COMBINATORIAL FUNCTIONS AND REGRESSIVE ISOLS

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1. Introduction. It is assumed that the reader is familiar with the notions: regressive function, regressive set, regressive isol, cosimple isol, combinatorial function and its canonical extension. The first four are defined in [2], the last two in [3]. Denote the set of all numbers (nonnegative integers) by \( \varepsilon \), the collection of all isols by \( A \), the collection of all regressive isols by \( A_R \) and the collection of all cosimple isols by \( A_\lambda \). The following four propositions will be used.

1. Let \( \tau = \rho t \) and \( \tau^* = \rho t^* \), where \( t_* \) and \( t_*^* \) are regressive functions. Then \( \tau \equiv \tau^* \iff t_* \equiv t_*^* \).

2. \( B \subseteq A \& A \in A_R \implies B \in A_R \).

3. Let \( F(T) \) be the canonical extension to \( A \) of the recursive, combinatorial function \( f(n) \). Then \( T \in A_R \implies F(T) \in A_R \).

4. \( B \subseteq A \& A \in A_\lambda \implies B \in A_\lambda \).

The first three are Propositions 3, 9(b) and Theorem 3(a) of [2] respectively. The fourth is Theorem 56(b) of [1].

**Definition.** Let \( f(n) \) be a one-to-one function from \( \varepsilon \) into \( \varepsilon \) and let \( T \in A_R - \varepsilon \). Then

\[
\phi_f(T) = \text{Req } \rho t_{f(n)},
\]

where \( t_* \) is any regressive function ranging over any set in \( T \).

Using (1) it is readily seen that \( \phi_f \) is a well defined function from \( A_R - \varepsilon \) into \( A - \varepsilon \). The main result of this paper is as follows: Let \( f(n) \) be a strictly increasing, recursive, combinatorial function; let \( F(X) \) be its canonical extension to \( A \), and let \( T \in A_R - \varepsilon \); then \( \phi_f(F(T)) = T \).

2. The operation \( \phi_f \).

**Proposition 1.** Let \( f(n) \) be a strictly increasing, recursive function and let \( T \in A_R - \varepsilon \). Then

\[
\phi_f(T) \leq T \quad \text{and} \quad \phi_f(T) \in A_R.
\]
If in addition $T \in A$, then $\phi_f(T) \in \Lambda \cdot A$.

Proof. In view of (2) and (4), it suffices to show only that $\phi_f(T) \leq T$. Let $t_n$ be a regressive function such that $\rho t = \tau \in T$. Put $\alpha = \rho f$ and suppose $p(x)$ is a regressing function of $t_n$. Define

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)] \text{ for } x \in \delta p.$$  

Then $p^*(t_n) = n$ and

$$\rho t_f \subseteq \{x \in \delta p^* \mid p^*(x) \in \alpha\},$$  

$$\tau - \rho t_f \subseteq \{x \in \delta p^* \mid p^*(x) \notin \alpha\}.$$  

Since $\alpha$ is recursive it follows that $\rho t_f$ is separable from $\tau - \rho t_f$. Hence $\phi_f(T) \leq T$.

It is known (by an unpublished result of Dekker) that $\Lambda_R$ is neither closed under addition nor under multiplication. We do, however, have some closure properties for isols of the type $\phi_f(T)$, where $T \in \Lambda_R - \varepsilon$ and $f(n)$ is a strictly increasing, recursive function.

**Proposition 2.** Let $f(n)$ and $g(n)$ be strictly increasing, recursive function and let $T \in \Lambda_R - \varepsilon$. Then

(a) $\phi_f(\phi_g(T)) \in \Lambda_R - \varepsilon$,  
(b) $\phi_f(T) \cdot \phi_g(T) \in \Lambda_R - \varepsilon$,  
(c) $\phi_f(T) + \phi_g(T) \in \Lambda_R - \varepsilon$.

Proof. In view of Proposition 1,

$$\phi_f(\phi_g(T)) \leq \phi_g(T) \leq T.$$  

This implies (a). To verify (a) one could also observe that $\phi_f(\phi_g(T)) = \phi_{fg}(T)$. Combining $\phi_f(T) \leq T$ and $\phi_g(T) \leq T$, we obtain by [1, Cor. of Thm. 77]

$$\phi_f(T) \cdot \phi_g(T) \leq T^2.$$  

However, $T^2 \in \Lambda_R - \varepsilon$ by (3). Hence (b) follows by (2). Finally, it is readily seen that

$$\phi_f(T) + \phi_g(T) \leq \phi_f(T) \cdot \phi_g(T),$$  

since $\phi_f(T)$ and $\phi_g(T)$ are $\geq 2$ (in fact, infinite). Thus (c) follows from (2) and (b).

3. The main result. We first state and prove two lemmas which might be of interest for their own sake. Let $\rho_{\alpha}, \rho_{\beta}, \cdots$ be the canonical enumeration of the class $Q$ of all finite sets defined by
\( \rho_0 = 0 \)
\[ \rho_{n+1} = (y_1, \ldots, y_k) \text{ where } y_1, \ldots, y_k \text{ are the distinct numbers such that } x + 1 = 2^{y_1} + \cdots + 2^{y_k}. \]

We denote the cardinality of \( \rho_x \) by \( r_x \).

**Lemma 1.** Let \( f(n) \) be any combinatorial function and let \( C_\ell \) be the function from \( \varepsilon \) into \( \varepsilon \) such that \( f(n) = \sum_{i=0}^n c_i \binom{n}{i}. \) Then
\[ f(n) = \sum_{x=0}^{2^n-1} c_{r(x)}. \]

**Proof.** Since every \( n \)-element set has \( \binom{n}{i} \) subsets of cardinality \( i \), we have
\[ f(n) = \text{card} \{ j(x, y) \mid \rho_x \subseteq (0, 1, \ldots, n-1) \text{ and } y < c_{r(x)} \}. \]
It follows from the definition of \( \rho_x \) that
\[ \rho_x \subseteq (0, 1, \ldots, n-1) \implies x \leq 2^0 + 2^1 + \cdots + 2^{n-1} \]
\[ \implies x \leq 2^n - 1. \]
Combining this with (5) we obtain
\[ f(n) = \text{card} \{ j(x, y) \mid x \leq 2^n - 1 \text{ and } y < c_{r(x)} \} = \sum_{x=0}^{2^n-1} c_{r(x)}. \]

**Definition.** Let \( a(n) \) be a one-to-one function from \( \varepsilon \) into \( \varepsilon \). Then
\[ a'(n) = l_{a(0)} \cdot 2^{a(0)} + \cdots + l_{a(n)} \cdot 2^{a(n)}, \]
where \( l_{a(0)}, \ldots, l_{a(n)} \) is the sequence of zeros and ones such that
\[ n = l_{a(0)} \cdot 2^0 + \cdots + l_{a(n)} \cdot 2^n. \]

**Lemma 2.** (Dekker) Let \( a(n) \) be a one-to-one function from \( \varepsilon \) into \( \varepsilon \) with range \( \alpha \) and let \( A = \text{Req}(\alpha) \). Then \( a'(n) \) is also a one-to-one function from \( \varepsilon \) into \( \varepsilon \). Moreover,
\[ a'(2^n) = 2^{a(\alpha)}, \quad \rho_x \subseteq \alpha \text{ and } \rho_x \in 2^\alpha. \]
Finally, if \( a(n) \) is regressive, so is \( a'(n) \).

**Proof.** It is clear that \( a'(n) \) is a one-to-one function such that \( a'(2^n) = 2^{a(\alpha)}. \) We have \( \rho_x \subseteq \alpha = o \) while \( a(\rho_x) = a(o) = o; \) for \( n \geq 1 \)
\[ \rho_x = \{ i \mid 0 \leq i \leq n \text{ and } l_{a(i)} = 1 \}. \]
Hence for every number \( n \)
\[ \rho_{a'(n)} = \{a(i) \mid 0 \leq i \leq n \ & l_{ni} = 1\} \]
\[ = a\{i \mid 0 \leq i \leq n \ & l_{ni} = 1\} = a(\rho_n). \]

Thus, if \( n \) ranges over \( \varepsilon \), \( \rho_n \) ranges over the class \( Q \) of all finite sets, \( \rho_{a'(n)} = a(\rho_n) \) over the class of all finite subsets of \( \alpha \). We conclude that \( \rho a' \in 2^\alpha \). Finally, assume that \( a(n) \) is a regressive function. Using the three facts that

\[ a'(n + 1) = l_{n+1,0} \cdot 2^{a(0)} + \cdots + l_{n+1,n+1} \cdot 2^{a(n+1)}, \]
\[ a'(n) = l_{n0} \cdot 2^{a(0)} + \cdots + l_{nn} \cdot 2^{a(n)}, \]
\[ \max \{i \mid l_{ni} = 1\} \leq \max \{i \mid l_{n+1,i} = 1\}, \]

we infer that \( a'(n) \) is a regressive function.

**Theorem.** Let \( f(n) \) be a strictly increasing, recursive combinatorial function, let \( F(X) \) be its canonical extension to \( A \) and let \( T \epsilon A \epsilon A - \varepsilon \). Then \( \phi_T(F(T)) = T \).

**Proof.** Let \( f(n) = \sum_{i=0}^n c_i \binom{n}{i} \) be the strictly increasing, recursive, combinatorial function. Then \( c_i > 0 \) since \( f(n) \) is strictly increasing, and \( c_i \) is a recursive function of \( i \), since \( f(n) \) is recursive. Let \( \tau \epsilon T \epsilon A - \varepsilon \) and assume that \( t_n \) is a regressive function ranging over \( \tau \). Put \( g(n) = t'(n) \). By Lemma 2 we have \( \rho_{g(n)} = t(\rho_n) \); thus, if \( n \) assumes successively the values 0, 1, 2, 3, 4, 5, 6, \( \cdots \), \( \rho_{g(n)} \) assumes successively the “values”

\[ 0, (t_0), (t_1), (t_0, t_1), (t_2), (t_0, t_2), (t_1, t_2), (t_0, t_1, t_2), \cdots. \]

We have by definition

\[ F(T) = \text{Req} \{j(x, y) \mid \rho_x \subset \tau \ & y < c_{r(x)}\} . \]

Since \( g(n) \) ranges without repetitions over \( \{n \mid \rho_n \subset \tau\} \), it follows that

\[ F(T) = \text{Req} \{j(g(x), y) \mid y < c_{r(x)}\} . \]

We shall use \( w_2 \) to denote the function which for 0, 1, \( \cdots \) takes on the values of the array

\[ j(g(0), 0), \cdots, j(g(0), c_{r(0)} - 1) \]
\[ j(g(1), 0), \cdots, j(g(1), c_{r(1)} - 1) \]
\[ j(g(2), 0), \cdots, j(g(2), c_{r(2)} - 1) \]
\[ \vdots \ 

reading from the left to the right in each row and from the top row down; it is understood that every row which starts with \( j(g(k), 0) \) for
some \( k \) with \( c_{r(k)} = 0 \) is to be deleted. From the definitions of \( \rho_n \) and \( r(k) \) we see that
\[
k \in (2^0, 2^1, 2^2, \ldots) \implies r(k) = 1 \implies c_{r(k)} = c_1 > 0 .
\]
The function \( g(n) = \tau'(n) \) is regressive by Lemma 2. Taking into account that \( c_i \) is a recursive function, it readily follows that \( w_n \) is a regressive function. In view of (6) we have \( \rho w_n \in F(T) \) it therefore suffices to prove that \( \rho w_{f(n)} \in T \). By Lemma 1
\[
f(n) = \sum_{\sigma = 0}^{2^n - 1} c_{r(\sigma)} ,
\]
hence
\[
f(0) = c_{r(0)}, \quad f(1) = c_{r(0)} + c_{r(1)}, \quad f(2) = c_{r(0)} + c_{r(1)} + c_{r(2)} + c_{r(3)}, \ldots \]
and
\[
w_{f(0)} = j(g(1), 0), \quad w_{f(1)} = j(g(2), 0), \ldots, \quad w_{f(n)} = j(g(2^n), 0), \ldots .
\]
We conclude that \( w_{f(n)} \equiv g(2^n) \). However, by Lemma 2
\[
g(2^n) = \tau'(2^n) \equiv t(n) .
\]
Thus \( w_{f(n)} \equiv t_n \) and \( \rho w_{f(n)} \in T \). This completes the proof.

References


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