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**REPRESENTATION OF A POINT OF A SET AS SUM OF
TRANSFORMS OF BOUNDARY POINTS**

T. S. MOTZKIN AND ERNST GABOR STRAUS

REPRESENTATION OF A POINT OF A SET AS SUM OF TRANSFORMS OF BOUNDARY POINTS

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In a previous paper [1] we established a condition (Theorem I) for real numbers such that, in a linear space of dimension at least 2, every point of a 2-bounded set can always be represented as a sum of boundary points of the set, multiplied by these numbers. It is natural to ask for the corresponding condition in the case of complex numbers. Multiplication of a point by a real or complex number can be regarded as a special similarity. A more general theorem in which these similarities are replaced by linear transformations, or operators, will be proved in the present paper.

DEFINITION. Let B be a real Banach space with conjugate space B' . Let $S \subset B$ and $x' \in B'$, $\|x'\| = 1$. The x' -width of S is

$$w_{x'}(S) = \sup_{x, y \in S} (x - y)x', \quad w_{x'}(\phi) = -\infty.$$

The width of S is $w(S) = \inf w_{x'}(S)$.

Let \mathfrak{A} be a linear transformation of B and \mathfrak{A}^* the adjoint operation on B' defined by $x(x'\mathfrak{A}^*) = (x\mathfrak{A})x'$. Then $x'\mathfrak{A}^* = 0$ or we can define $x'_{\mathfrak{A}} = x'\mathfrak{A}^* / \|x'\mathfrak{A}^*\|$.

In the following all sets are assumed to be in a real Banach space.

LEMMA 1. (1) *If S is bounded then $w_{x'}(S)$ is a continuous function of x' .*

(2) $w_x(S + T) = w_x(S) + w_x(T)$ (with the proviso that $-\infty$ added to anything—even $+\infty$ —is $-\infty$).

(3) *If S has interior points then $w(S) > 0$.*

$$(4) \quad w_{x'}(S\mathfrak{A}) = \begin{cases} 0 & \text{if } x'\mathfrak{A}^* = 0; \\ w_{x'_{\mathfrak{A}}}(S) \cdot \|x'\mathfrak{A}^*\| & \text{if } x'\mathfrak{A}^* \neq 0. \end{cases}$$

The proofs are all obvious.

LEMMA 2. *Let T be a connected set so that no translate of $-T$ is contained in the interior of S , then $S + T \subset T + \text{bd } S$.*

Proof. Let $s \in S$, $t \in T$; then $s + t - T$ contains $s \in S$ but is not contained in the interior of S . Hence $(s + t - T) \cap \text{bd } S$ is not empty and $s + T \subset T + \text{bd } S$.

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LEMMA 3. *If S is bounded and $-\text{cl}S \subset \text{int } T$ then no translate of $-\text{cl}T$ is contained in $\text{int } S$.*

Proof. For one-dimensional spaces this is obvious since the hypothesis implies $\text{diam } S < \text{diam } T$. If the lemma were false then $a - \text{cl}T \subset \text{int } S$ for some point a . The mapping $x \rightarrow a - x$ leaves the lines through $a/2$ invariant and the contradiction follows from the fact that the inclusion is false for the intersection of the sets with such lines l for which $l \cap \text{int } S \neq \phi$.

LEMMA 4. *Let $w_x(S) < \infty$, let T be a connected set, and let $U = (S + T) \setminus (T + \text{bd } S)$, then*

$$w_x(U) \leq w_x(S) - w_x(T).$$

Proof. If $w_x(T) = \infty$ then $S + T \subset T + \text{bd } S$ by Lemma 2. If $w_x(T) < \infty$ let $a = \inf_{s \in S} sx'$, $b = \sup_{s \in S} sx'$, $c = \inf_{t \in T} tx'$, $d = \sup_{t \in T} tx'$. If $s \in S$, $t \in T$ so that $(s + t)x' < a + d$ then $s + t - T$ contains s in S and $\inf_{t_1 \in T} (s + t - t_1)x' < a$ so that $s + t - T$ contains points in the complement of S . Since $s + t - T$ is connected it follows that $(s + t - T) \cap \text{bd } S \neq \phi$ or $s + t \in T + \text{bd } S$. Thus $\inf_{u \in U} ux' \geq a + d$.

Similarly, if $s \in S$, $t \in T$ and $(s + t)x' > b + c$ then $s + t - T$ contains $s \in S$ while $\sup_{t_1 \in T} (s + t - t_1)x' > b$ so that $s + t - T$ contains points in the complement of S . Hence $(s + t - T) \cap \text{bd } S \neq \phi$ and $s + t \in T + \text{bd } S$. Thus $\sup_{u \in U} ux' \leq b + c$, and hence

$$\begin{aligned} w_x(U) &= \sup_{u \in U} ux' - \inf_{u \in U} ux' \leq (b + c) - (a + d) = (b - a) - (d - c) \\ &= w_x(S) - w_x(T). \end{aligned}$$

DEFINITION. Let S be a bounded connected set in B . The *outer set*, oS , of S is the complement of the unbounded component of the complement of S and the *outer boundary*, $\text{obd } S$, of S is the boundary of oS . Clearly $\text{obd } S \subset \text{bd } S$ and if $\dim B \geq 2$ then $\text{obd } S$ is connected.

THEOREM 1. *Let S_1, S_2, \dots, S_n be bounded connected sets in B with $\dim B \geq 2$ so that no translate of $-\text{cl } oS_1$ is contained in $\text{int } oS_i$ ($i = 2, \dots, n$). Then*

$$\begin{aligned} w_x((S_1 + S_2 + \dots + S_n) \setminus (\text{obd } S_1 + \text{obd } S_2 + \dots + \text{obd } S_n)) \\ \leq w_x(S_1) - w_x(S_2) - \dots - w_x(S_n). \end{aligned}$$

Proof. By repeated application of Lemma 2 we have $S_1 + \dots + S_n \subset oS_1 + \dots + oS_n \subset oS_1 + \text{obd } S_2 + \dots + \text{obd } S_n$ and the theorem follows from Lemma 4 where oS_1 plays the role of S and $\text{obd } S_2 + \dots + \text{obd } S_n$ that of T .

COROLLARY. If S_1, \dots, S_n satisfy the conditions of Theorem 1 and in addition for each i there is an x'_i so that $w_{x'_i}(S_i) < \sum_{j \neq i} w_{x'_i}(S_j)$ then $S_1 + \dots + S_n \subset \text{obd } S_1 + \dots + \text{obd } S_n$.

DEFINITION. Let B be a real Banach space with $\dim B \geq 2$. A set of bounded linear operators $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is *admissible* if for every bounded set $S \subset B$ and every point $p \in S$ there exist outer boundary points $x_1, \dots, x_n \in \text{obd } S$ such that

$$p = x_1\mathfrak{A}_1 + \dots + x_n\mathfrak{A}_n .$$

THEOREM 2. If a set \mathfrak{A} of operators $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is admissible then

- (i) $\mathfrak{A}_1 + \dots + \mathfrak{A}_n = \mathcal{I}$, the identity.
- (ii) For each i there exists an $x' \in B'$, $x' \neq 0$ such that

$$\|x'\mathfrak{A}_i^*\| \leq \sum_{j \neq i} \|x'\mathfrak{A}_j^*\| .$$

If B is finite dimensional, $\dim B \geq 2$, and \mathfrak{A} satisfies (i) and

$$(ii') \quad \|x'\mathfrak{A}_i^*\| \leq \sum \|x'\mathfrak{A}_j^*\| , \quad i = 1, \dots, n$$

for all $x' \in B'$ then \mathfrak{A} is admissible.

Proof. The necessity of (i) and (ii) is nearly obvious. If $\mathfrak{A}_1 + \dots + \mathfrak{A}_n \neq \mathcal{I}$, let $p \in B$ be a point which is not invariant under $\mathfrak{A}_1 + \dots + \mathfrak{A}_n$ and let $S = \{p\}$.

If S is the unit ball of B and

$$0 = x_1\mathfrak{A}_1 + \dots + x_n\mathfrak{A}_n , \quad \|x_1\| = \dots = \|x_n\| = 1$$

then

$$\|x_i\mathfrak{A}_i x'\| \leq \sum_{j \neq i} \|x_j\mathfrak{A}_j x'\|$$

or

$$\|x_i x'\mathfrak{A}_i^*\| \leq \sum_{j \neq i} \|x_j x'\mathfrak{A}_j^*\| .$$

Now if $\inf_{\|x\|=1} \|x\mathfrak{A}_i\| = 0$, then for every $\varepsilon > 0$ there exists an x' with $\|x'\| = 1$ and $\|x'\mathfrak{A}_i^*\| < \varepsilon$ and (ii) is trivial. If $\inf_{\|x\|=1} \|x\mathfrak{A}_i\| > 0$ then \mathfrak{A}_i^* is onto and we can pick x' so that $\|x_i x'\mathfrak{A}_i^*\| = \|x'\mathfrak{A}_i^*\|$ and hence $\|x'\mathfrak{A}_i^*\| \leq \sum_{j \neq i} \|x_j x'\mathfrak{A}_j^*\| \leq \sum_{j \neq i} \|x'\mathfrak{A}_j^*\|$.

To prove the sufficiency of (i) and (ii') we may restrict attention to connected sets since we may consider the component of p in S . Let $S_i = S\mathfrak{A}_i$. If for each S_i there is an S_j so that $j \neq i$ and no translate of $-\text{cl } S_j$ is contained in $\text{int } S_i$ then according to Lemma 2 we have

$$\begin{aligned} S \subset S_1 + \dots + S_n &\subset oS_1 + \dots + oS_n \\ &\subset \text{obd } S_1 + (oS_2 + \dots + oS_n) \\ &\subset \text{obd } S_1 + \text{obd } S_2 + (oS_3 + \dots + oS_n) \subset \dots \\ &\subset \text{obd } S_1 + \dots + \text{obd } S_n . \end{aligned}$$

Since B is finite dimensional we have $\text{obd } S_i = (\text{obd } S)\mathfrak{A}_i$ so that

$$S \subset (\text{obd } S)\mathfrak{A}_1 + \dots + (\text{obd } S)\mathfrak{A}_n$$

which was to be proved. We may therefore assume that $-\text{cl } S_j$ has a translate in $\text{int } S_1$ for each $j = 2, \dots, n$. Then according to Lemma 3 and Theorem 1

$$\begin{aligned} (1) \quad w_{x'}((S_1 + \dots + S_n) \setminus (\text{obd } S_1 + \dots + \text{obd } S_n)) \\ \leq w_{x'}(S_1) - w_{x'}(S_2) - \dots - w_{x'}(S_n) . \end{aligned}$$

Since S_1 has an interior \mathfrak{A}_1 , and hence \mathfrak{A}_1^* , are regular and we can choose x' so that $w_{x'}(S) = w(S)$ where $x'_1 = x'\mathfrak{A}_1^*/\|x'\mathfrak{A}_1^*\|$. By part (4) of Lemma 1 we have $w_{x'}(S_j) \geq w(S) \cdot \|x'\mathfrak{A}_j\|$. Thus (1) becomes

$$\begin{aligned} w_{x'}((S_1 + \dots + S_n) \setminus (\text{obd } S_1 + \dots + \text{obd } S_n)) &\leq w(S)(\|x'\mathfrak{A}_1^*\| - \sum_{j \neq 1} \|x'\mathfrak{A}_j^*\|) \\ &\leq 0 \end{aligned}$$

so that $(S_1 + \dots + S_n) \setminus (\text{obd } S_1 + \dots + \text{obd } S_n)$ has no interior points and is therefore empty since $\text{obd } S_1 + \dots + \text{obd } S_n$ is closed. So we have again

$$\begin{aligned} S \subset S_1 + \dots + S_n &\subset \text{obd } S_1 + \dots + \text{obd } S_n \\ &= (\text{obd } S)\mathfrak{A}_1 + \dots + (\text{obd } S)\mathfrak{A}_n . \end{aligned}$$

REMARK. The hypothesis that B is finite dimensional can be dropped if we assume that the mappings \mathfrak{A}_i are onto. If the \mathfrak{A}_i are similarities of B onto itself then (ii) and (ii') have the same simple form

$$(ii'') \quad \|\mathfrak{A}_i\| \leq \sum_{j \neq i} \|\mathfrak{A}_j\| \quad i = 1, \dots, n .$$

We thus have the following:

THEOREM 2'. *A set of similarities $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ of a Banach space B of dimension at least 2 onto itself is admissible if and only if it satisfies conditions (i) and (ii'').*

In the manner analogous to that used in [1] we can generalize the validity of Theorem 2 to a class of linear spaces which we define as follows.

DEFINITIONS. Let B be a linear space and let \mathcal{F} be a family of linear transformations of B onto itself so that \mathcal{F} is transitive on the nonzero elements of B . A B -space S is a linear subspace of a (finite or infinite) direct product of copies of B that is closed under simultaneous application of \mathcal{F} to the components of a point. If $x, y \in S$ and $y \neq 0$ then $\{x + yF \mid F \in \mathcal{F}\}$ is a B -subspace of S . The B -subspaces can be given the topology of B by the association $x + yF \leftrightarrow zF$, $z \in B$, $z \neq 0$ where the choice of z is arbitrary due to the transitivity of \mathcal{F} . We can therefore define boundedness in B -subspaces (if boundedness is defined in B) and a set in S is B -bounded if through every point of the set there is a B -subspace whose intersection with the set is bounded.

THEOREM 3. *Theorem 2 remains valid for B -bounded sets in a B -space where B satisfies the conditions stated in Theorem 2. If B is one-dimensional then the same theorem holds for sets which are 2-bounded (in the sense of [1]) and satisfy the other conditions of Theorem 2.*

This is an immediate consequence of Theorem 2 if we consider the bounded intersection of S with a B -subspace through a point p of S .

Theorem 3 applied to the conditions of Theorem 2' subsums the results of [1]. As one application we give the following:

THEOREM 4. *Let $f(z)$ be analytic in a proper subdomain D of the Riemann sphere and continuous in $\text{cl } D$. Let $\alpha_1, \dots, \alpha_n$ be complex numbers satisfying*

$$(i) \quad \alpha_1 + \dots + \alpha_n = 1$$

and

$$(ii) \quad |\alpha_i| \leq \sum_{j \neq i} |\alpha_j|.$$

Then for every $z_0 \in D$ there exist z_1, \dots, z_n in $\text{bd } D$ such that

$$f(z_0) = \alpha_1 f(z_1) + \dots + \alpha_n f(z_n).$$

REFERENCE

1. T. S. Motzkin and E. G. Straus, *Representation of a point of a set as a linear combination of boundary points*, Proceedings of the Symposium on Convexity, Seattle 1961.

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