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**MAXIMUM MODULUS ALGEBRAS AND LOCAL
APPROXIMATION IN C^n**

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MAXIMUM MODULUS ALGEBRAS AND LOCAL APPROXIMATION IN C^n

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1. In [4] W. Rudin established an important result concerning maximum modulus algebras A of continuous complex-valued functions defined on the closure K of a Jordan domain in the complex plane (see also [5]). Rudin's result states, under the assumptions (a) A contains a function ψ which is schlicht on K , and (b) A contains a non-constant function ϕ which is analytic in the interior, $\text{int } K$, of K , that every function in A is analytic in $\text{int } K$. In this note we will establish conditions under which assumption (b) alone yields the desired conclusion in a slightly more general setting. We assume that K is a compact set, with interior, of a Riemann surface, but also assume that $\text{int } K$ is essentially open in the maximal ideal space Σ_A of A (A being regarded as a Banach algebra with the sup norm $\|f\| = \sup_{p \in K} |f(p)|$; see [2]). This means that each point of $\text{int } K$, excepting a set of points having no limit point in $\text{int } K$, has a neighborhood in $\text{int } K$ which is open in Σ_A under the natural mapping of K into Σ_A . Under these assumptions it is easy to show, using the Local Maximum Modulus Principle of H. Rossi [3; Theorem 6.1] and Rudin's results, that (b) is sufficient to guarantee that A consists only of analytic functions. Our main purpose, however, is to establish the result by a geometric method, independent of Rudin's work, which is based on an appropriate local approximation in C^n . Unfortunately the geometric approach being used here only allows us to make the desired conclusion for twice continuously differentiable functions in A whereas the use of Rudin's results would give a proof valid for any function in A . However it is hoped that our method will be of some interest in itself.

The basic idea of the proof is as follows. For simplicity let K be the unit circle $\{z \in C: |z| \leq 1\}$ in the complex plane, and let f and g be nonconstant functions in the maximum modulus algebra A . Suppose that $\Sigma_A = K$. Use f and g to map K into C^2 (the space of 2 complex variables) in the obvious way. If f and g are twice continuously differentiable in the neighborhood of a given point in $\text{int } K$ then the image of this neighborhood in C^2 will be a two (real) dimensional surface possessing a tangent plane at the image p of the point. Let π be the two (real) dimensional tangent plane to this surface at p . If this plane is nonanalytic (Definition 1) then we can find a polynomial in the coordinates w_1 and w_2 of C^2 which locally peaks [3] at p when

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restricted to the surface. The results of Rossi and the Arens-Calderon Theorem [1] then show that this will contradict the maximum modulus property. Thus π cannot be nonanalytic. This gives a relation between the complex derivatives of f and g which, in particular, implies that both functions are analytic at the pre-image of p if one of them is analytic there. In § 2 the essential geometric lemma is established and in § 3 it is used to prove the main result.

2. Let $F: M \rightarrow R^n$ be an immersion (a regular map in the sense of Whitney [6]) of a two (real) dimensional twice continuously differentiable manifold M into real Euclidean n -space R^n . Let $p \in M$ and let (U, h) give local coordinates about p , where U is an open set in M , and h is a homeomorphism from U onto $D = \{(u, v) \in R^2: u^2 + v^2 < 1\}$, with $h(p) = (0, 0)$. If x_j ($j = 1, \dots, n$) is a coordinate function in R^n then the functions $\phi_j(u, v) = x_j \circ F \circ h^{-1}(u, v)$ ($j = 1, \dots, n$) are differentiable and give a map $\Phi: D \rightarrow R^n$ defined by $\Phi(u, v) = (\phi_1(u, v), \dots, \phi_n(u, v))$. Since F is an immersion, the 2 by n matrix

$$\left(\frac{\partial \phi_j}{\partial u}, \frac{\partial \phi_j}{\partial v} \right) = \begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \vdots & \vdots \\ \frac{\partial \phi_n}{\partial u} & \frac{\partial \phi_n}{\partial v} \end{pmatrix}$$

has rank 2 and the mapping Φ is one-to-one in some disc $V = \{(u, v) \in R^2: u^2 + v^2 < r^2 < 1\}$. Further, the set $\Phi(V)$ is a surface element having a tangent plane at $\Phi(0, 0)$.

We can suppose for our purposes that $\Phi(0, 0)$ is the origin 0 in R^n . The tangent plane π to $\Phi(V)$ at 0 is then given parametrically by

$$(2.1) \quad x_j = \frac{\partial \phi_j}{\partial u} u + \frac{\partial \phi_j}{\partial v} v \quad (j = 1, \dots, n),$$

where the derivatives are evaluated at $u = v = 0$. A change of local parameters from u and v to $u' = u'(u, v)$ and $v' = v'(u, v)$ with $\partial(u', v')/\partial(u, v) \neq 0$ (the inverse transformation being given by $u = u(u', v')$ and $v = v(u', v')$ in some neighborhood of $u = v = 0$) yield new functions $\phi'_j(u', v') = \phi_j(u(u', v'), v(u', v'))$ and a new parametrization of the tangent plane, namely,

$$\begin{aligned} x_j &= \frac{\partial \phi'_j}{\partial u'} u' + \frac{\partial \phi'_j}{\partial v'} v' \\ &= \left(\frac{\partial \phi_j}{\partial u} \frac{\partial u}{\partial u'} + \frac{\partial \phi_j}{\partial v} \frac{dv}{\partial u'} \right) u' + \left(\frac{\partial \phi_j}{\partial u} \frac{\partial u}{\partial v'} + \frac{\partial \phi_j}{\partial v} \frac{\partial v}{\partial v'} \right) v' \end{aligned} \quad (j = 1, \dots, n).$$

Note that the rank of the matrix $(\partial\phi_j/\partial u, \partial\phi_j/\partial v)$ is the same as that of $(\partial\phi'_j/\partial u', \partial\phi'_j/\partial v')$ since $\partial(u', v')/\partial(u, v) \neq 0$.

Now u and v parametrize both the surface element $\Phi(V)$ and the tangent plane (given by (2.1)). Let η_j and η'_j ($j = 1, 2, \dots, n$) denote the coordinates in R^n of the points B and B' on π and $\Phi(V)$, respectively, corresponding to the parameters u and v ($u^2 + v^2 < r^2$). For sufficiently small u and v ,

$$\eta'_j = \frac{\partial\phi_j}{\partial u} u + \frac{\partial\phi_j}{\partial v} v + \frac{1}{2} \left(\frac{\partial^2\phi_j}{\partial u^2} u^2 + 2 \frac{\partial^2\phi_j}{\partial u\partial v} uv + \frac{\partial^2\phi_j}{\partial v^2} v^2 \right)$$

where the first derivatives are evaluated at $u = v = 0$ and the second derivatives are evaluated at $u' = \theta u, v' = \theta v$ for some θ satisfying $0 < \theta < 1$. Since M is twice continuously differentiable, the second derivatives of ϕ_j are bounded in absolute value in some sufficiently small neighborhood of $(0, 0)$ and we obtain

$$\sum_1^m (\eta_j - \eta'_j)^2 \leq K(|u| + |v|)^4$$

and so

$$(2.2) \quad |\eta_j - \eta'_j| \leq L(|u| + |v|)^2$$

where K and L are constants depending on these bounds and on n , and u and v are sufficiently small. These estimates will be used later.

Suppose now that $n = 2m$. One can define complex coordinates $w_j = x_{2j-1} + ix_{2j}$ making R^n into complex Euclidean space C^m . Also the (u, v) -plane can be formally complexified by writing $z = u + iv, \bar{z} = u - iv$. We then have a mapping $\Psi: V \rightarrow C^m$ defined by $\Psi(z, \bar{z}) = (w_1, \dots, w_m)$ where

$$w_j = \Psi_j(z, \bar{z}) = \phi_{2j-1} \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) + i\phi_{2j} \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) \quad (j = 1, \dots, m).$$

An elementary computation shows that in this formalism the tangent plane π to $\Psi(V)$ at the origin 0 is given parametrically by

$$w_j = \frac{\partial\Psi_j}{\partial z} z + \frac{\partial\Psi_j}{\partial \bar{z}} \bar{z} \quad (j = 1, \dots, m)$$

where the derivatives are evaluated at $z = \bar{z} = 0$. Furthermore, under a change of local coordinates in the parameter plane from z and \bar{z} to $z' = u' + iv'$ and $\bar{z}' = u' - iv'$, the tangent plane is given parametrically by

$$\begin{aligned} w_j &= \frac{\partial\Psi'_j}{\partial z'} z' + \frac{\partial\Psi'_j}{\partial \bar{z}'} \bar{z}' \\ &= \left(\frac{\partial\Psi_j}{\partial z} \frac{\partial z}{\partial z'} + \frac{\partial\Psi_j}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial z'} \right) z' + \left(\frac{\partial\Psi_j}{\partial z} \frac{\partial z}{\partial \bar{z}'} + \frac{\partial\Psi_j}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \bar{z}'} \right) \bar{z}' \end{aligned}$$

where $\Psi'_j(z', \bar{z}') = \Psi_j(z(z'), \bar{z}(\bar{z}'))$. Since, as a short calculation shows,

$$\frac{\partial(z, \bar{z})}{\partial(z', \bar{z}')} = \frac{\partial(u, v)}{\partial(u', v')} \neq 0,$$

the complex rank of $(\partial\Psi_j/\partial z, \partial\Psi_j/\partial \bar{z})$ remains unaffected by a parameter change. We now make the following definition.

Definition 1. The two (real) dimensional plane in C^m defined parametrically by $w_j = \alpha_j z + \beta_j \bar{z}$ ($j = 1, 2, \dots, m$) is said to be *non-analytic* if the rank of the 2 by m (complex) matrix (α_j, β_j) is 2 .

The preceding remarks show that if π is nonanalytic in one coordinate parametrization then it remains so under any change of coordinates in the parameter plane. We want to establish the following.

LEMMA. *Suppose the tangent plane π to $\Psi(V)$ at the origin in C^m is nonanalytic. Then there is a polynomial in the coordinates w_j whose absolute value takes on a local maximum at the origin when restricted to $\Psi(V)$.*

Proof. Since π is nonanalytic there exist new coordinates $w'_i = \sum_{j=1}^m \gamma_{ij} w_j$ ($i = 1, \dots, m$), where the matrix (γ_{ij}) is nonsingular, such that in the w'_i -coordinates π is given parametrically by $w'_1 = z, w'_2 = \bar{z}$, and $w'_j = 0$ ($m \geq j \geq 3$). Now let B and B' be points on π and $\Psi(V)$, respectively, corresponding to the parameters u and v . Let η_j and η'_j ($j = 1, \dots, 2n$) be the real coordinates of B and B' (with C^m regarded as R^{2n}) in the new coordinate system. Clearly $\eta_1 = u, \eta_2 = v, \eta_3 = u, \eta_4 = -v$, and $\eta_j = 0$ for $5 \leq j \leq 2n$. Let $\eta'_j - \eta_j = \varepsilon_j$ ($j = 1, \dots, 4$).

Now consider the function $P(w_i) = 1 - w'_1 w'_2$ (a polynomial in w_1, \dots, w_m). When restricted to π , $P(w_i)$ is real-valued and has a maximum in absolute value at the origin. We would like to show that $|P(w_i)|$ also has a local maximum at the origin when restricted to a sufficiently small neighborhood of the origin on $\Psi(V)$. This will be true essentially because π has a contact of order at least 1 with $\Psi(V)$ at the origin (here we will use the estimates (2.2)).

We have, at the point B' ,

$$\begin{aligned} |P(B')|^2 &= |1 - (\eta'_1 + i\eta'_2)(\eta'_3 + i\eta'_4)|^2 \\ &= 1 - 2(u^2 + v^2) + (u^2 + v^2) + 2Q(u, v)[u^2 + v^2 - 1] \\ &\quad + [Q(u, v)]^2 + [u(\varepsilon_2 + \varepsilon_4) + v(\varepsilon_3 - \varepsilon_1)]^2 \end{aligned}$$

where

$$Q(u, v) = u(\varepsilon_1 + \varepsilon_3) + v(\varepsilon_2 - \varepsilon_4) + \varepsilon_1 \varepsilon_3 - \varepsilon_2 \varepsilon_4.$$

Using inequalities (2.2) for $|\varepsilon_j|$ ($j = 1, \dots, 4$) we obtain

$$\begin{aligned} |P(B')|^2 &\leq 1 - 2(u^2 + v^2) + M(|u| + |v|)^3 \\ &\leq 1 - [1 - M(|u| + |v|)](|u| + |v|)^2 \end{aligned}$$

for u and v sufficiently small and some constant M . If $|u| + |v| < 1/M$ we see that $|P(B')| < 1$ unless $B' = 0$.

3. Let K be a compact subset, with nonempty interior, of a Riemann surface M .

DEFINITION 2. An algebra A of continuous functions on K is said to be a maximum modulus algebra on K if for every $f \in A$ there is a point p on the boundary ∂K of K such that $|f(q)| \leq |f(p)|$ for all $q \in K$.

As remarked in [4], we can suppose without loss of generality that A is uniformly closed and contains the constants and so is a Banach algebra with identity and norm $\|f\| = \sup_{p \in K} |f(p)|$. It is well known that there is a natural continuous mapping $i: K \rightarrow \Sigma_A$, where Σ_A is the maximal ideal space of A (with the usual Gelfand topology), defined by point evaluation (which is not 1 : 1 unless A separates points in Σ_A).

THEOREM. Let A be a uniformly closed algebra of continuous functions, containing the constants, on the compact subset K (with nonempty interior) of the Riemann surface M . Suppose that there is a set D of points in $\text{int } K$ having no limit point in $\text{int } K$, such that each $p \in \text{int } K - D$ has a neighborhood U for which $i(U)$ is open in Σ_A . Suppose further that A contains one nonconstant analytic function $g = g^1 + ig^2$. Then any function $f = f^1 + if^2$ in A such that f^1 and f^2 are twice continuously differentiable is analytic in $\text{int } K$.

Proof. Let S be the discrete subset of $\text{int } K$ on which the differential dg vanishes. For any point p in $\text{int } K - S$ there is a neighborhood containing p and contained in $\text{int } K - S$ and in which g is one-to-one. Thus for any point $p \in \text{int } K - (D \cup S)$ there exists a neighborhood U containing p which is mapped homeomorphically by i onto an open set W in Σ_A and hence local coordinates in U may be transferred to W . Define the mapping $F: \Sigma_A \rightarrow C^2$ by $F(q') = (f(q'), g(q'))$, $q' \in \Sigma_A$ (where we have used the letters f and g to denote the extension, via the Gelfand representation, of f and g , defined on $i(K)$, to Σ_A). For any point q' in W we have $f(q') = f(i^{-1}(q'))$ and $g(q') = g(i^{-1}(q'))$ so that F can be regarded as a mapping defined on U by $F(q) = (f(q), g(q))$, $q \in U$. F defines an immersion of W since in the local coordinates $z = u + iv$ the matrix

$$\begin{pmatrix} f_u^1 & f_u^2 & g_u^1 & g_u^2 \\ f_v^1 & f_v^2 & g_v^1 & g_v^2 \end{pmatrix}$$

(here the subscripts u and v denote partial differentiation) is of rank 2 due to the nonvanishing of the differential $dg = \partial g/\partial z dz$ —apply the Cauchy-Riemann equations to the matrix

$$\begin{pmatrix} g_u^1 & g_u^2 \\ g_v^1 & g_v^2 \end{pmatrix}.$$

Since A contains the constants we can suppose without loss of generality that $F(p)$ is the origin 0 in C^2 . We have thus a mapping $F: \Sigma_A \rightarrow C^2$ which maps a neighborhood W of $i(p)$ onto a two-dimensional surface element $F(W)$ having a tangent plane π at 0.

We now note that π cannot be nonanalytic. For if this were the case then by the lemma of § 2 there would be a polynomial in the coordinates w_1 and w_2 of C^2 taking on a local maximum in absolute value at 0 when restricted to $F(W)$. By the Arens-Calderon theorem [1; Theorem 3.3] there would then be a function $k \in A$ taking on a local maximum at $i(p)$, and finally, by Rossi's Local Peak-Point Theorem [3, Theorem 4.1] there would be a function $\tilde{k} \in A$ taking on its maximum value exactly at $i(p)$, contradicting the fact that A is a maximum modulus algebra.

Thus the rank of

$$\begin{pmatrix} \frac{\partial g}{\partial z} & 0 \\ \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}} \end{pmatrix}$$

(the derivatives being evaluated in the local coordinates at p) must be 1 and this implies that $\partial f/\partial \bar{z} = 0$. The same conclusion could be drawn for any $p \in \text{int } K - (D \cup S)$ and so by the theorem of Riemann on removable singularities, f is analytic in $\text{int } K$.

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