

# Pacific Journal of Mathematics

**ON DENSITIES OF SETS OF LATTICE POINTS**

BETTY KVARDA

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**1. Introduction.** Let  $A$  be a set of positive integers, and for any positive integer  $x$  denote by  $A(x)$  the number of integers of  $A$  which are not greater than  $x$ . Then the Schnirelmann density of  $A$  is defined [4] to be the quantity

$$\alpha = \operatorname{glb}_x \frac{A(x)}{x}.$$

For any  $k$  sets  $A_1, \dots, A_k$  of positive integers,  $k \geq 2$ , let the sum set  $A_1 + \dots + A_k$  be the set of all nonzero sums  $a_1 + \dots + a_k$  for which each  $a_i, i = 1, \dots, k$ , is either contained in  $A_i$  or is 0. Let  $kA$  be the set  $A + \dots + A$  with  $k$  summands.

Schnirelmann [4] and Landau [2] have shown that if  $A$  and  $B$  are two sets of positive integers with  $C = A + B$ , and if  $\alpha, \beta, \gamma$  are the Schnirelmann densities of  $A, B, C$ , respectively, then  $\gamma \geq \alpha + \beta - \alpha\beta$ , and if  $\alpha + \beta \geq 1$  then  $\gamma = 1$ . They have also shown that if  $A$  is a set of positive integers whose Schnirelmann density is positive then  $A$  is a basic sequence for the set of positive integers, or, in other words, there exists a positive integer  $k$  such that every positive integer can be written as the sum of at most  $k$  elements of  $A$ .

We will show that by using extensions of the methods employed by Schnirelmann and Landau the above results can be generalized to certain sets of vectors in a discrete lattice (for definition and discussion see [3, pp. 28–31] or [5, pp. 141–145]). Without loss of generality it may be assumed that the components of the vectors in such a lattice are rational integers. The usual identification of algebraic integers with lattice points then gives an immediate extension of these results to algebraic integers.

**2. Notation and definitions.** Let  $Q_n$  be the set of all  $n$ -dimensional lattice points  $(x_1, \dots, x_n)$ ,  $n \geq 1$ , for which each  $x_i, i = 1, \dots, n$ , is a nonnegative integer and at least one  $x_i$  is positive. Define the sum of subsets of  $Q_n$  in the same manner as was done for sets of positive integers, and for any subsets  $A$  and  $B$  of  $Q_n$  let  $A - B$  denote the set of all elements of  $A$  which are not in  $B$ . If  $A$  and  $S$  are subsets of  $Q_n$  and  $S$  is finite let  $A(S)$  be the number of elements in  $A \cap S$ .

**DEFINITION 1.** A finite nonempty subset  $R$  of  $Q_n$  will be called a

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*fundamental subset* of  $Q_n$  or, briefly, a *fundamental set*, if whenever an element  $(r_1, \dots, r_n)$  is in  $R$  then all elements  $(x_1, \dots, x_n)$  of  $Q_n$  such that  $x_i \leq r_i, i = 1, \dots, n$ , are also in  $R$ .

DEFINITION 2. Let  $A$  be any subset of  $Q_n$ . The *density* of  $A$  is defined to be the quantity

$$\alpha = \text{glb} \frac{A(R)}{Q_n(R)}$$

taken over all fundamental sets  $R$ .

3. **Extension of the Landau-Schnirelmann results.** Throughout this section we let  $A$  and  $B$  be subsets of  $Q_n$  with  $C = A + B$ , and let  $\alpha, \beta, \gamma$  be the densities of  $A, B, C$ , respectively.

THEOREM 1. *If  $\alpha + \beta \geq 1$  then  $\gamma = 1$ .*

*Proof.* Assume  $\gamma < 1$ . Then there exists a fundamental set  $R$  for which  $C(R) < Q_n(R)$ , which in turn implies that there exists an element  $(x_1^0, \dots, x_n^0)$  in  $Q_n - C$ . Let  $R_0$  be the set of all elements  $(x_1, \dots, x_n)$  in  $Q_n$  for which  $x_i \leq x_i^0, i = 1, \dots, n$ . Then for any  $(x_1, \dots, x_n)$  in  $R_0$  either  $(x_1, \dots, x_n)$  is in  $A$ , or  $(x_1, \dots, x_n) = (x_1^0, \dots, x_n^0) - (b_1, \dots, b_n)$  for some  $(b_1, \dots, b_n)$  in  $B \cap R_0$ , or neither, but not both. In particular,  $(x_1^0, \dots, x_n^0)$  is neither. Hence,

$$A(R_0) + B(R_0) \leq Q_n(R_0) - 1,$$

and

$$\alpha + \beta \leq \frac{A(R_0) + B(R_0)}{Q_n(R_0)} < 1$$

which is a contradiction. Therefore  $\gamma = 1$ .

THEOREM 2.  $\gamma \geq \alpha + \beta - \alpha\beta$ .

*Proof.* Let  $\omega_i, 1 \leq i \leq n$ , be that vector in  $Q_n$  for which the  $i$ th component is 1 and the other components, if any, are 0. If any one of the vectors  $\omega_1, \dots, \omega_n$  is missing from  $A$  then  $\alpha = 0$  and the theorem is trivial. Hence we assume all the vectors  $\omega_1, \dots, \omega_n$  are in  $A$ . We must show

$$(1) \quad \frac{C(R)}{Q_n(R)} \geq \alpha + \beta - \alpha\beta$$

for all fundamental sets  $R$ . If  $C(R) = Q_n(R)$  then (1) holds, since

$(1 - \alpha)(1 - \beta) \geq 0$  implies  $1 \geq \alpha + \beta - \alpha\beta$ . Therefore we assume  $C(R) < Q_n(R)$  and, consequently,  $A(R) < Q_n(R)$ .

Let  $H = R - A$ . We will show that there exist vectors  $a^{(1)}, \dots, a^{(s)}$  in  $A$  and sets  $L_1, \dots, L_s$  with the following properties.

- (i)  $L_i \subseteq H$  and  $L_i$  is not empty,  $i = 1, \dots, s$ .
- (ii) The sets  $L'_i = \{x - a^{(i)} \mid x \in L_i\}$  are fundamental sets.
- (iii)  $L_i \cap L_j = \phi$  for  $i \neq j$ .
- (iv)  $H = L_1 \cup \dots \cup L_s$ .

Let the elements of  $R$  be ordered so that  $(x_1, \dots, x_n) > (x'_1, \dots, x'_n)$  if  $x_1 > x'_1$  or if  $x_1 = x'_1, \dots, x_p = x'_p, x_{p+1} > x'_{p+1}$ . For every  $h = (h_1, \dots, h_n)$  in  $H$ , let  $A_h$  be the set of all  $(a_1, \dots, a_n)$  in  $A$  such that each  $a_i \leq h_i$ . The sets  $A_h$  are not empty since  $\omega_i \in A$  for  $i = 1, \dots, n$ . The  $A_h$  are finite sets, hence they contain (in our ordering) a largest vector. Let  $a^{(1)}, \dots, a^{(s)}$  be all the distinct vectors that are largest vectors in any  $A_h$ . Let  $L_i$  be the set of all vectors  $x$  in  $H$  such that  $a^{(i)}$  is the largest vector in  $A_x$ .

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the  $L_i$ . To prove (ii) consider a vector  $y = (y_1, \dots, y_n)$  such that

$$(2) \quad x_j \geq y_j \geq a_j^{(i)},$$

where  $x = (x_1, \dots, x_n)$  is in  $L_i$  and  $y \neq a^{(i)}$ . Suppose  $y \in L_k$ ,  $k \neq i$ . Then

$$(3) \quad x_j \geq y_j \geq a_j^{(k)}$$

and  $a^{(k)} \geq a^{(i)}$ . But (2) and (3) and  $x \in L_i$  imply  $a^{(k)} \leq a^{(i)}$ , hence  $a^{(k)} = a^{(i)}$ . Similarly,  $y \in A$  implies  $y = a^{(i)}$ . This proves (ii).

If  $b \in B \cap L'_i$  then  $a^{(i)} + b$  is in  $C \cap L_i$ , hence in  $C - A$ . Therefore,

$$\begin{aligned} C(R) &\supseteq A(R) + B(L'_1) + \dots + B(L'_s) \\ &\supseteq A(R) + \beta[Q_n(L'_1) + \dots + Q_n(L'_s)] \\ &= A(R) + \beta[Q_n(L_1) + \dots + Q_n(L_s)] \\ &= A(R) + \beta[Q_n(H)] \\ &= A(R) + \beta[Q_n(R) - A(R)] \\ &= (1 - \beta)A(R) + \beta[Q_n(R)] \\ &\supseteq (1 - \beta)\alpha[Q_n(R)] + \beta[Q_n(R)], \end{aligned}$$

and

$$\frac{C(R)}{Q_n(R)} \supseteq \alpha + \beta - \alpha\beta,$$

which completes the proof.

**COROLLARY 1.** *Let  $A_1, \dots, A_k$  be any  $k$  subsets of  $Q_n$ ,  $k \geq 2$ , let  $\alpha_i$  be the density of  $A_i$  for  $i = 1, \dots, k$ , and let  $d(A_1 + \dots + A_k)$  be the density of  $A_1 + \dots + A_k$ . Then*

$$1 - d(A_1 + \dots + A_k) \leq (1 - \alpha_1) \cdots (1 - \alpha_k).$$

*Proof.* If  $k = 2$  then Theorem 2 implies that  $1 - d(A_1 + A_2) \leq 1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 = (1 - \alpha_1)(1 - \alpha_2)$ . Hence assume  $1 - d(A_1 + \dots + A_{k-1}) \leq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})$ . Then

$$\begin{aligned} 1 - d(A_1 + \dots + A_{k-1} + A_k) &\leq [1 - d(A_1 + \dots + A_{k-1})](1 - \alpha_k) \\ &\leq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})(1 - \alpha_k). \end{aligned}$$

**COROLLARY 2.** *If  $A$  is any subset of  $Q_n$  with density  $\alpha > 0$  then there exists an integer  $k > 0$  such that  $kA = Q_n$ .*

*Proof.* There exists an integer  $m > 0$  such that  $(1 - \alpha)^m \leq 1/2$ . Let  $d(mA)$  be the density of  $mA$ . Then Corollary 1 implies that  $1 - d(mA) \leq (1 - \alpha)^m \leq 1/2$ , or  $d(mA) \geq 1/2$ . From Theorem 1,  $d(mA) + d(mA) \geq 1$  implies  $d(2mA) = 1$ , or  $2mA = Q_n$ .

**4. Remark.** We may identify  $Q_2$  with the set of nonzero Gaussian integers  $x + yi$  for which  $x$  and  $y$  are both nonnegative rational integers. Luther Cheo [1] defined density for subsets of this  $Q_2$  as follows, using our notation.

**DEFINITION 3.** Let  $x_0 + y_0i$  be any element of  $Q_2$  and  $S$  the set of all  $x + yi$  in  $Q_2$  such that  $x \leq x_0$  and  $y \leq y_0$ . Then for any subset  $A$  of  $Q_2$  the *density* of  $A$  is the quantity

$$\alpha_c = \text{glb}_S \frac{A(S)}{Q_2(S)}.$$

Cheo proved Theorem 1 for his density and also a theorem which implies that if  $ji$  is in  $A$  for all  $j = 1, 2, \dots$ , and if  $\alpha_c, \beta_c, \gamma_c$  are the Cheo densities of  $A, B, C = A + B$ , respectively, then

$$\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c.$$

We cannot remove the requirement that all  $ji$  be in  $A$  by means of an argument like that used to establish Theorem 2 since it would be necessary to partition  $H$  in such a way that the sets  $L'_j$  are of the type  $S$  used in defining the Cheo density, and this is not always possible. Consider, for example, the set  $R = \{x + yi: x + yi \text{ is in } Q_2, x \leq 4, y \leq 3\}$ , and let  $A \cap R = \{1, i, 3 + 3i\}$ . Then  $H = R - A$  cannot be so partitioned, as the reader can easily verify.

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