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ON DENSITIES OF SETS OF LATTICE POINTS

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1. Introduction. Let A be a set of positive integers, and for any positive integer x denote by $A(x)$ the number of integers of A which are not greater than x . Then the Schnirelmann density of A is defined [4] to be the quantity

$$\alpha = \operatorname{glb}_x \frac{A(x)}{x}.$$

For any k sets A_1, \dots, A_k of positive integers, $k \geq 2$, let the sum set $A_1 + \dots + A_k$ be the set of all nonzero sums $a_1 + \dots + a_k$ for which each $a_i, i = 1, \dots, k$, is either contained in A_i or is 0. Let kA be the set $A + \dots + A$ with k summands.

Schnirelmann [4] and Landau [2] have shown that if A and B are two sets of positive integers with $C = A + B$, and if α, β, γ are the Schnirelmann densities of A, B, C , respectively, then $\gamma \geq \alpha + \beta - \alpha\beta$, and if $\alpha + \beta \geq 1$ then $\gamma = 1$. They have also shown that if A is a set of positive integers whose Schnirelmann density is positive then A is a basic sequence for the set of positive integers, or, in other words, there exists a positive integer k such that every positive integer can be written as the sum of at most k elements of A .

We will show that by using extensions of the methods employed by Schnirelmann and Landau the above results can be generalized to certain sets of vectors in a discrete lattice (for definition and discussion see [3, pp. 28–31] or [5, pp. 141–145]). Without loss of generality it may be assumed that the components of the vectors in such a lattice are rational integers. The usual identification of algebraic integers with lattice points then gives an immediate extension of these results to algebraic integers.

2. Notation and definitions. Let Q_n be the set of all n -dimensional lattice points $(x_1, \dots, x_n), n \geq 1$, for which each $x_i, i = 1, \dots, n$, is a nonnegative integer and at least one x_i is positive. Define the sum of subsets of Q_n in the same manner as was done for sets of positive integers, and for any subsets A and B of Q_n let $A - B$ denote the set of all elements of A which are not in B . If A and S are subsets of Q_n and S is finite let $A(S)$ be the number of elements in $A \cap S$.

DEFINITION 1. A finite nonempty subset R of Q_n will be called a

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fundamental subset of Q_n or, briefly, a *fundamental set*, if whenever an element (r_1, \dots, r_n) is in R then all elements (x_1, \dots, x_n) of Q_n such that $x_i \leq r_i, i = 1, \dots, n$, are also in R .

DEFINITION 2. Let A be any subset of Q_n . The *density* of A is defined to be the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q_n(R)}$$

taken over all fundamental sets R .

3. **Extension of the Landau-Schnirelmann results.** Throughout this section we let A and B be subsets of Q_n with $C = A + B$, and let α, β, γ be the densities of A, B, C , respectively.

THEOREM 1. If $\alpha + \beta \geq 1$ then $\gamma = 1$.

Proof. Assume $\gamma < 1$. Then there exists a fundamental set R for which $C(R) < Q_n(R)$, which in turn implies that there exists an element (x_1^0, \dots, x_n^0) in $Q_n - C$. Let R_0 be the set of all elements (x_1, \dots, x_n) in Q_n for which $x_i \leq x_i^0, i = 1, \dots, n$. Then for any (x_1, \dots, x_n) in R_0 either (x_1, \dots, x_n) is in A , or $(x_1, \dots, x_n) = (x_1^0, \dots, x_n^0) - (b_1, \dots, b_n)$ for some (b_1, \dots, b_n) in $B \cap R_0$, or neither, but not both. In particular, (x_1^0, \dots, x_n^0) is neither. Hence,

$$A(R_0) + B(R_0) \leq Q_n(R_0) - 1,$$

and

$$\alpha + \beta \leq \frac{A(R_0) + B(R_0)}{Q_n(R_0)} < 1$$

which is a contradiction. Therefore $\gamma = 1$.

THEOREM 2. $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof. Let $\omega_i, 1 \leq i \leq n$, be that vector in Q_n for which the i th component is 1 and the other components, if any, are 0. If any one of the vectors $\omega_1, \dots, \omega_n$ is missing from A then $\alpha = 0$ and the theorem is trivial. Hence we assume all the vectors $\omega_1, \dots, \omega_n$ are in A . We must show

$$(1) \quad \frac{C(R)}{Q_n(R)} \geq \alpha + \beta - \alpha\beta$$

for all fundamental sets R . If $C(R) = Q_n(R)$ then (1) holds, since

$(1 - \alpha)(1 - \beta) \geq 0$ implies $1 \geq \alpha + \beta - \alpha\beta$. Therefore we assume $C(R) < Q_n(R)$ and, consequently, $A(R) < Q_n(R)$.

Let $H = R - A$. We will show that there exist vectors $\alpha^{(1)}, \dots, \alpha^{(s)}$ in A and sets L_1, \dots, L_s with the following properties.

- (i) $L_i \subseteq H$ and L_i is not empty, $i = 1, \dots, s$.
- (ii) The sets $L'_i = \{x - \alpha^{(i)} \mid x \in L_i\}$ are fundamental sets.
- (iii) $L_i \cap L_j = \phi$ for $i \neq j$.
- (iv) $H = L_1 \cup \dots \cup L_s$.

Let the elements of R be ordered so that $(x_1, \dots, x_n) > (x'_1, \dots, x'_n)$ if $x_1 > x'_1$ or if $x_1 = x'_1, \dots, x_p = x'_p, x_{p+1} > x'_{p+1}$. For every $h = (h_1, \dots, h_n)$ in H , let A_h be the set of all (a_1, \dots, a_n) in A such that each $a_i \leq h_i$. The sets A_h are not empty since $\omega_i \in A$ for $i = 1, \dots, n$. The A_h are finite sets, hence they contain (in our ordering) a largest vector. Let $\alpha^{(1)}, \dots, \alpha^{(s)}$ be all the distinct vectors that are largest vectors in any A_h . Let L_i be the set of all vectors x in H such that $\alpha^{(i)}$ is the largest vector in A_x .

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the L_i . To prove (ii) consider a vector $y = (y_1, \dots, y_n)$ such that

$$(2) \quad x_j \geq y_j \geq \alpha_j^{(i)},$$

where $x = (x_1, \dots, x_n)$ is in L_i and $y \neq \alpha^{(i)}$. Suppose $y \in L_k, k \neq i$. Then

$$(3) \quad x_j \geq y_j \geq \alpha_j^{(k)}$$

and $\alpha^{(k)} \geq \alpha^{(i)}$. But (2) and (3) and $x \in L_i$ imply $\alpha^{(k)} \leq \alpha^{(i)}$, hence $\alpha^{(k)} = \alpha^{(i)}$. Similarly, $y \in A$ implies $y = \alpha^{(i)}$. This proves (ii).

If $b \in B \cap L'_i$ then $\alpha^{(i)} + b$ is in $C \cap L_i$, hence in $C - A$. Therefore,

$$\begin{aligned} C(R) &\geq A(R) + B(L'_1) + \dots + B(L'_s) \\ &\geq A(R) + \beta[Q_n(L'_1) + \dots + Q_n(L'_s)] \\ &= A(R) + \beta[Q_n(L_1) + \dots + Q_n(L_s)] \\ &= A(R) + \beta[Q_n(H)] \\ &= A(R) + \beta[Q_n(R) - A(R)] \\ &= (1 - \beta)A(R) + \beta[Q_n(R)] \\ &\geq (1 - \beta)\alpha[Q_n(R)] + \beta[Q_n(R)], \end{aligned}$$

and

$$\frac{C(R)}{Q_n(R)} \geq \alpha + \beta - \alpha\beta,$$

which completes the proof.

COROLLARY 1. *Let A_1, \dots, A_k be any k subsets of Q_n , $k \geq 2$, let α_i be the density of A_i for $i = 1, \dots, k$, and let $d(A_1 + \dots + A_k)$ be the density of $A_1 + \dots + A_k$. Then*

$$1 - d(A_1 + \dots + A_k) \leq (1 - \alpha_1) \dots (1 - \alpha_k).$$

Proof. If $k = 2$ then Theorem 2 implies that $1 - d(A_1 + A_2) \leq 1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 = (1 - \alpha_1)(1 - \alpha_2)$. Hence assume $1 - d(A_1 + \dots + A_{k-1}) \leq (1 - \alpha_1) \dots (1 - \alpha_{k-1})$. Then

$$\begin{aligned} 1 - d(A_1 + \dots + A_{k-1} + A_k) &\leq [1 - d(A_1 + \dots + A_{k-1})](1 - \alpha_k) \\ &\leq (1 - \alpha_1) \dots (1 - \alpha_{k-1})(1 - \alpha_k). \end{aligned}$$

COROLLARY 2. *If A is any subset of Q_n with density $\alpha > 0$ then there exists an integer $k > 0$ such that $kA = Q_n$.*

Proof. There exists an integer $m > 0$ such that $(1 - \alpha)^m \leq 1/2$. Let $d(mA)$ be the density of mA . Then Corollary 1 implies that $1 - d(mA) \leq (1 - \alpha)^m \leq 1/2$, or $d(mA) \geq 1/2$. From Theorem 1, $d(mA) + d(mA) \geq 1$ implies $d(2mA) = 1$, or $2mA = Q_n$.

4. Remark. We may identify Q_2 with the set of nonzero Gaussian integers $x + yi$ for which x and y are both nonnegative rational integers. Luther Cheo [1] defined density for subsets of this Q_2 as follows, using our notation.

DEFINITION 3. Let $x_0 + y_0i$ be any element of Q_2 and S the set of all $x + yi$ in Q_2 such that $x \leq x_0$ and $y \leq y_0$. Then for any subset A of Q_2 the density of A is the quantity

$$\alpha_c = \operatorname{glb}_S \frac{A(S)}{Q_2(S)}.$$

Cheo proved Theorem 1 for his density and also a theorem which implies that if ji is in A for all $j = 1, 2, \dots$, and if $\alpha_c, \beta_c, \gamma_c$ are the Cheo densities of $A, B, C = A + B$, respectively, then

$$\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c.$$

We cannot remove the requirement that all ji be in A by means of an argument like that used to establish Theorem 2 since it would be necessary to partition H in such a way that the sets L'_j are of the type S used in defining the Cheo density, and this is not always possible. Consider, for example, the set $R = \{x + yi: x + yi \text{ is in } Q_2, x \leq 4, y \leq 3\}$, and let $A \cap R = \{1, i, 3 + 3i\}$. Then $H = R - A$ cannot be so partitioned, as the reader can easily verify.

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