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In a previous paper [1] we established a condition (Theorem I) for real numbers such that, in a linear space of dimension at least 2, every point of a 2-bounded set can always be represented as a sum of boundary points of the set, multiplied by these numbers. It is natural to ask for the corresponding condition in the case of complex numbers. Multiplication of a point by a real or complex number can be regarded as a special similarity. A more general theorem in which these similarities are replaced by linear transformations, or operators, will be proved in the present paper.

DEFINITION. Let $B$ be a real Banach space with conjugate space $B'$. Let $S \subseteq B$ and $x' \in B', \|x'\| = 1$. The $x'$-width of $S$ is

$$w_x(S) = \sup_{x, y \in S} (x - y)x', \quad w_x(\phi) = -\infty.$$  

The width of $S$ is $w(S) = \inf w_x(S)$.

Let $A$ be a linear transformation of $B$ and $A^*$ the adjoint operation on $B'$ defined by $x(x'A^*) = (xA)x'$. Then $x'A^* = 0$ or we can define $x'A^* = x'A^*/\|x'A^*\|$.

In the following all sets are assumed to be in a real Banach space.

**LEMMA 1. (1)** If $S$ is bounded then $w_x(S)$ is a continuous function of $x'$.

(2) $w_x(S + T) = w_x(S) + w_x(T)$ (with the proviso that $-\infty$ added to anything—even $+\infty$—is $-\infty$).

(3) If $S$ has interior points then $w(S) > 0$.

(4) $w_{x'}(S\mathcal{A}) = \begin{cases} 0 & \text{if } x'A^* = 0 \\ w_{x'A^*}(S) \cdot \|x'A^*\| & \text{if } x'A^* \neq 0. \end{cases}$

The proofs are all obvious.

**LEMMA 2.** Let $T$ be a connected set so that no translate of $-T$ is contained in the interior of $S$, then $S + T \subseteq T + \text{bd} S$.

Proof. Let $s \in S$, $t \in T$; then $s + t - T$ contains $s \in S$ but is not contained in the interior of $S$. Hence $(s + t - T) \cap \text{bd} S$ is not empty and $s + T \subseteq T + \text{bd} S$.
Lemma 3. If $S$ is bounded and $-\text{cl}S \subset \text{int} T$ then no translate of $-\text{cl}T$ is contained in $\text{int} S$.

Proof. For one-dimensional spaces this is obvious since the hypothesis implies $\text{diam } S < \text{diam } T$. If the lemma were false then $a - \text{cl} T \subset \text{int } S$ for some point $a$. The mapping $x \to a - x$ leaves the lines through $a/2$ invariant and the contradiction follows from the fact that the inclusion is false for the intersection of the sets with such lines $l$ for which $l \cap \text{int } S \neq \emptyset$.

Lemma 4. Let $w_x(S) < \infty$, let $T$ be a connected set, and let $U = (S + T) \setminus (T + \text{bd } S)$, then

$$w_x(U) \leq w_x(S) - w_x(T).$$

Proof. If $w_x(T) = \infty$ then $S + T \subset T + \text{bd } S$ by Lemma 2. If $w_x(T) < \infty$ let $a = \inf_{s \in S} sx'$, $b = \sup_{s \in S} sx'$, $c = \inf_{t \in T} tx'$, $d = \sup_{t \in T} tx'$. If $s \in S$, $t \in T$ so that $(s + t)x' < a + d$ then $s + t - T$ contains $s$ in $S$ and $\inf_{t \in T} (s + t - t)x' < a$ so that $s + t - T$ contains points in the complement of $S$. Since $s + t - T$ is connected it follows that $(s + t - T) \cap \text{bd } S \neq \emptyset$ or $s + t \in T + \text{bd } S$. Thus $\inf_{u \in U} ux' \leq a + d$.

Similarly, if $s \in S$, $t \in T$ and $(s + t)x' > b + c$ then $s + t - T$ contains $s \in S$ while $\sup_{t \in T} (s + t - t)x' > b$ so that $s + t - T$ contains points in the complement of $S$. Hence $(s + t - T) \cap \text{bd } S \neq \emptyset$ and $s + t \in T + \text{bd } S$. Thus $\sup_{u \in U} ux' \leq b + c$, and hence

$$w_x(U) = \sup_{u \in U} ux' - \inf_{u \in U} ux' \leq (b + c) - (a + d) = (b - a) - (d - c) = w_x(S) - w_x(T).$$

Definition. Let $S$ be a bounded connected set in $B$. The outer set, $oS$, of $S$ is the complement of the unbounded component of the complement of $S$ and the outer boundary, $\text{obd } S$, of $S$ is the boundary of $oS$. Clearly $\text{obd } S \subset \text{bd } S$ and if $\dim B \geq 2$ then $\text{obd } S$ is connected.

Theorem 1. Let $S_1, S_2, \cdots, S_n$ be bounded connected sets in $B$ with $\dim B \geq 2$ so that no translate of $-\text{cl } oS_i$ is contained in $\text{int } oS_i (i = 2, \cdots, n)$. Then

$$w_x((S_1 + S_2 + \cdots + S_n) \setminus (\text{obd } S_1 + \text{obd } S_2 + \cdots + \text{obd } S_n)) \leq w_x(S_1) - w_x(S_2) - \cdots - w_x(S_n).$$

Proof. By repeated application of Lemma 2 we have $S_1 + \cdots + S_n \subset oS_1 + \cdots + oS_n \subset oS_1 + \text{obd } S_2 + \cdots + \text{obd } S_n$ and the theorem follows from Lemma 4 where $oS_1$ plays the role of $S$ and $\text{obd } S_1 + \cdots + \text{obd } S_n$ that of $T$. 
COROLLARY. If \( S_1, \ldots, S_n \) satisfy the conditions of Theorem 1 and in addition for each \( i \) there is an \( x'_i \) so that \( w_{x'_i}(S_i) < \sum_{i \neq j} w_{x'_j}(S_j) \) then \( S_1 + \cdots + S_n \subset \text{obd } S_1 + \cdots + \text{obd } S_n \).

DEFINITION. Let \( B \) be a real Banach space with \( \dim B \geq 2 \). A set of bounded linear operators \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) is admissible if for every bounded set \( S \subset B \) and every point \( p \in S \) there exist outer boundary points \( x_1, \ldots, x_n \in \text{obd } S \) such that
\[
p = x_1 \mathcal{A}_1 + \cdots + x_n \mathcal{A}_n.
\]

THEOREM 2. If a set \( \mathcal{A} \) of operators \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) is admissible then
\[\begin{align*}
(\text{i}) & \quad \mathcal{A}_1 + \cdots + \mathcal{A}_n = \mathcal{I}, \text{ the identity.} \\
(\text{ii}) & \quad \text{For each } i \text{ there exists an } x' \in B', x' \neq 0 \text{ such that}
\quad ||x' \mathcal{A}_i^*|| \leq \sum_{j \neq i} ||x' \mathcal{A}_j^*||.
\end{align*}\]

If \( B \) is finite dimensional, \( \dim B \geq 2 \), and \( \mathcal{A} \) satisfies (i) and
\[\begin{align*}
(\text{ii}') & \quad ||x' \mathcal{A}_i^*|| \leq \sum_{j \neq i} ||x' \mathcal{A}_j^*||, \\
& \quad i = 1, \ldots, n
\end{align*}\]
for all \( x' \in B' \) then \( \mathcal{A} \) is admissible.

Proof. The necessity of (i) and (ii) is nearly obvious. If \( \mathcal{A}_1 + \cdots + \mathcal{A}_n \neq \mathcal{I} \), let \( p \in B \) be a point which is not invariant under \( \mathcal{A}_1 + \cdots + \mathcal{A}_n \) and let \( S = \{p\} \).

If \( S \) is the unit ball of \( B \) and
\[
0 = x_1 \mathcal{A}_1 + \cdots + x_n \mathcal{A}_n, \quad ||x_1|| = \cdots = ||x_n|| = 1
\]
then
\[
||x_i \mathcal{A}_i x'|| \leq \sum_{j \neq i} ||x_j \mathcal{A}_j x'||
\]
or
\[
||x_i x' \mathcal{A}_i^*|| \leq \sum_{j \neq i} ||x_j x' \mathcal{A}_j^*||.
\]

Now if \( \inf_{||x||=1} ||x \mathcal{A}_i|| = 0 \), then for every \( \varepsilon > 0 \) there exists an \( x' \) with \( ||x'|| = 1 \) and \( ||x' \mathcal{A}_i^*|| < \varepsilon \) and (ii) is trivial. If \( \inf_{||x||=1} ||x \mathcal{A}_i|| > 0 \) then \( \mathcal{A}_i^* \) is onto and we can pick \( x' \) so that \( ||x_i x' \mathcal{A}_i^*|| = ||x' \mathcal{A}_i^*|| \) and hence \( ||x' \mathcal{A}_i^*|| \leq \sum_{j \neq i} ||x_j x' \mathcal{A}_j^*|| \leq \sum_{j \neq i} ||x' \mathcal{A}_j^*|| \).

To prove the sufficiency of (i) and (ii) we may restrict attention to connected sets since we may consider the component of \( p \) in \( S \). Let \( S_i = \mathcal{A} \mathcal{S}_i \). If for each \( S_i \) there is an \( S_j \) so that \( j \neq i \) and no translate of \( -\text{cl } S_j \) is contained in \( \text{int } S_i \) then according to Lemma 2 we have
Since $B$ is finite dimensional we have $\text{obd} S_i = (\text{obd} S) \mathcal{A}_i$ so that

$$S \subseteq (\text{obd} S) \mathcal{A}_i + \cdots + (\text{obd} S) \mathcal{A}_n$$

which was to be proved. We may therefore assume that $-\text{cl} S_j$ has a translate in $\text{int} S_i$ for each $j = 2, \cdots, n$. Then according to Lemma 3 and Theorem 1

$$w_x((S_i + \cdots + S_n)(\text{obd} S_i + \cdots + \text{obd} S_n)) \leq w_x(S_i) - w_x(S_2) - \cdots - w_x(S_n).$$

Since $S_i$ has an interior $\mathcal{A}_i$, and hence $\mathcal{A}_i^*$, are regular and we can choose $x'$ so that $w_{x'}(S) = w(S)$ where $x' = x'\mathcal{A}_i^* / \| x' \mathcal{A}_i^* \|$. By part (4) of Lemma 1 we have $w_{x'}(S_j) \geq w(S) \cdot \| x' \mathcal{A}_j^* \|$. Thus (1) becomes

$$w_x((S_i + \cdots + S_n)(\text{obd} S_i + \cdots + \text{obd} S_n)) \leq w(S)(\| x' \mathcal{A}_i^* \| - \sum_{j \neq i} \| x' \mathcal{A}_j^* \|) \leq 0$$

so that $(S_i + \cdots + S_n)(\text{obd} S_i + \cdots + \text{obd} S_n)$ has no interior points and is therefore empty since $\text{obd} S_i + \cdots + \text{obd} S_n$ is closed. So we have again

$$S \subseteq S_i + \cdots + S_n \subseteq \text{obd} S_i + \cdots + \text{obd} S_n$$

$$= (\text{obd} S) \mathcal{A}_i + \cdots + (\text{obd} S) \mathcal{A}_n.$$

**Remark.** The hypothesis that $B$ is finite dimensional can be dropped if we assume that the mappings $\mathcal{A}_i$ are onto. If the $\mathcal{A}_i$ are similarities of $B$ onto itself then (ii) and (ii') have the same simple form

$$(\text{ii}'') \quad \| \mathcal{A}_i \| \leq \sum_{j \neq i} \| \mathcal{A}_j \| \quad i = 1, \cdots, n.$$  

We thus have the following:

**Theorem 2'.** A set of similarities $\mathcal{A}_1, \cdots, \mathcal{A}_n$ of a Banach space $B$ of dimension at least 2 onto itself is admissible if and only if it satisfies conditions (i) and (ii'').

In the manner analogous to that used in [1] we can generalize the validity of Theorem 2 to a class of linear spaces which we define as follows.
DEFINITIONS. Let $B$ be a linear space and let $\mathcal{F}$ be a family of linear transformations of $B$ onto itself so that $\mathcal{F}$ is transitive on the nonzero elements of $B$. A $B$-space $S$ is a linear subspace of a (finite or infinite) direct product of copies of $B$ that is closed under simultaneous application of $\mathcal{F}$ to the components of a point. If $x, y \in S$ and $y \neq 0$ then $\{x + yF | F \in \mathcal{F}\}$ is a $B$-subspace of $S$. The $B$-subspaces can be given the topology of $B$ by the association $x + yF \rightarrow zF$, $z \in B$, $z \neq 0$ where the choice of $z$ is arbitrary due to the transitivity of $\mathcal{F}$. We can therefore define boundedness in $B$-subspaces (if boundedness is defined in $B$) and a set in $S$ is $B$-bounded if through every point of the set there is a $B$-subspace whose intersection with the set is bounded.

THEOREM 3. Theorem 2 remains valid for $B$-bounded sets in a $B$-space where $B$ satisfies the conditions stated in Theorem 2. If $B$ is one-dimensional then the same theorem holds for sets which are 2-bounded (in the sense of [1]) and satisfy the other conditions of Theorem 2.

This is an immediate consequence of Theorem 2 if we consider the bounded intersection of $S$ with a $B$-subspace through a point $p$ of $S$.

Theorem 3 applied to the conditions of Theorem 2' subsums the results of [1]. As one application we give the following:

THEOREM 4. Let $f(z)$ be analytic in a proper subdomain $D$ of the Riemann sphere and continuous in $\text{cl } D$. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers satisfying

(i) \[ \alpha_1 + \cdots + \alpha_n = 1 \]

and

(ii) \[ |\alpha_i| \leq \sum_{j \neq i} |\alpha_j| . \]

Then for every $z_0 \in D$ there exist $z_1, \ldots, z_n$ in $\text{bd } D$ such that

\[ f(z_0) = \alpha_1 f(z_1) + \cdots + \alpha_n f(z_n) . \]

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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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