

Pacific Journal of Mathematics

COMBINATORIAL FUNCTIONS AND REGRESSIVE ISOLS

F. J. SANSONE

COMBINATORIAL FUNCTIONS AND REGRESSIVE ISOLS

F. J. SANSONE

1. **Introduction.** It is assumed that the reader is familiar with the notions: regressive function, regressive set, regressive isol, co-simple isol, combinatorial function and its canonical extension. The first four are defined in [2], the last two in [3]. Denote the set of all numbers (nonnegative integers) by ε , the collection of all isols by \mathcal{A} , the collection of all regressive isols by \mathcal{A}_R and the collection of all cosimple isols by \mathcal{A}_1 . The following four propositions will be used.

- (1) $\left\{ \begin{array}{l} \text{Let } \tau = \rho t \text{ and } \tau^* = \rho t^*, \text{ where } t_n \text{ and } t_n^* \text{ are regressive} \\ \text{functions. Then } \tau \cong \tau^* \iff t_n \cong t_n^* . \end{array} \right.$
- (2) $B \leq A \ \& \ A \in \mathcal{A}_R \implies B \in \mathcal{A}_R .$
- (3) $\left\{ \begin{array}{l} \text{Let } F(T) \text{ be the canonical extension to } \mathcal{A} \text{ of the recursive,} \\ \text{combinatorial function } f(n). \text{ Then } T \in \mathcal{A}_R \implies F(T) \in \mathcal{A}_R . \end{array} \right.$
- (4) $B \leq A \ \& \ A \in \mathcal{A}_1 \implies B \in \mathcal{A}_1 .$

The first three are Propositions 3, 9(b) and Theorem 3(a) of [2] respectively. The fourth is Theorem 56(b) of [1].

DEFINITION. Let $f(n)$ be a one-to-one function from ε into ε and let $T \in \mathcal{A}_R - \varepsilon$. Then

$$\phi_f(T) = \text{Req } \rho t_{f(n)} ,$$

where t_n is any regressive function ranging over any set in T .

Using (1) it is readily seen that ϕ_f is a well defined function from $\mathcal{A}_R - \varepsilon$ into $\mathcal{A} - \varepsilon$. The main result of this paper is as follows: *Let $f(n)$ be a strictly increasing, recursive, combinatorial function; let $F(X)$ be its canonical extension to \mathcal{A} , and let $T \in \mathcal{A}_R - \varepsilon$; then $\phi_f(F(T)) = T$.*

2. The operation ϕ_f .

PROPOSITION 1. *Let $f(n)$ be a strictly increasing, recursive function and let $T \in \mathcal{A}_R - \varepsilon$. Then*

$$\phi_f(T) \leq T \quad \text{and} \quad \phi_f(T) \in \mathcal{A}_R .$$

February 15, 1963. This paper was written while the author was a National Defense Graduate Fellow at Rutgers University and is a portion of a thesis directed by Professor J. C. E. Dekker to submitted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

If in addition $T \in \mathcal{A}_1$, then $\phi_f(T) \in \mathcal{A}_R \cdot \mathcal{A}_1$.

Proof. In view of (2) and (4), it suffices to show only that $\phi_f(T) \leq T$. Let t_n be a regressive function such that $\rho t = \tau \in T$. Put $\alpha = \rho f$ and suppose $p(x)$ is a regressing function of t_n . Define

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)] \quad \text{for } x \in \delta p.$$

Then $p^*(t_n) = n$ and

$$\begin{aligned} \rho t_f &\subset \{x \in \delta p^* \mid p^*(x) \in \alpha\}, \\ \tau - \rho t_f &\subset \{x \in \delta p^* \mid p^*(x) \notin \alpha\}. \end{aligned}$$

Since α is recursive it follows that ρt_f is separable from $\tau - \rho t_f$. Hence $\phi_f(T) \leq T$.

It is known (by an unpublished result of Dekker) that \mathcal{A}_R is neither closed under addition nor under multiplication. We do, however, have some closure properties for isols of the type $\phi_f(T)$, where $T \in \mathcal{A}_R - \varepsilon$ and $f(n)$ is a strictly increasing, recursive function.

PROPOSITION 2. *Let $f(n)$ and $g(n)$ be strictly increasing, recursive function and let $T \in \mathcal{A}_R - \varepsilon$. Then*

- (a) $\phi_f(\phi_g(T)) \in \mathcal{A}_R - \varepsilon$,
- (b) $\phi_f(T) \cdot \phi_g(T) \in \mathcal{A}_R - \varepsilon$,
- (c) $\phi_f(T) + \phi_g(T) \in \mathcal{A}_R - \varepsilon$.

Proof. In view of Proposition 1,

$$\phi_f(\phi_g(T)) \leq \phi_g(T) \leq T.$$

This implies (a). To verify (a) one could also observe that $\phi_f(\phi_g(T)) = \phi_{\rho f}(T)$. Combining $\phi_f(T) \leq T$ and $\phi_g(T) \leq T$, we obtain by [1, Cor. of Thm. 77]

$$\phi_f(T) \cdot \phi_g(T) \leq T^2.$$

However, $T^2 \in \mathcal{A}_R - \varepsilon$ by (3). Hence (b) follows by (2). Finally, it is readily seen that

$$\phi_f(T) + \phi_g(T) \leq \phi_f(T) \cdot \phi_g(T),$$

since $\phi_f(T)$ and $\phi_g(T)$ are ≥ 2 (in fact, infinite). Thus (c) follows from (2) and (b).

3. The main result. We first state and prove two lemmas which might be of interest for their own sake. Let ρ_0, ρ_1, \dots be the canonical enumeration of the class Q of all finite sets defined by

$$\rho_0 = o$$

$$\rho_{x+1} = \left\{ \begin{array}{l} (y_1, \dots, y_k) \text{ where } y_1, \dots, y_k \text{ are the distinct numbers} \\ \text{such that } x + 1 = 2^{y_1} + \dots + 2^{y_k} . \end{array} \right.$$

We denote the cardinality of ρ_x by r_x .

LEMMA 1. Let $f(n)$ be any combinatorial function and let C_i be the function from ε into ε such that $f(n) = \sum_{i=0}^n c_i \binom{n}{i}$. Then

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

Proof. Since every n -element set has $\binom{n}{i}$ subsets of cardinality i , we have

$$(5) \quad f(n) = \text{card} \{j(x, y) \mid \rho_x \subset (0, 1, \dots, n - 1) \ \& \ y < c_{r(x)}\} .$$

It follows from the definition of ρ_x that

$$\begin{aligned} \rho_x \subset (0, 1, \dots, n - 1) &\iff x \leq 2^0 + 2^1 + \dots + 2^{n-1} \\ &\iff x \leq 2^n - 1 . \end{aligned}$$

Combining this with (5) we obtain

$$f(n) = \text{card} \{j(x, y) \mid x \leq 2^n - 1 \ \& \ y < c_{r(x)}\} = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

DEFINITION. Let $a(n)$ be a one-to-one function from ε into ε . Then

$$a'(n) = l_{n0} \cdot 2^{a(0)} + \dots + l_{nn} \cdot 2^{a(n)} ,$$

where l_{n0}, \dots, l_{nn} is the sequence of zeros and ones such that

$$n = l_{n0} \cdot 2^0 + \dots + l_{nn} \cdot 2^n .$$

LEMMA 2. (Dekker) Let $a(n)$ be a one-to-one function from ε into ε with range α and let $A = \text{Req}(\alpha)$. Then $a'(n)$ is also a one-to-one function from ε into ε . Moreover,

$$a'(2^n) = 2^{a(n)} , \quad \rho_{a'(n)} = a(\rho_n) \text{ and } \rho a' \in 2^A .$$

Finally, if $a(n)$ is regressive, so is $a'(n)$.

Proof. It is clear that $a'(n)$ is a one-to-one function such that $a'(2^n) = 2^{a(n)}$. We have $\rho_{a'(0)} = \rho_0 = o$ while $a(\rho_0) = a(o) = o$; for $n \geq 1$

$$\rho_n = \{i \mid 0 \leq i \leq n \ \& \ l_{ni} = 1\} .$$

Hence for every number n

$$\begin{aligned}\rho_{a'(n)} &= \{a(i) \mid 0 \leq i \leq n \ \& \ l_{ni} = 1\} \\ &= a\{i \mid 0 \leq i \leq n \ \& \ l_{ni} = 1\} = a(\rho_n).\end{aligned}$$

Thus, if n ranges over ε , ρ_n ranges over the class Q of all finite sets, $\rho_{a'(n)} = a(\rho_n)$ over the class of all finite subsets of α . We conclude that $\rho a' \in 2^A$. Finally, assume that $a(n)$ is a regressive function. Using the three facts that

$$\begin{aligned}a'(n+1) &= l_{n+1,0} \cdot 2^{a(0)} + \cdots + l_{n+1,n+1} \cdot 2^{a(n+1)}, \\ a'(n) &= l_{n0} \cdot 2^{a(0)} + \cdots + l_{nn} \cdot 2^{a(n)}, \\ \max \{i \mid l_{ni} = 1\} &\leq \max \{i \mid l_{n+1,i} = 1\},\end{aligned}$$

we infer that $a'(n)$ is a regressive function.

THEOREM. *Let $f(n)$ be a strictly increasing, recursive combinatorial function, let $F(X)$ be its canonical extension to Δ and let $T \in \Delta_R - \varepsilon$. Then $\phi_r(F(T)) = T$.*

Proof. Let $f(n) = \sum_{i=0}^n c_i \binom{n}{i}$ be the strictly increasing, recursive, combinatorial function. Then $c_1 > 0$ since $f(n)$ is strictly increasing, and c_i is a recursive function of i , since $f(n)$ is recursive. Let $\tau \in T \in \Delta_R - \varepsilon$ and assume that t_n is a regressive function ranging over τ . Put $g(n) = t'(n)$. By Lemma 2 we have $\rho_{g(n)} = t(\rho_n)$; thus, if n assumes successively the values $0, 1, 2, 3, 4, 5, 6, 7, \dots$, $\rho_{g(n)}$ assumes successively the "values"

$$0, (t_0), (t_1), (t_0, t_1), (t_2), (t_0, t_2), (t_1, t_2), (t_0, t_1, t_2), \dots$$

We have by definition

$$F(T) = \text{Req} \{j(x, y) \mid \rho_x \subset \tau \ \& \ y < c_{r(x)}\}.$$

Since $g(n)$ ranges without repetitions over $\{n \mid \rho_n \subset \tau\}$, it follows that

$$(6) \quad F(T) = \text{Req} \{j(g(x), y) \mid y < c_{r(x)}\}.$$

We shall use w_n to denote the function which for $0, 1, \dots$ takes on the values of the array

$$\begin{array}{c}j(g(0), 0), \dots, j(g(0), c_{r(0)} - 1) \\ j(g(1), 0), \dots, j(g(1), c_{r(1)} - 1) \\ j(g(2), 0), \dots, j(g(2), c_{r(2)} - 1) \\ \vdots \\ \vdots\end{array}$$

reading from the left to the right in each row and from the top row down; it is understood that every row which starts with $j(g(k), 0)$ for

some k with $c_{r(k)} = 0$ is to be deleted. From the definitions of ρ_k and $r(k)$ we see that

$$k \in (2^0, 2^1, 2^2, \dots) \implies r(k) = 1 \implies c_{r(k)} = c_1 > 0 .$$

The function $g(n) = t'(n)$ is regressive by Lemma 2. Taking into account that c_i is a recursive function, it readily follows that w_n is a regressive function. In view of (6) we have $\rho w_n \in F(T)$ it therefore suffices to prove that $\rho w_{f(n)} \in T$. By Lemma 1

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} ,$$

hence

$$f(0) = c_{r(0)} , \quad f(1) = c_{r(0)} + c_{r(1)} , \quad f(2) = c_{r(0)} + c_{r(1)} + c_{r(2)} + c_{r(3)} , \dots$$

and

$$w_{f(0)} = j(g(1), 0) , \quad w_{f(1)} = j(g(2), 0) , \dots , \quad w_{f(n)} = j(g(2^n), 0) , \dots .$$

We conclude that $w_{f(n)} \cong g(2^n)$. However, by Lemma 2

$$g(2^n) = t'(2^n) \cong t(n) .$$

Thus $w_{f(n)} \cong t_n$ and $\rho w_{f(n)} \in T$. This completes the proof.

REFERENCES

1. J. C. E. Dekker and J. Myhill, *Recursive Equivalence Types*, University of California Publications in Mathematics (New Series), 3, No. 3 (1960), 67-214.
2. ———, *Infinite Series of Isols*, Proc. of the Symp. on Recursive Function Theory, Amer. Math. Soc., Providence, R. I., (1962), 77-96.
3. J. Myhill, *Recursive Equivalence Types and Combinatorial Functions*, Bull. Amer. Math. Soc., **64** (1958), 373-376.
4. ———, *Recursive Equivalence Types and Combinatorial Functions*, Poo. of the Symp. on Logic, Methodology and Philosophy of Science, Stanford, 1962, pp. 46-55.

RUTGERS, THE STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 13, No. 2

April, 1963

Rafael Artzy, <i>Solution of loop equations by adjunction</i>	361
Earl Robert Berkson, <i>A characterization of scalar type operators on reflexive Banach spaces</i>	365
Mario Borelli, <i>Divisorial varieties</i>	375
Raj Chandra Bose, <i>Strongly regular graphs, partial geometries and partially balanced designs</i>	389
R. H. Bruck, <i>Finite nets. II. Uniqueness and imbedding</i>	421
L. Carlitz, <i>The inverse of the error function</i>	459
Robert Wayne Carroll, <i>Some degenerate Cauchy problems with operator coefficients</i>	471
Michael P. Drazin and Emilie Virginia Haynsworth, <i>A theorem on matrices of 0's and 1's</i>	487
Lawrence Carl Eggan and Eugene A. Maier, <i>On complex approximation</i>	497
James Michael Gardner Fell, <i>Weak containment and Kronecker products of group representations</i>	503
Paul Chase Fife, <i>Schauder estimates under incomplete Hölder continuity assumptions</i>	511
Shaul Foguel, <i>Powers of a contraction in Hilbert space</i>	551
Neal Eugene Foland, <i>The structure of the orbits and their limit sets in continuous flows</i>	563
Frank John Forelli, Jr., <i>Analytic measures</i>	571
Robert William Gilmer, Jr., <i>On a classical theorem of Noether in ideal theory</i>	579
P. R. Halmos and Jack E. McLaughlin, <i>Partial isometries</i>	585
Albert Emerson Hurd, <i>Maximum modulus algebras and local approximation in C^n</i>	597
James Patrick Jans, <i>Module classes of finite type</i>	603
Betty Kvarda, <i>On densities of sets of lattice points</i>	611
H. Larcher, <i>A geometric characterization for a class of discontinuous groups of linear fractional transformations</i>	617
John W. Moon and Leo Moser, <i>Simple paths on polyhedra</i>	629
T. S. Motzkin and Ernst Gabor Straus, <i>Representation of a point of a set as sum of transforms of boundary points</i>	633
Rajakularaman Ponnuswami Pakshirajan, <i>An analogue of Kolmogorov's three-series theorem for abstract random variables</i>	639
Robert Ralph Phelps, <i>Čebyšev subspaces of finite codimension in $C(X)$</i>	647
James Dolan Reid, <i>On subgroups of an Abelian group maximal disjoint from a given subgroup</i>	657
William T. Reid, <i>Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems</i>	665
Georg Johann Rieger, <i>Some theorems on prime ideals in algebraic number fields</i>	687
Gene Fuerst Rose and Joseph Silbert Ullian, <i>Approximations of functions on the integers</i>	693
F. J. Sansone, <i>Combinatorial functions and regressive isols</i>	703
Leo Sario, <i>On locally meromorphic functions with single-valued moduli</i>	709
Takayuki Tamura, <i>Semigroups and their subsemigroup lattices</i>	725
Pui-kei Wong, <i>Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equations</i>	737
Fawzi Mohamad Yaqub, <i>Free extensions of Boolean algebras</i>	761