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ČEBYŠEV SUBSPACES OF FINITE CODIMENSION IN $C(X)$

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1. Introduction. Suppose that X is a compact Hausdorff space and that M is an n dimensional linear subspace of $C(X)$, the Banach space of all real-valued continuous functions f on X, with supremum norm. If f is in $C(X)$, the local compactness of M guarantees the existence of at least one function g in M such that $||f - g|| = d(f, M) = \inf\{||f - h||: h \in M\}$, i.e., f has a *nearest point* g in M. A well-known extension of a classical theorem of Haar (see e.g. [2, Theorem 3.6]) states that M is a *Čebyšev subspace* of $C(X)$ (that is, there is a *unique* nearest point in M to f for every f in $C(X)$) if and only if each nontrivial function in M has most $n - 1$ zeros.[1] In the present note we intend to investigate infinite dimensional closed subspaces M of $C(X)$ in the hope of characterizing those having the Čebyšev property. Except for Proposition 3, our attention will be restricted to closed M of *finite codimension*, that is, those M for which the factor space $C(X)/M$ is finite dimensional. (The dimension of this factor space is the same as the dimension of the annihilator $M^{\perp}$ of M, the subspace of $C(X)^*$ consisting of all those continuous linear functionals on $C(X)$ which vanish on M.) There is, of course, an additional problem when dealing with infinite dimensional M. We have no assurance that a function f in $C(X)$ has even *one* nearest point in M. A subspace M with the property that each f in $C(X)$ contains at least one nearest point in M will be called a *Haar subspace* (or be said to have the *Haar property*). We know of no characterization of the Haar subspaces of $C(X)$. (A general necessary condition is given in Proposition 2, but we show by example that it is not sufficient.) Thus, most of our results are devoted to characterizing the Čebyšev subspaces from among the Haar subspaces of finite codimension.

Mairhuber was the first to show (see the discussion and references in [2, p. 253]) that if $C(X)$ contains a Čebyšev subspace of finite dimension $n, n > 1$, then X must be homeomorphic with a subset of the circle $|z| = 1$ in the complex z-plane. We show that if $C(X)$ contains a Čebyšev subspace of finite *codimension* $n, n > 1$, then X is totally disconnected; we also prove that X can contain at most countably many isolated points. The examples in § 4 show that, for certain X, $C(X)$ contains Čebyšev subspaces of codimension n, for $n = 1, 2, 3, \cdots$. In these examples, X is always *extremally* disconnected (that is, the

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[1] What we call "Čebyšev" subspaces were called "Haar" subspaces in [2], but we defer here to a more common usage. We now use the term "Haar subspace" (below) to replace the term "proximinal" used in [2].

closure of every open set in X is open), but we don't know whether this property is necessary for the existence of Čebyšev subspaces in $C(X)$. To obtain our examples we make use of the well known fact [1, p. 445] that the space L^∞ can be realized as $C(X)$ (for a certain extremally disconnected X).

As usual, we identify the space $C(X)^*$ with $rca(X)$, the space of regular countably additive real-valued finite measures μ on the Borel subsets of X [1, p. 265]. (Throughout the rest of this paper we will refer to an element of $rca(X)$ as simply "a measure on X.") If μ is a measure on X, then $\mu = \mu^+ - \mu^-$ (where μ^+ and μ^- are nonnegative measures on X), $|\mu| = \mu^+ + \mu^-$ and $||\mu|| = |\mu|(X)$. For f in $C(X)$, the value of μ at f is given by $(f, \mu) = \int_X f\,d\mu$. The *support* $S(\mu)$ of a measure μ is the closed set which equals the complement of the union of all open sets U for which $|\mu|(U) = 0$. Most of what we say about Čebyšev subspaces M will be in terms of $S(\mu)$ for μ in $M^\perp$; for instance, if M is a Haar subspace of finite codimension n in $C(X)$, then in order that M be a Čebyšev subspace it is sufficient that $S(\mu) = X$ for each μ in $M^\perp \sim \{0\}$, and it is necessary (for each μ) that $X \sim S(\mu)$ contain at most $n - 1$ points. (Since $X \sim S(\mu)$ is in some sense the "zero set" of μ, this latter property is dual to that discovered by Haar.) The above sufficient condition is necessary if X contains no isolated points, and the above necessary condition is sufficient if X contains n or more isolated points. Examples in § 4 show that for $n = 2$ and X having one isolated point, these converse statements are false.

2. General results. In Lemma 1 we give a well-known characterization of the Haar subspaces of codimension one in $C(X)$ (see, e.g. [3, p. 165] and references cited there; see [5] for the complex case) which is basic to much of what follows. In general, if E is a normed linear space and M is a subspace of codimension one (so that $M = L^{-1}(0)$ for some continuous linear functional L on E and $M^\perp = RL$, the one dimensional space of all real multiples of L), an application of the Hahn-Banach theorem shows that M is a Haar subspace if and only if there exists f in E such that $||f|| = 1$ and $L(f) = ||L||$; equivalently, L attains its supremum on the unit ball of E. The set of all linear functionals having this latter property is denoted by P. In the case $E = C(X)$, then, we are interested in measures which represent such functionals; the set of such measures is also denoted by P.

LEMMA 1. *A measure μ is in P if and only if the supports $S(\mu^+)$ and $S(\mu^-)$ are disjoint.*

Proof. If $S(\mu^+)$ and $S(\mu^-)$ are disjoint, then we choose f in $C(X)$,

$\|f\| = 1$, such that $f = 1$ on $S(\mu^+)$, $f = -1$ on $S(\mu^-)$, and hence $(f, \mu) = \|\mu\|$. Conversely, suppose $\|f\| = 1$ and $(f, \mu) = \|\mu\| = \mu^+(X) + \mu^-(X) = \|\mu^+\| + \|\mu^-\|$. If $f(x) < 1$ for some x in $S(\mu^+)$, then $f < 1$ in some open set U containing x; since $\mu^+(U) > 0$, it follows that $(f, \mu^+) < \|\mu\|$ (and $-(f, \mu^-) \leqq \|\mu^-\|$) so that $(f, \mu) < \|\mu\|$, a contradiction. Thus, $f = 1$ on $S(\mu^+)$ and (similarly) $f = -1$ on $S(\mu^-)$, so that these sets must be disjoint.

It is clear that a measure μ is in P if and only if every real multiple of μ is in P, so that a subspace M of codimension one in $C(X)$ has the Haar property if and only if $M^\perp \subset P$. The conjecture that this latter relation characterizes Haar subspaces of any finite codimension in $C(X)$ is disproved by Example 3 in § 4. One implication remains valid, however, as we now show.

PROPOSITION 2. *Suppose that M is a Haar subspace of finite codimension in the normed linear space E. Then $M^\perp \subset P$.*

Proof. If $L \in M^\perp$ we may represent L as a linear functional on E/M by $L(f + M) = L(f)$. Since E/M is finite dimensional there exists an element F of E/M such that $\|F\| = 1$ and $L(F) = \|L\|$. Since F is a translate of M and since M has the Haar property, there exists an element f in F of least norm, i.e., there exists f in E such that $F = f + M$ and $\|f\| = \|F\| = 1$. It follows that $L(f) = \|L\|$, which completes the proof.

(Note that the above proof is valid under the weaker assumption that E/M is a reflexive Banach space.)

In the next proposition the Čebyšev property of a subspace M of $C(X)$ is formulated in terms of functions in M and measures in $M^\perp$. The central idea is a slight extension of a construction due to V. Pták [4].

PROPOSITION 3. *Suppose that M is a Haar subspace of $C(X)$. Then M fails to have the Čebyšev property if and only if there exist f in $M \sim \{0\}$ and μ in $M^\perp \cap P \sim \{0\}$ such that $f(S(\mu)) = 0$.*

Proof. If the Haar subspace M does not have the Čebyšev property there exists h in $C(X)$ and f in $M \sim \{0\}$ with $d(h, M) = 1 = \|h\| = \|h - f\|$. By the Hahn-Banach theorem we can choose μ in $M^\perp$ such that $(h, \mu) = (h - f, \mu) = 1 = \|\mu\|$. As in the proof of Lemma 1 we see that $h = h - f = 1$ on $S(\mu^+)$ and $h = h - f = -1$ on $S(\mu^-)$. It follows that $f = 0$ on $S(\mu) = S(\mu^+) \cup S(\mu^-)$. To prove the converse, suppose there exist μ and f with the stated properties; we can assume $\|\mu\| = 1 = \|f\|$. Choose g in $C(X)$ such that $g = 1$ on $S(\mu^+)$, $g = -1$ on $S(\mu^-)$ and $\|g\| = 1$. Let $h = g(1 - |f|)$; since $h = g$ on $S(\mu)$, we have $(h, \mu) = 1$. Furthermore, $\|h\| = 1$ and $|h| + |f| = |g| + (1 - |g|)|f| \leqq 1$, so $||h - f|| = 1$. Finally, if $e \in M$, then $1 = (h, \mu) = (h - e, \mu) \leqq ||h - e||$, so $d(h, M) = 1$ and hence M does not have the Čebyšev property.

COROLLARY 4. *If M is a Haar subspace of $C(X)$ such that $S(\mu) = X$ for each μ in $M^{\perp} \cap P \sim \{0\}$, then M has the Čebyšev property.*

COROLLARY 5. *Suppose that M is a closed subspace of $C(X)$ of codimension one. Then M is a Čebyšev subspace if and only if $M^{\perp} \subset P$ and $S(\mu) = X$ for each μ in $M^{\perp} \sim \{0\}$.*

Proof. By Lemma 1, $M^{\perp} \subset P$ is equivalent to the fact that M is a Haar subspace. If $S(\mu) \neq X$ for some μ in $M^{\perp} \sim \{0\}$, then there exists f in $C(X) \sim \{0\}$ which vanishes on $S(\mu)$, and hence $f \in \{g: (g, \mu) = 0\} = M$, which shows (by Proposition 3) that M is not a Čebyšev subspace. The remainder of the proof is a consequence of Corollary 4.

(A result of the above nature was pointed out to us (without proof) by O. Hustad, who noted that it leads to an easy counterexample to the sufficiency portion of Theorem 3.4 of [2].)

PROPOSITION 6. *Suppose that M is a Čebyšev subspace of codimension $n > 0$ in $C(X)$. Then the set $X \sim S(\mu)$ contains at most $n - 1$ points for each μ in $M^{\perp} \sim \{0\}$*

Proof. Suppose that for some μ in $M^{\perp} \sim \{0\}$ the set $X \sim S(\mu)$ contains n or more points. Denote by N the subspace of $C(X)$ consisting of all functions which vanish on $S(\mu)$; N must have dimension n or greater. Choose a basis $\mu_1, \mu_2, \cdots, \mu_n$ for $M^{\perp}$, with $\mu_1 = \mu$. The subspace $M_1 = \{g: (g, \mu_i) = 0, i = 2, 3, \cdots, n\}$ has codimension $n - 1$, hence there exists f in $M_1 \cap N \sim \{0\}$. Since we also have $(f, \mu_1) = 0$, it follows that $f \in M \sim \{0\}$; by Proposition 3, M is not a Čebyšev subspace.

3. Main results.

THEOREM 7. *Suppose that $C(X)$ contains a Čebyšev subspace M of finite codimension n, $n \geqq 2$. Then X is totally disconnected.*

Proof. Suppose that X contains a connected subset K such that K contains more than one point (and hence contains infinitely many points). Note that we must have $K \subset S(\mu)$ for each μ in $M^{\perp} \sim \{0\}$. [Indeed, if $A = K \sim S(\mu)$ were nonempty, it would (by Proposition 6) be a finite set and therefore K (being infinite) would intersect both $S(\mu)$ and A. But A and $S(\mu)$ are disjoint closed sets whose union contains K, an impossibility.] Since, by Proposition 2, $S(\mu^+)$ and $S(\mu^-)$ are disjoint closed sets, we must have $K \subset S(\mu^+)$ or $K \subset S(\mu^-)$, so that

$\mu(K) \neq 0$ for each μ in $M^{\perp} \sim \{0\}$. But if μ_1 and μ_2 are linearly independent measures in $M^{\perp}$, we see that the nontrivial measure $\mu_1(K)\mu_2 - \mu_2(K)\mu_1$ vanishes on K, a contradiction which completes the proof.

THEOREM 8. *If $C(X)$ contains a Čebyšev subspace M of finite codimension n $(n \geqq 1)$, then X contains at most countably many isolated points.*

Proof. Choose μ in $M^{\perp} \sim \{0\}$; by Proposition 6, $X \sim S(\mu)$ contains only finitely many points. Now, $|\mu|(\{x\}) > 0$ for any isolated point x in X for which $x \in S(\mu)$. Since $|\mu|$ is a countably additive finite measure, its support can not contain uncountably many pairwise disjoint sets of positive measure. This shows that $S(\mu)$ (and hence X) contains at most countably many isolated points of X.

An example of a space $C(X)$ which contains no Čebyšev subspace of finite codimension may be obtained by letting X be a compactification of an uncountable discrete set.

THEOREM 9. *Suppose that X contains n or more isolated points. A Haar subspace M of codimension n $(n \geqq 1)$ in $C(X)$ is a Čebyšev subspace if and only if $X \sim S(\mu)$ contains at most $n - 1$ points for each μ in $M^{\perp} \sim \{0\}$.*

Proof. The necessity portion follows from Proposition 6. To prove the sufficiency, suppose that M is not a Čebyšev subspace of $C(X)$; we will produce a measure ν in $M^{\perp} \sim \{0\}$ such that $X \sim S(\nu)$ contains n or more points. By Proposition 3 there exists f in $M \sim \{0\}$ and μ in $M^{\perp} \cap P \sim \{0\}$ such that $f = 0$ on $S(\mu)$. Thus, $X \sim S(\mu)$ is nonempty, and if it contained n or more points, our proof would be complete. Suppose that $X \sim S(\mu)$ contains fewer than n points. We will show that if $\nu_0 \in M^{\perp} \sim \{0\}$ is such that $S(\nu_0) \subset S(\mu)$ and $X \sim S(\nu_0)$ contains fewer than n points, then there exists ν_1 in $M^{\perp} \sim \{0\}$ such that $S(\nu_1)$ is a proper subset of $S(\nu_0)$. (Once we have shown this, an obvious induction will complete the proof.) Let X_0 denote the set $X \sim S(\nu_0)$; by assumption, X_0 contains exactly k points, $1 \leqq k \leqq n - 1$. We first obtain an element ν_2 in $M^{\perp}$ for which $S(\nu_2) \subset S(\nu_0)$ and which is linearly independent of ν_0; this is done as follows: Choose a basis $\nu_0, \mu_1, \cdots, \mu_{n-1}$ for $M^{\perp}$ and let M_0 be the subspace of $C(X_0)^*$ spanned by the restrictions of the measures $\mu_1, \cdots, \mu_{n-1}$ to X_0. Since $S(\nu_0) \subset S(\mu)$, we have $f = 0$ on $S(\nu_0)$, while $(f, \mu_i) = 0$ for $i = 1, \cdots, n - 1$; furthermore the restriction of f to X_0 is not identically zero. This shows that $M_0^{\perp}$ is a proper subspace of the k dimensional space $C(X_0)^*$, so that $M_0^{\perp}$ has dimension at most $k - 1 \leqq n - 2$. Hence there exists a nontrivial linear combination ν_2 of the measures μ_i which vanishes at each point of X_0, so that

$S(\nu_2) \subset S(\nu_0)$. By hypothesis, X contains at least n isolated points, so one of them, say x, is in $S(\nu_0)$. Since ν_0 and ν_2 are linearly independent, the measure $\nu_1 = \nu_2(\{x\})\nu_0 - \nu_0(\{x\})\nu_2$ is nontrivial, has support in $S(\nu_0)$, and vanishes at x. The latter property shows that $S(\nu_1) \neq S(\nu_0)$, which completes the proof.

Example 2 of § 4 shows that this theorem is invalid if the codimension of M is greater than the number of isolated points of X.

THEOREM 10. *Suppose that X contains no isolated points. A Haar subspace M of finite codimension in $C(X)$ is a Čebyšev subspace if and only if $S(\mu) = X$ for each μ in $M^\perp \sim \{0\}$.*

Proof. The sufficiency portion follows from Corollary 4. To complete the proof, suppose that M is a Čebyšev subspace and there exists μ in $M^\perp \sim \{0\}$ such that $X \sim S(\mu)$ is nonempty. By Proposition 6, this set contains only finitely many points; since their union is open, they are isolated points, a contradiction.

Example 1 of § 4 shows that this theorem fails to be true if X contains an isolated point. It is interesting to note that an argument similar to (but simpler than) the one in Theorem 7 shows that if $C(X)$ contains a Haar subspace M of finite codimension n $(n \geqq 2)$ such that $S(\mu) = X$ for each μ in $M^\perp \sim \{0\}$, then X contains no isolated point.

4. **Examples.** As mentioned in the introduction, our examples are obtained by exploiting the fact that the space L^∞ can be "realized" as $C(X)$. The connections between X and the measure space on which L^∞ is defined are certainly well known, but we know of no explicit reference to them, so the next few paragraphs are devoted to a sketch of the material we need.

Suppose that (T, Σ, λ) is a σ-finite measure space; then there exists a compact Hausdoff space X_T and an isometry J from $L^\infty(T, \Sigma, \lambda)$ onto $C(X_T)$ which is linear, multiplicative (i.e., if $h = fg$ a.e. in L^∞, then $Jh = JfJg$ in $C(X_T)$) and carries a.e. nonnegative elements of L^∞ into nonnegative functions in $C(X_T)$ (see, e.g. [1, p. 445[). Hence, if χ_E is the characteristic function of the measurable set E in T, then $J\chi_E$ is a characteristic function in $C(X_T)$ (since $J\chi_E = J(\chi_E^2) = (J\chi_E)^2$). Writing $J\chi_E = \chi_{\phi E}$, we have defined a map from the σ-algebra Σ (modulo sets of measure zero) onto the family of all open and closed subsets of X_T (the proof is the same as in [1, p. 312]). It is readily seen that ϕ maps the atoms of (T, Σ, λ) (i.e. those A in Σ of finite positive measure such that $B \subset A$ and B in Σ imply $\lambda(B) = 0$ or $\lambda(B) = \lambda(A)$) in a one-to-one fashion onto the isolated points of X_T; this is a consequence of the fact that the elements of $L^\infty(T, \Sigma, \lambda)$ are constant a.e. on each atom.

If $f \in L^1(T, \Sigma, \lambda)$, then f defines (in the natural way) a continuous linear functional on L^∞. Since $(L^\infty)^*$ and $C(X_T)^*$ are isometric there exists a unique measure $\nu(f)$ on X_T corresponding to the functional defined on L^∞ by f. (This correspondence may be described by the equation

$$\int_T gf d\lambda = \int_{X_T} Jg d\nu(f) , \quad g \in L^\infty .$$

Since J carries nonnegative elements of L^∞ onto nonnegative functions in $C(X_T)$, it follows that $\nu(|f|) = |\nu(f)|$. If we define the *support* $S(f)$ of f in L^1 to be the complement in T of the set on which $|f| = 0$ a.e., then $\phi(S(f)) = S(\nu(f))$.

We may now obtain Haar subspaces of $C(X_T)$ as follows: If N is a closed subspace of L^1, its annihilator M in L^∞ is weak*-closed and hence is a Haar subspace [2, p. 239]; it follows that $J(M)$ is a Haar subspace of $C(X_T)$. If N is finite dimensional, then we may identify N (by means of the natural embedding of L^1 into $(L^\infty)^*$) with $M^\perp = (N^\perp)^\perp$ in $(L^\infty)^*$. It follows that $(JM)^\perp$ in $C(X_T)^*$ consists of those measures of the form $\nu(f)$, for f in N. We now apply these remarks to the construction of two examples.

EXAMPLE 1. There exists a compact Hausdorff space X for which the following is true:

(i) X contains exactly one isolated point

(ii) $C(X)$ contains a Čebyšev subspace M of codimension 2, but $S(\mu) \neq X$ for some μ in $M^\perp \sim \{0\}$.

Proof. We will obtain X as the space X_T corresponding to (T, Σ, λ), where $T = [0, 1] \cup \{2\}$, Σ is the family of Borel subsets of T, and λ is Lebesque measure on the Borel subsets of $[0, 1]$, but $\lambda(\{2\}) = 1$. We define f_0 and f_1 in $L^1(T, \Sigma, \lambda)$ by $f_0 = 1$ on $T, f_1(x) = x$ if $0 \leqq x \leqq 1$, $f_1(2) = 0$. If N is the two dimensional space spanned by f_0 and f_1, then, as noted above, its annihilator M in L^∞ is a Haar subspace, and we may consider N and $M^\perp$ to be the same subspace of $C(X_T)^*$. The atom $\{2\}$ corresponds to an isolated point of X_T, and it is not in the support of the measure corresponding to f_1 (since it is not in $S(f_1)$. The subspace M is a Čebyšev subspace, however. If it were not, then by Proposition 3 there would exist g in $M \sim \{0\}$ such that $g = 0$ on the support of some measure in $M^\perp \sim \{0\}$. Equivalently, there would be g in $M \sim \{0\} \subset L^\infty$ and constants a_0 and a_1 such that $g = 0$ a.e. on $S(a_0 f_0 + a_1 f_1)$. If $a_0 = 0$, this implies $g = 0$ a.e. on $[0, 1]$; since $0 = (g, f_0) = \int_T g f_0 d\lambda = g(2)$, we see that $g = 0$ a.e. If $a_0 \neq 0$, then $S(a_0 f_0 + a_1 f_1) = T$ and therefore $g = 0$ a.e. Thus, no such g exists, which completes the proof.

EXAMPLE 2. There exists a compact Hausdorff space X for which the following is true:

(i) X contains exactly one isolated point.

(ii) For each $n \geqq 2$, $C(X)$ contains a Haar subspace M of codimension n which is not a Čebyšev subspace, although $X \sim S(\mu)$ is one point for each μ in $M^{\perp} \sim \{0\}$.

Proof. We let $X = X_T$, where (T, Σ, λ) is defined as in Example 1. Let N be the linear subspace of $L^1(T, \Sigma, \lambda)$ spanned by the functions $f_0, f_1, \cdots, f_{n-1}$, where $f_k(x) = x^k$ for $0 \leqq x \leqq 1$, and $f_k(2) = 0$, $k = 0, 1, \cdots, n-1$. As before, the annihilator M of N in L^∞ is a Haar subspace of codimension n; it is not a Čebyšev subspace, however, since it contains the function g which is zero on $[0, 1]$, while $g(2) = 1$. (This function is zero on $S(f_0)$, say.) Clearly, the isolated point of X_T corresponding to the atom $\{2\}$ is in $X_T \sim S(\mu)$ for each μ in $M^{\perp}$.

EXAMPLE 3. There exists a compact Hausdorff space X and a closed subspace M of codimension 2 in $C(X)$ such that $M^{\perp} \subset P$ but M is not a Haar subspace.

Proof. We take $C(X)$ to be the space c of all convergent sequences $f = \{f_n\}_{n=1}^\infty$ of real numbers (so that X is the one-point compactification of the integers). The space $C(X)^*$ is isometric with the space l of absolutely summable sequences, under the following correspondence: If $f \in c$ and if $\mu = \{\mu_n\}_{n=1}^\infty \in l$, then $(f, \mu) = \sum_{n=1}^\infty f_n \mu_n + \mu_0 \lim f_n$. Define measures μ^1 and μ^2 by $\mu_n^1 = 2^{-n}$, $n = 0, 1, \cdots$ and $\mu_n^2 = 4^{-n}$, $n = 1, 2, 3, \cdots$, $\mu_0^2 = 0$. For any real number a the sequence $(\mu^1 + a\mu^2)_n$ is eventually positive (and equals 1 at $n = 0$), so that the measure $\mu^1 + a\mu^2$ has disjoint positive and negative supports. It follows that the same is true for $b\mu^1 + a\mu^2$, a, b real. Let $M = \{f: (f, \mu^1) = 0 = (f, \mu^2)\}$; M is a closed subspace of c, and the previous remarks show that $M^{\perp} \subset P$. To see that M is not a Haar subspace it suffices to show that the translate M_1 of M defined by $M_1 = \{f: (f, \mu^1) = 1 = (f, \mu^2)\}$ does not contain a point of least norm. Let $m = \inf\{\|f\|: f \in M_1\}$, and suppose that there exists f in M_1 such that $\|f\| = m$. We can choose μ in $M^{\perp}$ such that $\|\mu\| = 1$ and $(f, \mu) = m$; letting $g = m^{-1}f$, we see that $\|g\| = 1 = (g, \mu)$. It follows that $g = 1$ on $S(\mu^+)$, $g = -1$ on $S(\mu^-)$ and $|g| \leqq 1$ elsewhere. Since $\mu \in M^{\perp}$, the sequence $\{\mu_n\}$ is eventually positive (and $\mu_0 > 0$) or it is eventually negative (and $\mu_0 < 0$). Letting $\varepsilon = \operatorname{sgn} \mu_0$, we see that there exists an integer $N > 0$ such that $f_n = \varepsilon m$ if $n \geqq N$, while $|f_n| \leqq m$ if $n < N$. Since $(\mu^1 - \mu^2)_n$ is eventually positive, we may assume that N is so large that $\mu_n^1 - \mu_n^2 > 0$ for $n \geqq N$. By assumption,

$$1 = (f, \mu^i) = \varepsilon m\Big(\mu_0^i + \sum_{n=N}^{\infty} \mu_n^i\Big) + \sum_{n=1}^{N-1} f_n \mu_n^i \qquad (i = 1, 2) .$$

Subtracting and dividing by ε, we obtain

$$\begin{aligned} m\Big[1 + \sum_{n=N}^{\infty} (\mu_n^1 - \mu_n^2)\Big] &\leqq m \sum_{n=1}^{N-1} |\mu_n^1 - \mu_n^2| \\ &\leqq m \sum_{n=1}^{\infty} (2^{-n} - 4^{-n}) < m , \end{aligned}$$

a contradiction which completes the proof.

The connection between $L^\infty(T, \Sigma, \lambda)$ and $C(X_T)$ described above may be used to obtain new proofs of Theorems 2.2 and 2.3 of [2]; they are immediate corollaries of Theorems 10 and 9 (respectively). For instance, Theorem 9 yields the following result, which is stronger than Theorem 2.3 of [2].

COROLLARY. *Suppose that (T, Σ, λ) is a σ-finite measure space containing at least n atoms, and that N is a subspace of dimension n in $L^1(T, \Sigma, \lambda)$. Then $N^\perp$ is a Čebyšev subspace of $L^\infty(T, \Sigma, \lambda)$ if and only if each f in $N \sim \{0\}$ vanishes on at most $n-1$ atoms.*

Finally, the fact that for $n = 1, 2, 3, \cdots$ the space $L^1(T, \Sigma, \lambda)$ ($T = [0, 1]$, $\Sigma =$ Borel sets, $\lambda =$ Lesbegue measure) contains subspaces N of dimension n such that each f in $N \sim \{0\}$ is a.e. nonzero shows that $C(X_T)$ contains Čebyšev subspaces of each finite codimension.

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