

Pacific Journal of Mathematics

MODULE CLASSES OF FINITE TYPE

JAMES PATRICK JANS

MODULE CLASSES OF FINITE TYPE

J. P. JANS

1. **Finite type.** In this paper we consider only rings with minimum condition on left and on right ideals. Also, we only consider finitely generated modules over these rings (such modules always possess a composition series of submodules).

There have been several papers [3, 4, 5, 10, 11, 12] on the problem of constructing indecomposable modules over such rings. Most of these papers are devoted to showing that certain rings have an infinite number of non-isomorphic modules of a given composition length for each of an infinite number of composition lengths. In this paper we shall consider a finiteness condition, not on the class of all finitely generated modules but on certain subclasses of that class.

DEFINITION. If C is a class of modules over the ring R we shall say that C is of finite type if for each integer n there are only a finite number of non-isomorphic modules in C of composition length less than n .

We shall study conditions under which the following classes of modules are of finite type:

1. LT the class of left modules which are submodules of projectives. From the results of [1], it is clear that these are the torsionless modules.
2. LW the class of left W -modules, these modules A for which $\text{Ext}_R^1(A, R) = 0$
3. LN the non-torsionless left modules
4. LQ the torsionless left modules which are not duals of right modules.
5. LD the class of duals of right modules.
6. LR the class of reflexive left modules [1].
7. LTW the class of torsionless W -modules.

In the above definitions the dual of a module A is $\text{Hom}_R(A, R)$ denoted by A^* . Also, A is reflexive if the natural homomorphism $A \rightarrow A^{**}$ is an isomorphism. See [7].

The corresponding classes of right modules (RT, RW , etc.) are defined analogously. All the theorems we prove go through with left and right interchanged.

A useful tool in our study is the following theorem proved by Morita and Tachikawa in [9] and also mentioned by Brauer in [2].

THEOREM A. *If P is projective and if the diagram*

$$(1) \quad \begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & & & & & \downarrow \theta \\ 0 & \rightarrow & Y & \rightarrow & P & \rightarrow & B \rightarrow 0 \end{array}$$

has exact rows and if θ is an isomorphism, then the diagram can be embedded in the commutative diagram

$$(1') \quad \begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & \mu \downarrow & & \rho \downarrow & & \downarrow \theta \\ 0 & \rightarrow & Y & \rightarrow & P & \rightarrow & B \rightarrow 0 \end{array}$$

where ρ and μ are also isomorphisms.

It should be noted that the proof of Theorem A requires our standing hypothesis that every module under consideration has a composition series.

Before we deduce some corollaries from Theorem A, we need some additional information. Let l be the left composition length of the ring R and let r be the right composition length. Note that l and r need not be equal. Let $C(A)$ be the composition length of the module A .

LEMMA 1.1. *If the left module A has $C(A) = n$ then there exists a free module F_n , the direct sum of n copies of R considered as a left module, of composition length ln such that $F_n \rightarrow A \rightarrow 0$ is exact.*

The proof, an induction on n , is essentially the same as the proof of Lemma 2.6 of [6]. By dualizing the above sequence we obtain

LEMMA 1.2. *If the left module A has $C(A) = n$ then A^* has composition length $\leq nr$.*

Proof. The sequence of Lemma 1.1 induces $0 \rightarrow A^* \rightarrow F_n^*$ exact. The module F_n^* is a direct sum of copies of R considered as a right module [7] and hence $C(F_n^*) = nr$. Since A^* is a submodule of F_n^* , $C(A) \leq nr$.

LEMMA 1.3. *If the left module A is torsionless with $C(A) = n$, then A can be embedded in a free module F such that $C(F) \leq nrl$.*

Proof. By Lemma 1.2 $C(A^*) \leq nr$ and by Lemma 1.1 there exists a free right module F_0 (the direct sum of nr copies of R) of composition length nr^2 such that $F_0 \rightarrow A^* \rightarrow 0$ is exact. This dualizes to

$$0 \rightarrow A^{**} \rightarrow F_0^* \quad \text{exact,}$$

whereby the proof of Lemma 1.2 $C(F_0^*) = nrl$. But since $A \rightarrow A^{**}$ is a monomorphism, this can be used to embed A in F_0^* . The idea of the above proof is due to Bass [1], although, being in a more general situation he was not concerned there with composition length.

It should be noted that the inequalities of Lemma 1.2 and 1.3 are, for most rings, quite crude. Using the above lemmas, and Theorem A have the following results.

THEOREM 1.4. *If LN is of finite type then so is LQ .*

Proof. Suppose that for some n there were an infinite number of non-isomorphic modules $\{T_\alpha\}$ in LQ all of composition length n . Then by Lemma 1.3 we can embed them *all* as submodules of a free module F of composition length nlr . Consider the infinite collection of factors $\{F/T_\alpha\}$. By [1, 7] these are modules in LN all having composition length $n(lr - 1)$.

But the hypotheses of the theorem require that there are only a finite number of non-isomorphic modules in LN of each composition length. Thus for some $\alpha \neq \beta$ $F/T_\alpha \cong F/T_\beta$ and by Theorem A we have $T_\alpha \cong T_\beta$. This contradicts the assumption that the collection $\{T_\alpha\}$ consists of non-isomorphic modules.

The following theorem is modeled on the duality Theorem 1.1 of [7].

THEOREM 1.5. *LT is of finite type if and only if RT is of finite type.*

Proof. By right-left symmetry it is sufficient to prove the statement in one direction only.

Suppose that RT is of finite type and $\{T_\alpha\}$ is an infinite collection of non-isomorphic torsionless left modules all of composition length n . By Lemma 1.1 there is a free module F of composition length ln and an infinite collection of short exact sequences

$$F \xrightarrow{\mu_\alpha} T_\alpha \longrightarrow 0.$$

Now form the dual exact sequences.

$$0 \longrightarrow T_\alpha^* \xrightarrow{\mu_\alpha^*} F^* \longrightarrow F^*/T_\alpha^* \longrightarrow 0.$$

The right modules F/T_α^* are torsionless right modules [1; statements 4.2 and 4.4] and each of these modules has composition length less than rn . Since RT is of finite type there exist two indices α and β such that $F^*/T_\alpha^* \xrightarrow{\theta} F^*/T_\beta^*$ is an isomorphism. Using Theorem A we construct the exact commuting diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_\alpha^* & \xrightarrow{\mu_\alpha} & F^* & \longrightarrow & F^*/T_\alpha^* \longrightarrow 0 \\
 & & \downarrow \mu & & \downarrow \rho & & \downarrow \theta \\
 0 & \longrightarrow & T_\beta^* & \xrightarrow{\mu_\beta} & F^* & \longrightarrow & F^*/T_\beta^* \longrightarrow 0
 \end{array}$$

with vertical isomorphisms. This gives the commutative diagram

$$\begin{array}{ccc}
 F^{**} & \xrightarrow{\mu_\alpha^{**}} & T_\beta^{**} \\
 \downarrow \rho^* & & \downarrow \mu^* \\
 F^{**} & \xrightarrow{\mu_\beta^{**}} & T_\alpha^{**} .
 \end{array}$$

In this situation $Im\mu_\beta^{**}$ coincides with the natural image of T_β in T_β^{**} and the similar situation holds for the subscript α . Then commutativity then implies that T_α is isomorphic with T_β via the isomorphism μ^* . This contradicts the assumption that the collection $\{T_\alpha\}$ consisted of non-isomorphic modules.

2. A dual to Theorem A. A dual to Theorem A would state that if two submodules of a free module F were isomorphic, then the isomorphism can be extended to an automorphism of F . This is not, in general, true as we shall show by an example. However, by assuming enough extra conditions we can obtain the desired conclusion. Recall that X is a W -module if $Ext_R^1(X, R) = 0$; see [8].

THEOREM 2.1. *If in the diagram*

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \rightarrow & F & \rightarrow & F/A \rightarrow 0 \\
 & & & & \downarrow \theta & & \\
 0 & \rightarrow & B & \rightarrow & F & \rightarrow & F/B \rightarrow 0
 \end{array}$$

θ is an isomorphism, F is a free module and F/A and F/B are W modules, then the diagram can be embedded in a commutative diagram.

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \rightarrow & F & \rightarrow & F/A \rightarrow 0 \\
 & & \downarrow \theta & & \downarrow \rho^* & & \downarrow \mu \\
 0 & \rightarrow & B & \rightarrow & F & \rightarrow & F/B \rightarrow 0
 \end{array}$$

with all the vertical maps isomorphisms.

Proof. Consider the dual sequences

$$\begin{array}{ccccccc}
 & & 0 & \rightarrow & (F/A)^* & \rightarrow & F^* \rightarrow A^* \rightarrow 0 \\
 (*) & & & & & & \uparrow \theta^* \\
 & & 0 & \rightarrow & (F/B)^* & \rightarrow & F^* \rightarrow B^* \rightarrow 0
 \end{array}$$

The exactness at A^* and B^* comes from the fact that F/A and F/B are W -modules. Also θ^* is an isomorphism because θ is one. By Theorem A there exists an automorphism ρ of F^* so that the diagram

$$\begin{array}{ccccc}
 F^* & \rightarrow & A^* & \rightarrow & 0 \\
 \uparrow \rho & & \uparrow \theta^* & & \\
 F^* & \rightarrow & B^* & \rightarrow & 0
 \end{array}$$

is commutative.

Now dualize again to obtain the commutative diagram

$$\begin{array}{ccccc}
 0 & \rightarrow & A^{**} & \rightarrow & F^{***} \\
 & & \downarrow \theta^{**} & & \downarrow \rho^* \\
 0 & \rightarrow & B^{**} & \rightarrow & F^{***}
 \end{array}$$

Since both A, B are torsionless and F is reflexive [1, 7] we can identify A and B with their images in A^{**} and B^{**} . Also the mappings with two stars on them, when restricted to these images, coincide with the original maps. Thus, identifying F with F^{***} , we have the commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \rightarrow & F & \rightarrow & F/A \rightarrow 0 \\
 & & \downarrow \theta & & \downarrow \rho^* & & \downarrow \mu \\
 0 & \rightarrow & B & \rightarrow & F & \rightarrow & F/B \rightarrow 0
 \end{array}$$

where ρ^* induces μ on F/A to F/B . All the vertical maps are isomorphisms.

COROLLARY 2.2. *If LT is of finite type then so is LW .*

Proof. Suppose $\{W_\alpha\}$ is an infinite collection of nonisomorphic W -modules such that $C(W_\alpha) = n$. By Lemma 1.1 they are all epimorphic images of a free module $F, F \xrightarrow{\pi_\alpha} W_\alpha \rightarrow 0$ and $C(F) = ln$. The submodules $\text{Ker } \pi_\alpha$ of F all satisfy $C(\text{ker } \pi_\alpha) = (l - 1)n$ and by the assumption that LT is of finite type there exist two indices $\alpha \neq \beta$

such that $\text{Ker } \pi_\alpha \cong \text{Ker } \pi_\beta$. Now Theorem 2.1 implies that $W_\alpha \cong W_\beta$ contradicting the assumption that the elements in the collection $\{W_\alpha\}$ were non-isomorphic.

COROLLARY 2.3. *LTW is of finite type if and only if LR is of finite type.*

Proof. For the "if" part of the proof we proceed exactly as in the proof of Corollary 2.2. We use this fact, proved in [3], that if W is a torsionless W -module and

$$0 \rightarrow \text{Ker } \pi \rightarrow F \rightarrow W \rightarrow 0$$

is exact with F free then $\text{Ker } \pi$ is reflexive. Then the proof of 2.2 with the class LR replacing LT works here.

Conversely, if LTW is of finite type and if $\{Q_\alpha\}$ is an infinite collection of reflexives with $C(Q_\alpha) = n$, then by Lemma 1.3 they can all be embedded in a free module F with $C(F) \leq lnr$,

$$0 \rightarrow Q_\alpha \rightarrow F.$$

But by [8] this embedding of the reflexive Q_α results in F/Q_α being a torsionless W -module. Hence by assumption there exists $\alpha \neq \beta$ such that $F/Q_\alpha \cong F/Q_\beta$. Then Theorem A implies $Q_\alpha \cong Q_\beta$ contradicting the assumption that the collection $\{Q_\alpha\}$ consists of non-isomorphic modules.

We conclude with an example which shows that Theorem 2.1. does not hold without the hypothesis that F/A and F/B are W -modules. Let R be the ring of matrices

$$\begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \end{pmatrix}$$

with x, y, z in a field K having more than 2 elements. R is commutative and is an indecomposable free module over itself. The radical N of R is the direct sum of two simple modules, $N = S_1 \oplus S_2$. If α, β are two distinct nonzero elements of K there is an automorphism θ of N which is "multiplication by α on S_1 and multiplication by β on S_2 ". Any extension of θ to a K -linear transformation on R will have two distinct eigenvalues. However, since R is indecomposable every module endomorphism (or automorphism) has only one eigenvalue, therefore θ cannot be extended to R .

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi primary rings*, Trans. Amer. Math. Soc., **95** (1960), 466-488.
2. R. Brauer, *Some remarks on associative rings and algebras*, National Research Council Publication **502** (1957), 4-11.
3. A. Heller, and I. Reiner, *Indecomposable representations*, Illinois J. of Math., **5** (1961), 314-323.
4. J. P. Jans, *On the indecomposable representations of algebras*, Annals of Math., **66** (1957), 418-429.
5. ———, *The representation type of algebras and subalgebras*, Canadian J. of Math., **X** (1958), 39-49.
6. ———, *Some generalizations of finite projective dimension*, Illinois J. of Math., **5** (1961) 334-344.
7. ———, *Duality in Noetherian rings*, Proc. Amer. Math. Soc., **12** (1961), 829-835.
8. ———, *On finitely generated modules over Noetherian rings*, To appear in Trans. Amer. Math. Soc.
9. K. Morita, and H. Tachikawa, *Character modules, submodules of a free module and quasi-Frobenius rings*, Math. Zeitschr, **65** (1956), 414-428.
10. H. Tachikawa, *A note on algebras of unbounded representation type*, Proc. Japan Acad., **36** (1960), 59-61.
11. T. Yoshii, *Note on algebras of strongly unbounded representation type*, Proc. Japan Acad., **32** (1956), 383-387.
12. ———, *On algebras of unbounded representation type*, Osaka Math. J., **8** (1956), 51-105.

UNIVERSITY OF WASHINGTON

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Rafael Artzy, <i>Solution of loop equations by adjunction</i>	361
Earl Robert Berkson, <i>A characterization of scalar type operators on reflexive Banach spaces</i>	365
Mario Borelli, <i>Divisorial varieties</i>	375
Raj Chandra Bose, <i>Strongly regular graphs, partial geometries and partially balanced designs</i>	389
R. H. Bruck, <i>Finite nets. II. Uniqueness and imbedding</i>	421
L. Carlitz, <i>The inverse of the error function</i>	459
Robert Wayne Carroll, <i>Some degenerate Cauchy problems with operator coefficients</i>	471
Michael P. Drazin and Emilie Virginia Haynsworth, <i>A theorem on matrices of 0's and 1's</i>	487
Lawrence Carl Eggan and Eugene A. Maier, <i>On complex approximation</i>	497
James Michael Gardner Fell, <i>Weak containment and Kronecker products of group representations</i>	503
Paul Chase Fife, <i>Schauder estimates under incomplete Hölder continuity assumptions</i>	511
Shaul Foguel, <i>Powers of a contraction in Hilbert space</i>	551
Neal Eugene Foland, <i>The structure of the orbits and their limit sets in continuous flows</i>	563
Frank John Forelli, Jr., <i>Analytic measures</i>	571
Robert William Gilmer, Jr., <i>On a classical theorem of Noether in ideal theory</i>	579
P. R. Halmos and Jack E. McLaughlin, <i>Partial isometries</i>	585
Albert Emerson Hurd, <i>Maximum modulus algebras and local approximation in C^n</i>	597
James Patrick Jans, <i>Module classes of finite type</i>	603
Betty Kvarda, <i>On densities of sets of lattice points</i>	611
H. Larcher, <i>A geometric characterization for a class of discontinuous groups of linear fractional transformations</i>	617
John W. Moon and Leo Moser, <i>Simple paths on polyhedra</i>	629
T. S. Motzkin and Ernst Gabor Straus, <i>Representation of a point of a set as sum of transforms of boundary points</i>	633
Rajakularaman Ponnuswami Pakshirajan, <i>An analogue of Kolmogorov's three-series theorem for abstract random variables</i>	639
Robert Ralph Phelps, <i>Čebyšev subspaces of finite codimension in $C(X)$</i>	647
James Dolan Reid, <i>On subgroups of an Abelian group maximal disjoint from a given subgroup</i>	657
William T. Reid, <i>Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems</i>	665
Georg Johann Rieger, <i>Some theorems on prime ideals in algebraic number fields</i> ...	687
Gene Fuerst Rose and Joseph Silbert Ullian, <i>Approximations of functions on the integers</i>	693
F. J. Sansone, <i>Combinatorial functions and regressive isols</i>	703
Leo Sario, <i>On locally meromorphic functions with single-valued moduli</i>	709
Takayuki Tamura, <i>Semigroups and their subsemigroup lattices</i>	725
Pui-kei Wong, <i>Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equations</i>	737
Fawzi Mohamad Yaqub, <i>Free extensions of Boolean algebras</i>	761