partial v)\) is the same as that of \((\partial \phi_j/\partial u', \partial \phi_j/\partial v')\) since \(\partial(u', v')/\partial(u, v) \neq 0\).

Now \(u\) and \(v\) parametrize both the surface element \(\Phi(V)\) and the tangent plane (given by (2.1)). Let \(\eta_j\) and \(\eta'_j\) \((j = 1, 2, \cdots n)\) denote the coordinates in \(R^n\) of the points \(B\) and \(B'\) on \(\pi\) and \(\Phi(V)\), respectively, corresponding to the parameters \(u\) and \(v\) \((u^2 + v^2 < r^2)\). For sufficiently small \(u\) and \(v\),

\[
\eta'_j = \frac{\partial \phi_j}{\partial u} u + \frac{\partial \phi_j}{\partial v} v + \frac{1}{2} \left( \frac{\partial^2 \phi_j}{\partial u^2} u^2 + 2 \frac{\partial^2 \phi_j}{\partial u \partial v} uv + \frac{\partial^2 \phi_j}{\partial v^2} v^2 \right)
\]

where the first derivatives are evaluated at \(u = v = 0\) and the second derivatives are evaluated at \(u' = \theta u, v' = \theta v\) for some \(\theta\) satisfying \(0 < \theta < 1\). Since \(M\) is twice continuously differentiable, the second derivatives of \(\phi_j\) are bounded in absolute value in some sufficiently small neighborhood of \((0, 0)\) and we obtain

\[
\sum_j (\eta_j - \eta'_j)^2 \leq K(|u| + |v|)^i
\]

and so

\[
(2.2) \quad |\eta_j - \eta'_j| \leq L(|u| + |v|)^i
\]

where \(K\) and \(L\) are constants depending on these bounds and on \(n\), and \(u\) and \(v\) are sufficiently small. These estimates will be used later.

Suppose now that \(n = 2m\). One can define complex coordinates \(w_j = x_{2j-1} + ix_{2j}\) making \(R^n\) into complex Euclidean space \(C^m\). Also the \((u, v)\)-plane can be formally complexified by writing \(z = u + iv, \bar{z} = u - iv\). We then have a mapping \(\Psi: V \rightarrow C^m\) defined by \(\Psi(z, \bar{z}) = (w_1, \cdots, w_m)\) where

\[
w_j = \Psi_j(z, \bar{z}) = \phi_{2j-1} \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) + i \phi_{2j} \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right)
\]

\((j = 1, \cdots m)\).

An elementary computation shows that in this formalism the tangent plane \(\pi\) to \(\Psi(V)\) at the origin \(0\) is given parametrically by

\[
w_j = \frac{\partial \Psi_j}{\partial z} z + \frac{\partial \Psi_j}{\partial \bar{z}} \bar{z} \quad (j = 1, \cdots, m)
\]

where the derivatives are evaluated at \(z = \bar{z} = 0\). Furthermore, under a change of local coordinates in the parameter plane from \(z\) and \(\bar{z}\) to \(z' = u' + iv'\) and \(\bar{z}' = u' - iv'\), the tangent plane is given parametrically by

\[
w_j = \frac{\partial \Psi_j}{\partial z} z' + \frac{\partial \Psi_j}{\partial \bar{z}} \bar{z}'
\]

\[
= \left( \frac{\partial \Psi_j}{\partial z} \frac{\partial z}{\partial z'} + \frac{\partial \Psi_j}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial z'} \right) z' + \left( \frac{\partial \Psi_j}{\partial z} \frac{\partial z}{\partial \bar{z}'} + \frac{\partial \Psi_j}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \bar{z}'} \right) \bar{z}'
\]
where \( T'(z', \bar{z}') = T(z(z'', z'), \bar{z}(z', \bar{z})) \). Since, as a short calculation shows, 
\[
\frac{\partial(z, \bar{z})}{\partial(z', \bar{z}')} = \frac{\partial(u, v)}{\partial(u', v')} \neq 0,
\]
the complex rank of \((\partial T_j/\partial z, \partial T_j/\partial \bar{z})\) remains unaffected by a parameter change. We now make the following definition.

**Definition 1.** The two (real) dimensional plane in \( C^m \) defined parametrically by \( w_j = \alpha_j z + \beta_j \bar{z} \) \((j = 1, 2, \cdots, m)\) is said to be non-analytic if the rank of the 2 by m (complex) matrix \((\alpha_j, \beta_j)\) is 2.

The preceding remarks show that if \( \pi \) is nonanalytic in one coordinate parametrization then it remains so under any change of coordinates in the parameter plane. We want to establish the following.

**Lemma.** Suppose the tangent plane \( \pi \) to \( T(V) \) at the origin in \( C^m \) is nonanalytic. Then there is a polynomial in the coordinates \( w_j \) whose absolute value takes on a local maximum at the origin when restricted to \( T(V) \).

**Proof.** Since \( \pi \) is nonanalytic there exist new coordinates \( w'_i = \sum_{i=1}^m \gamma_{ij} w_j \) \((i = 1, \cdots, m)\), where the matrix \((\gamma_{ij})\) is nonsingular, such that in the \( w'_i \)-coordinates \( \pi \) is given parametrically by \( w'_i = z, \ w'_i = \bar{z}, \) and \( w'_j = 0 \) \((m \geq j \geq 3)\). Now let \( B \) and \( B' \) be points on \( \pi \) and \( T(V) \), respectively, corresponding to the parameters \( u \) and \( v \). Let \( \eta_j \) and \( \eta'_j \) \((j = 1, \cdots, 2n)\) be the real coordinates of \( B \) and \( B' \) (with \( C^m \) regarded as \( R^{2n} \)) in the new coordinate system. Clearly \( \eta_j = u, \ \eta_2 = v, \ \eta_3 = u, \ \eta_4 = -v, \ \eta_j = 0 \) for \( 5 \leq j \leq 2n \). Let \( \eta'_j - \eta_j = \varepsilon_j \) \((j = 1, \cdots, 4)\).

Now consider the function \( P(w_i) = 1 - w'_i w'_i \) (a polynomial in \( w_1, \cdots, w_m \)). When restricted to \( \pi \), \( P(w_i) \) is real-valued and has a maximum in absolute value at the origin. We would like to show that \( |P(w_i)| \) also has a local maximum at the origin when restricted to a sufficiently small neighborhood of the origin on \( T(V) \). This will be true essentially because \( \pi \) has a contact of order at least 1 with \( T(V) \) at the origin (here we will use the estimates (2.2)).

We have, at the point \( B' \),
\[
|P(B')|^2 = |1 - (\eta'_j + i\eta'_j)(\eta_j + i\eta_j)|^2
= 1 - 2(u^2 + v^2) + (u^2 + v^2) + 2Q(u, v)[u^2 + v^2 - 1]
+ [Q(u, v)]^2 + [u(\varepsilon_3 + \varepsilon_4) + v(\varepsilon_3 - \varepsilon_4)]^2
\]
where
\[
Q(u, v) = u(\varepsilon_1 + \varepsilon_3) + v(\varepsilon_3 - \varepsilon_4) + \varepsilon_3 \varepsilon_4 - \varepsilon_4 \varepsilon_3.
\]
Using inequalities (2.2) for \( |\varepsilon_j| \) \((j = 1, \cdots, 4)\) we obtain
\[ |P(B')|^{2} \leq 1 - 2(u^{2} + v^{2}) + M(|u| + |v|)^{2} \]

\[ \leq 1 - [1 - M(|u| + |v|)](|u| + |v|)^{2} \]

for \( u \) and \( v \) sufficiently small and some constant \( M \). If \( |u| + |v| < 1/M \) we see that \( |P(B')| < 1 \) unless \( B' = 0 \).

3. Let \( K \) be a compact subset, with nonempty interior, of a Riemann surface \( M \).

**Definition 2.** An algebra \( A \) of continuous functions on \( K \) is said to be a maximum modulus algebra on \( K \) if for every \( f \in A \) there is a point \( p \) on the boundary \( \partial K \) of \( K \) such that \( |f(q)| \leq |f(p)| \) for all \( q \in K \).

As remarked in [4], we can suppose without loss of generality that \( A \) is uniformly closed and contains the constants and so is a Banach algebra with identity and norm \( \|f\| = \sup_{p \in K} |f(p)| \). It is well known that there is a natural continuous mapping \( i: K \rightarrow \Sigma_{A} \), where \( \Sigma_{A} \) is the maximal ideal space of \( A \) (with the usual Gelfand topology), defined by point evaluation (which is not \( 1:1 \) unless \( A \) separates points in \( \Sigma_{A} \)).

**Theorem.** Let \( A \) be a uniformly closed algebra of continuous functions, containing the constants, on the compact subset \( K \) (with nonempty interior) of the Riemann surface \( M \). Suppose that there is a set \( D \) of points in \( \text{int } K \) having no limit point in \( \text{int } K \), such that each \( p \in \text{int } K - D \) has a neighborhood \( U \) for which \( i(U) \) is open in \( \Sigma_{A} \). Suppose further that \( A \) contains one nonconstant analytic function \( g = g^{1} + ig^{2} \). Then any function \( f = f^{1} + if^{2} \) in \( A \) such that \( f^{1} \) and \( f^{2} \) are twice continuously differentiable is analytic in \( \text{int } K \).

**Proof.** Let \( S \) be the discrete subset of \( \text{int } K \) on which the differential \( dg \) vanishes. For any point \( p \) in \( \text{int } K - S \) there is a neighborhood containing \( p \) and contained in \( \text{int } K - S \) and in which \( g \) is one-to-one. Thus for any point \( p \in \text{int } K - (D \cup S) \) there exists a neighborhood \( U \) containing \( p \) which is mapped homeomorphically by \( i \) onto an open set \( W \) in \( \Sigma_{A} \) and hence local coordinates in \( U \) may be transferred to \( W \). Define the mapping \( F: \Sigma_{A} \rightarrow C^{2} \) by \( F(q') = (f(q'), g(q')) \), \( q' \in \Sigma_{A} \) (where we have used the letters \( f \) and \( g \) to denote the extension, via the Gelfand representation, of \( f \) and \( g \), defined on \( i(K) \), to \( \Sigma_{A} \)). For any point \( q' \) in \( W \) we have \( f(q') = f(i^{-1}(q')) \) and \( g(q') = g(i^{-1}(q')) \) so that \( F \) can be regarded as a mapping defined on \( U \) by \( F(q) = (f(q), g(q)) \), \( q \in U \). \( F \) defines an immersion of \( W \) since in the local coordinates \( z = u + iv \) the matrix

\[
\begin{pmatrix}
 f_{u}' & f_{v}' & g_{u}' & g_{v}' \\
 f_{u}'' & f_{v}'' & g_{u}'' & g_{v}''
\end{pmatrix}
\]
(here the subscripts \( u \) and \( v \) denote partial differentiation) is of rank 2 due to the nonvanishing of the differential \( dg = \partial g/\partial z \, dz \)—apply the Cauchy-Riemann equations to the matrix

\[
\begin{pmatrix}
g_u^z & g_v^z \\
g_u^z & g_v^z
\end{pmatrix}.
\]

Since \( A \) contains the constants we can suppose without loss of generality that \( F(p) \) is the origin 0 in \( C^2 \). We have thus a mapping \( F: \Sigma_A \rightarrow C^2 \) which maps a neighborhood \( W \) of \( i(p) \) onto a two-dimensional surface element \( F(W) \) having a tangent plane \( \pi \) at 0.

We now note that \( \pi \) cannot be nonanalytic. For if this were the case then by the lemma of § 2 there would be a polynomial in the coordinates \( w_1 \) and \( w_2 \) of \( C^2 \) taking on a local maximum in absolute value at 0 when restricted to \( F(W) \). By the Arens-Calderon theorem [1; Theorem 3.3] there would then be a function \( k \in A \) taking on a local maximum at \( i(p) \), and finally, by Rossi’s Local Peak-Point Theorem [3, Theorem 4.1] there would be a function \( \tilde{k} \in A \) taking on its maximum value exactly at \( i(p) \), contradicting the fact that \( A \) is a maximum modulus algebra.

Thus the rank of

\[
\begin{pmatrix}
\frac{\partial g}{\partial z} & 0 \\
\frac{\partial f}{\partial z} & \frac{\partial f}{\partial \zeta}
\end{pmatrix}
\]

(the derivatives being evaluated in the local coordinates at \( p \)) must be 1 and this implies that \( \partial f/\partial \zeta = 0 \). The same conclusion could be drawn for any \( p \in \inf K - (D \cup S) \) and so by the theorem of Riemann on removable singularities, \( f \) is analytic in \( \text{int} \, K \).

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