ON LOCALLY MEROMORPHIC FUNCTIONS WITH SINGLE-VALUED MODULI

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1. A meromorphic function of bounded characteristic in a disk is the quotient of two bounded analytic functions. This classical theorem can be extended to open Riemann surfaces $W$ as follows. Consider the class $MB$ of meromorphic functions $w$ of bounded characteristic on $W$, defined in terms of capacity functions on subregions. Let $L$ be the class of harmonic functions on $W$, regular except for logarithmic singularities with integral coefficients. Then $w \in MB$ if and only if $\log |w|$ is the difference of two positive functions in $L$. We shall construct these functions directly on $W$, without making use of uniformization.

The proof offers no essential difficulties. If $\log |w|$ is regular at the singularity of the capacity functions, then the classical reasoning carries over almost verbatim. In the general case we introduce the extended class $M_e$ of locally meromorphic functions $e^{u+iw^*}$, $u \in L$, with single-valued moduli. This class seems to offer some interest in its own right.

2. The class $O_{MB}$ of Riemann surfaces not admitting nonconstant $M_eB$-functions coincides with the class $O_\sigma$ of parabolic surfaces. Regarding the subclass $MB \subset M_eB$ and the strict inclusion relations $O_{HB} < O_{MB} < O_{AB}$, we refer to the pioneering work on Lindelöfian maps by M. Heins [2, 3] and M. Parreau [4], and the doctoral dissertation of K. V. R. Rao [5].

§ 1. Definitions.

3. Let $W$ be an arbitrary open Riemann surface. Given $\zeta \in W$ let $\Omega, \zeta \in \Omega$, be a relatively compact subregion of $W$ whose boundary $\beta_\sigma$ consists of a finite number of analytic Jordan curves. The Green's function on $\Omega$ with pole at $\zeta$ is denoted by $g_\sigma(z, \zeta)$. For $\Omega_0 \subset \Omega$ we have $g_{\sigma_0} \leq g_\sigma$ in $\Omega_0$ and $\lim_{z \to \infty} g_\sigma(z, \zeta)$ either $\equiv \infty$ or else $=\text{the Green's function } g(z, \zeta)$ of $W$. By definition, the class $O_\sigma$ of parabolic Riemann surfaces consists of those $W$ on which no $g(z, \zeta)$ exists. An equivalent definition of $O_\sigma$ is that there are no nonconstant nonnegative superharmonic functions on $W$.

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4. The capacity function \( p_\Omega(z, \zeta) \) on \( \Omega \) with pole at \( \zeta \) is defined as the harmonic function with singularity
\[
p_\Omega(z, \zeta) = \log |z - \zeta| \to 0
\]
as \( z \to \zeta \) and such that
\[
p_\Omega(z, \zeta) = k_\Omega = \text{const. on } \beta_\Omega.
\]
It is known [1] that \( k_{\Omega_0} \leq k_\Omega \) and the limit \( k_\beta = \lim k_\Omega \) is thus well-defined. A necessary and sufficient condition for \( W \in O_\beta \) is \( k_\beta = \infty \).

5. Let \( M \) be the class of meromorphic functions \( w \) on \( W \). The proximity function of \( w \) is defined [7] as
\[
m(\Omega, w) = m(\Omega, \infty) = \frac{1}{2\pi} \int_{\beta_\Omega} \log |w| \, dp_\Omega^*.
\]
If \( \beta_h \) is the level line \( p_\Omega = h, -\infty \leq h \leq k_\Omega \), and \( n(h, \infty) \) signifies the number of poles of \( w \) in \( \bar{\Omega}_h \); \( p_\Omega \leq h \), counted with multiplicities, then the counting function is defined as
\[
N(\Omega, w) = N(\Omega, \infty) = \int_{-\infty}^{k_\Omega} (n(h, \infty) - n(-\infty, \infty)) dh + n(-\infty, \infty)k_\Omega.
\]

The characteristic function is, by definition,
\[
T(\Omega) = T(\Omega, w) = m(\Omega, w) + N(\Omega, w).
\]
The function \( w \) has at \( \zeta \) the Laurent expansion
\[
w(z) = c_\lambda (z - \zeta)^\lambda + c_{\lambda+1} (z - \zeta)^{\lambda+1} + \cdots,
\]
c\( _\lambda \neq 0 \), and the Jensen formula reads [7, 8]
\[
T(\Omega, w) = T(\Omega, w^{-1}) + \log |c_\lambda|.
\]

6. We shall need a class \( M_e \) more comprehensive than \( M \). We introduce:

**Definitions.** The class \( L \) consists of functions \( u \) on \( W \), harmonic except for logarithmic singularities \( \lambda_i \log |z - z_i| \) at \( z_i, i = 1, 2, \ldots \), with integral coefficients \( \lambda_i \). The subclass of nonnegative functions in \( L \) will be denoted by \( LP \).

The class \( M_e \) is defined to consist of (multiple-valued) functions of the form
\[
w = e^{u + i\pi}, \quad u \in L.
\]
The conjugate function $u^*$ has periods around $z_i$ and along some cycles in $W$. Every branch of $w$ is locally meromorphic, the branches differing by multiplicative constants $c$ with $|c| = 1$. The modulus $|w|$ is single-valued throughout $W$.

The quantities $m(\Omega, w)$, $N(\Omega, w)$, $T(\Omega, w)$, and the Jensen formula carry over to $M_\epsilon$ without modifications [7]. We further introduce:

**Definition.** The class $MB$ (or $M_\epsilon B$) consists of functions $w$ in $M$ (or $M_\epsilon$) with bounded characteristics,

\[(6) \quad T(\Omega) = O(1) .\]

Explicitly, one requires the existence of a bound $C < \infty$ independent of $\Omega$ such that $T(\Omega) < C$ for all $\Omega \subset W$. That (6) is independent of $\zeta$ will be a consequence of a decomposition theorem which we proceed to establish.

§ 2. The decomposition theorem.

7. We continue considering arbitrary open Riemann surfaces $W$.

**Theorem.** A necessary and sufficient condition for $w \in M_\epsilon B$ on $W$ is that

\[(7) \quad \log |w| = u - v ,\]

where $u, v \in LP$.

The proof will be given in nos. 8-18. As a corollary we observe that $w \in MB$ on $W$ if and only if (7) holds.

8. First we shall discuss in nos. 8-11 the case $w(\zeta) = 0$ or $\infty$.

Suppose $w \in M_\epsilon B$. We begin by showing that $W \notin O_\zeta$. If $w(\zeta) = \infty$, then

\[T(\Omega) \geq N(\Omega, w) \geq n(-\infty, \infty) k_\Omega \geq k_\Omega .\]

From $W \notin O_\zeta$ it would follow that $k_\Omega \to \infty$ as $\Omega \to W$ and consequently $T(\Omega) \to \infty$, a contradiction. We conclude that $W \notin O_\zeta$. If $w(\zeta) = 0$, then in Jensen's formula

\[T(\Omega, w) = T\left(\Omega, \frac{1}{w}\right) + O(1)\]

we have

\[T\left(\Omega, \frac{1}{w}\right) \geq N\left(\Omega, \frac{1}{w}\right) \geq n(-\infty, 0) k_\Omega \geq k_\Omega .\]
and arrive at the same conclusion $W \notin O_g$.

On the other hand, if condition (7) is true, the existence of nonnegative superharmonic functions $u, v$ implies $W \notin O_g$. Thus either condition of the theorem gives the hyperbolicity of $W$, and we may henceforth assume the existence of $g(z, \zeta)$ on $W$ if $w(\zeta) = 0$ or $\infty$.

9. The functions

$$\varphi(z) = e^{\lambda g(z, \zeta) + i \varphi(z, \zeta)},$$

$$w_1(z) = w(z)\varphi(z)$$

belong to $M$. We shall show:

**Lemma.** A necessary and sufficient condition for $w \in M_B$ is that $w_1 \in M_B$.

**Proof.** By definition,

$$T(\Omega, \varphi) = N(\Omega, \varphi) + m(\Omega, \varphi).$$

For $\lambda > 0$ we have trivially $N(\Omega, \varphi^{-1}) \equiv 0$, $m(\Omega, \varphi^{-1}) \equiv 0$, hence $T(\Omega, \varphi^{-1}) \equiv 0$, and it follows from Jensen’s formula that $T(\Omega, \varphi) = O(1)$. If $\lambda < 0$, then $N(\Omega, \varphi) \equiv m(\Omega, \varphi) \equiv 0$, and $T(\Omega, \varphi) \equiv 0$, hence $T(\Omega, \varphi^{-1}) = O(1)$. In both cases

$$T(\Omega, \varphi) = O(1), T(\Omega, \varphi^{-1}) = O(1).$$

The inequalities

$$T(\Omega, w) \leq T(\Omega, w_1) + T(\Omega, \varphi^{-1}) = T(\Omega, w_1) + O(1),$$

$$T(\Omega, w_1) \leq T(\Omega, w) + T(\Omega, \varphi) = T(\Omega, w) + O(1)$$

yield

$$T(\Omega, w) = T(\Omega, w_1) + O(1)$$

and the lemma follows.

10. The following intermediate result can now be established:

**Lemma.** A necessary and sufficient condition for

$$\log |w| = u - v$$

with $u, v \in LP$ is that

$$\log |w_1| = u_1 - v_1$$

with $u_1, v_1 \in LP$. 
Proof. We know that
\[ \log |w_1| = \log |w| + \lambda g = \log |w| + (n_0 - n_\omega)g , \]
where $n_0, n_\omega$ are the multiplicities of the zero or pole of $w(z)$ at $\zeta$. If (13) is true, then
\[ \log |w_1| = (u + n_0 g) - (v + n_\omega g) \]
and (14) follows. Conversely, (14) implies
\[ \log |w| = (u_1 + n_\omega g) - (v_1 + n_0 g) . \]
This proves the lemma.

11. We conclude that Theorem 7 will be proved for $w$ with $w(\zeta) = 0$ or $\infty$ if we establish it for $w_1$. Since $w_1(\zeta) \neq 0, \infty$, the proof for $w_1$ will also apply to $w$ with this property. Explicitly, we are to show that $w_1 \in M_\varepsilon B$ if and only if $\log |w_1| = u_1 - v_1, u_1, v_1 \in LP$.

12. Let $p_{\varepsilon z}$ be the capacity function in $\Omega$ with pole at $z$. For a harmonic function $h$ on $\bar{\Omega}$ it is known [7] that
\[ h(z) = \frac{1}{2\pi} \int_{\beta_\varepsilon} h \, dp_{\varepsilon z}^+ . \]

Denote by $a_\mu, b_\nu$ the zeros and poles of $w$ in $W$. Those in $W - \zeta$ are the zeros and poles of $w_1$ in $W$. Suppose first there is no $a_\mu, b_\nu$ on $\beta_\varepsilon$. Then the function
\[ h(z) = \log |w_1(z)| + \sum_{a_\mu \in \partial - \zeta} g_\partial(z, a_\mu) - \sum_{b_\nu \in \partial - \zeta} g_\partial(z, b_\nu) \]
is harmonic on $\bar{\Omega}$. Throughout this paper the zeros and poles are counted with their multiplicities. We set
\[ x_\partial(z, w_1) = \frac{1}{2\pi} \int_{\beta_\varepsilon} \log |w_1| \, dp_{\varepsilon z}^+ , \]
\[ y_\partial(z, w_1) = \sum_{b_\nu \in \partial - \zeta} g_\partial(z, b_\nu) , \]
and
\[ u_\partial(z, w_1) = x_\partial(z, w_1) + y_\partial(z, w_1) . \]
Then
\[ \log |w_1(z)| = u_\partial(z, w_1) - u_\partial(z, w_1^{-1}) . \]

Since all terms are continuous in $a_\mu, b_\nu$, the equation remains valid if there are zeros or poles of $w$ on $\beta_\varepsilon$. 
We observe that
\begin{align}
(24) \quad x_o(\zeta, w) &= m(\Omega, w), \\
(25) \quad y_o(\zeta, w) &= N(\Omega, w).
\end{align}

Here we shall only make use of the consequence
\begin{equation}
(26) \quad u_o(\zeta, w) = T(\Omega, w).
\end{equation}

13. We next show:

**Lemma.** For \( \Omega \subset \Omega \),
\begin{align}
(27) \quad u_{o_0}(z, w_1) &\leq u_o(z, w_1), \\
(27)' \quad u_{o_0}(z, w_1^{-1}) &\leq u_o(z, w_1^{-1}).
\end{align}

*Proof.* By (23),
\begin{equation}
(28) \quad \log |w(z)| \leq u_o(z, w_1)
\end{equation}
for every \( \Omega \). It follows that
\begin{align*}
x_{o_0}(z, w_1) &\leq \frac{1}{2\pi} \int_{\partial_\Omega} u_o(t, w_1) dp_{o_0}^* \\
&= \frac{1}{2\pi} \int_{\partial_\Omega} (u_o(t, w_1) - y_{o_0}(t, w_1)) dp_{o_0}^* \\
&= u_o(z, w_1) - y_{o_0}(z, w_1),
\end{align*}
because this difference is regular harmonic in \( \Omega_0 \). We have reached statement (27),
\[ x_{o_0}(z, w_1) + y_{o_0}(z, w_1) \leq u_o(z, w_1), \]
and inequality (27)' follows in the same fashion.

14. From (26) and (27) we infer that \( T(\Omega, w) \) increases with \( \Omega \). We can set
\begin{equation}
(29) \quad T(W, w) = \lim_{\Omega \to W} T(\Omega, w)
\end{equation}
and use alternatively the notations \( T(\Omega) = O(1) \) and \( T(W) < \infty \).

15. The convergence of \( u_o \) can now be established:

**Lemma.** If \( T(W, w) < \infty \), then the functions
\begin{equation}
(30) \quad u(z, w) = \lim_{\Omega \to W} u_o(z, w),
\end{equation}
(30) \[ u(z, w_i^{-1}) = \lim_{\varphi \to w} u_{\varphi}(z, w_i^{-1}) \]

are positive harmonic on \( W \) except for logarithmic poles of \( u(z, w_i) \) at the \( b_j \in W - \zeta \) and those of \( u(z, w_i^{-1}) \) at the \( a_k \in W - \zeta \).

**Proof.** By Harnack's principle the limit in (30) is either identically infinite or else harmonic on \( W - \{b_j\} \). That the latter alternative occurs is a consequence of
\[ \lim_{\varphi \to w} u_{\varphi}(\zeta, w_i^{-1}) = T(\Omega, w_i^{-1}) = T(\Omega, w_i) + O(1). \]

The statement for \( u_{\varphi}(z, w_i^{-1}) \) follows similarly from \( u_{\varphi}(\zeta, w_i^{-1}) = T(\Omega, w_i^{-1}) = T(\Omega, w_i) + O(1) \).

16. On combining the lemma with (23) we see that \( w_i \in M_eB \) has the asserted representation
\[ \log | w_i(z) | = u(z, w_i) - u(z, w_i^{-1}) \]
with the \( u \)-functions in \( L_P \). It remains to establish the converse.

17. Suppose
\[ \log | w_i(z) | = u_i(z) - v_i(z) \]

where \( u_i, v_i \in L_P \). The positive logarithmic poles of \( u_{\varphi}(z, w_i) \) are those of \( \log | w_i(z) | \) in \( \Omega \), hence among those of \( u_i(z) \). Consequently \( u_i(z) - u_{\varphi}(z, w_i) \) is superharmonic in \( \Omega \) and its minimum on \( \partial \Omega \) is reached on \( \beta_\varphi \), where \( u_i(z) - u_{\varphi}(z, w_i) = u_i(z) - \log | w_i(z) | \geq 0. \) One infers that \( u_i(z) \geq u_{\varphi}(z, w_i) \) in \( \partial \Omega \). At \( \zeta \) this means
\[ T(\Omega, w_i) = u_{\varphi}(\zeta, w_i) \leq u_i(\zeta). \]

If \( u_i(\zeta) < \infty \), the proof is complete.

18. If \( u_i(\zeta) = \infty \), then
\[ u_i(z) + \lambda_1 \log | z - \zeta | \]
is harmonic at \( \zeta \) for some positive integer \( \lambda_1 \). We set
\[ w_2 = w_i \cdot e^{-\lambda_1(\varphi + i\theta^*)} \in M_e, \]
where \( g = g(z, \zeta) \), and obtain
\[ \log | w_2 | = \log | w_i | - \lambda_1 g = (u_1 - \lambda_1 g) - v_i. \]
The function \( u_1 - \lambda_1 g_\varphi \) with \( g_\varphi = g_\varphi(z, \zeta) \) is superharmonic on \( \Omega \), hence its minimum on \( \partial \Omega \) is taken on \( \beta_\varphi \), where
From \( u_1 \geq \lambda g \) on \( \Omega \) it follows that
\[
(38) \quad u_1 - \lambda g = \lim_{\substack{a \to \infty}} (u_1 - \lambda g) \geq 0
\]
on \( W \). On setting
\[
(39) \quad u_2 = u_1 - \lambda g, \quad v_2 = v_1
\]
one gets
\[
(40) \quad \log |w_2| = u_2 - v_2
\]
with \( u_2, v_2 \in LP \).

The positive logarithmic poles of \( u_2(z, w_2) \) are those of \( \log |w_1| \)
on \( \Omega \), hence among those of \( u_2 \). The minimum of the superharmonic
function \( u_2(z) - u_2(z, w_2) \) on \( \overline{\Omega} \) is taken on \( \beta \), where it is
\[
\min_{\beta} (u_2 - \log |w_2|) \geq 0.
\]

One infers that
\[
(41) \quad T(\Omega, w_2) = u_2(\zeta, w_2) \leq u_2(\zeta) < \infty,
\]
that is, \( T(\Omega, w_2) = O(1) \). The reasoning leading to (12) yields
\[
(42) \quad T(\Omega, w_1) = T(\Omega, w_2) + O(1),
\]
and consequently \( T(\Omega, w_1) = O(1) \).

We have shown that (32) implies \( T(W, w_1) < \infty \). The proof of
Theorem 7 is complete.

19. As an immediate consequence we see that the property
\( T(\Omega, w) = O(1) \) and thus the class \( M_{\lambda}B \) is independent of \( \zeta \).

§ 3. Extremal decompositions.

20. Consider an arbitrary \( w \in M_{\lambda} \). In contrast with no. 12 we
now make no restrictive assumptions on \( w(\zeta) \) and form
\[
(43) \quad x_\alpha(z, w) = \frac{1}{2\pi} \int_{\beta} \text{log} |w| d\rho_\alpha^*,
\]
\[
(44) \quad y_\alpha(z, w) = \sum_{b_\gamma \in \beta} g_\beta(z, b_\gamma),
\]
\[
(45) \quad u_\alpha(z, w) = x_\alpha(z, w) + y_\alpha(z, w).
\]

It is seen as in no. 13 that \( u_\alpha \) increases with \( \Omega \) and that
(46)  \[ u(z, w) = \lim_{\varrho \to w} u_\varrho(z, w) \]

is either identically infinite or else positive harmonic on \( W \) except for logarithmic poles \( b_\nu \). The same is true of

(47)  \[ u(z, w^{-1}) = \lim_{\varrho \to w} u_\varrho(z, w^{-1}) \]

with singularities \( a_\mu \).

The functions (46) and (47) will now be shown to be extremal in all decompositions (7):

**Theorem.** If there is a decomposition

(48)  \[ \log |w(z)| = u_1(z) - u_2(z) \]

with \( u_1, u_2 \in LP \), then also

(49)  \[ \log |w(z)| = u(z, w) - u(z, w^{-1}) \]

and

(50)  \[
\begin{align*}
    u(z, w) & \leq u_1(z) \\
    u(z, w^{-1}) & \leq u_2(z).
\end{align*}
\]

**Proof.** One observes that the positive logarithmic poles of \( u_\varrho(z, w) \) are those of \( \log |w(z)| \) in \( \Omega \), hence among those of \( u_1(z) \) in \( \Omega \). The superharmonic function \( u_1(z) - u_\varrho(z, w) \) in \( \Omega \) dominates

\[ \min_{\beta_\varrho} (u_1(z) - \log |w(z)|) \geq 0 \]

and we find that \( u_1(z) - u(z, w) = \lim_{\varrho \to w}(u_1(z) - u_\varrho(z, w)) \geq 0 \) in \( W \). Similarly, the superharmonic function \( u_2(z) - u_\varrho(z, w^{-1}) \geq 0 \) on \( \Omega \), and \( u_2(z) \geq u(z, w^{-1}) \) on \( W \). By virtue of Harnack's principle, equality (49) then follows on letting \( \Omega \to W \) in

(51)  \[ \log |w(z)| = u_\varrho(z, w) - u_\varrho(z, w^{-1}) \].

21. The extremal functions \( u(z, w), u(z, w^{-1}) \) can in turn be decomposed:

**Theorem.** A function \( w \) on \( W \) belongs to \( M_cB \) if and only if

(52)  \[ \log |w| = (x(z, w) + y(z, w)) - (x(z, w^{-1}) + y(z, w^{-1})) \],

where the functions \( x \geq 0 \) are regular harmonic and the functions \( y \geq 0 \) have the representations
\[ y(z, w) = \sum g(z, b_v) \]
\[ y(z, w^{-1}) = \sum g(z, a_\mu) . \]

Here the sums are extended over all poles \( b_v \) and all zeros \( a_\mu \) of \( w \) on \( W \) respectively, each counted with its multiplicity.

22. Suppose indeed that \( w \in M_eB \). It is evident from the maximum principle that
\[ y_{\partial\alpha}(z, w) \leq y_{\alpha}(z, w) \]
for \( \Omega_0 \subset \Omega \). We know that
\[ \log |w| = u_1 - u_2 , \]
\( u_1, u_2 \in LP \), and the superharmonic function \( u_\alpha(z) - y_{\alpha}(z, w) \) on \( \Omega \) cannot exceed \( \min_{\partial\alpha} u_1 \geq 0 \). Hence \( y_{\alpha}(z, w) \leq u_\alpha(z) \) on \( \Omega \) and, by Harnack’s principle,
\[ y(z, w) = \lim_{\partial \rightarrow W} y_{\alpha}(z, w) \]
is positive harmonic on \( W \) except for logarithmic poles \( b_v \). Analogous reasoning shows that
\[ y(z, w^{-1}) = \lim_{\partial \rightarrow W} y_{\alpha}(z, w^{-1}) \]
is positive harmonic on \( W - \{ a_\mu \} \).

23. To prove (53) we must show that
\[ \lim_{\partial \rightarrow W} \sum_{b_v \in \partial} g_{\alpha}(z, b_v) = \sum_{b_v \in W} g(z, b_v) \]
and similarly for \( \sum g(z, a_\mu) \). First,
\[ \sum_{b_v \in \partial} g_{\alpha}(z, b_v) \leq \sum_{b_v \in \partial} g(z, b_v) \leq \sum_{b_v \in W} g(z, b_v) , \]
and we have
\[ \lim_{\partial \rightarrow W} \sum_{b_v \in \partial} g_{\alpha}(z, b_v) \leq \sum_{b_v \in W} g(z, b_v) . \]
Second, for \( \Omega_0 \subset \Omega \),
\[ \sum_{b_v \in A_0} g(z, b_v) = \lim_{\partial \rightarrow W} \sum_{b_v \in \partial} g_{\alpha}(z, b_v) \leq \lim_{\partial \rightarrow W} \sum_{b_v \in \partial} g_{\alpha}(z, b_v) \]
and a fortiori
\[ \sum_{b_v \in W} g(z, b_v) = \lim_{A_0 \rightarrow W} \sum_{b_v \in A_0} g(z, b_v) \leq \lim_{\partial \rightarrow W} \sum_{b_v \in \partial} g_{\alpha}(z, b_v) . \]
Statement (58) follows.

24. The convergence of \( x_\Omega(z, w) \) is obtained at once from

\[
x_\Omega(z, w) = u_\Omega(z, w) - y_\Omega(z, w),
\]

and the limiting function is

\[
x(z, w) = u(z, w) - y(z, w).
\]

The limit \( x(z, w^{-1}) \) of \( x_\Omega(z, w^{-1}) \) is obtained in the same way. Both limits are obviously positive and regular harmonic on \( W \).

Necessity of (52) for \( w \in M, B \) has thus been established. Sufficiency is a corollary of the main Theorem 7.


25. If only the \( x \)-terms in (52) are considered, the following corollary of Theorem 21 is obtained:

**Theorem.** If \( w \in M, B \) on \( W \), then

\[
\lim_{\nu \to w} \int_{\beta_\nu} \log |w| \, dp_\nu^z < \infty
\]

for any \( \zeta \).

Here \( p_\nu \) signifies, as before, the capacity function on \( \Omega \) with pole at \( \zeta \). For the proof we have

\[
\int_{\beta_\nu} \log |w| \, dp_\nu^z = \int_{\beta_\nu} \log |w| \, dp_\nu + \int_{\beta_\nu} \log |\frac{1}{w}| \, dp_\nu^z
\]

\[
= 2\pi(x_\nu(\zeta, w) + x_\nu(\zeta, w^{-1})),
\]

and this quantity tends to

\[
2\pi(x(\zeta, w) + x(\zeta, w^{-1})) < \infty.
\]

The limit (65) thus exists.

26. A consideration of the \( y \)-terms in (52) gives:

**Theorem.** Suppose \( w \in M, B \). Then the sum \( \Sigma g(z, z_i) \), with \( z_i \) ranging over all poles and zeros of \( w \), is harmonic on \( W - \{a_\nu\} - \{b_\nu\} \).

In fact,
For a sufficient condition the first terms of both \( x \)- and \( y \)-parts in (52) must be taken into account:

**Theorem.** If for some \( \zeta \in W \)

\[
\int_{\beta}^{+} \log |w| d\mu_{\omega} = O(1)
\]

and

\[
\sum_{b \in W} g(z, b) < \infty \text{ in } W - \{b\},
\]

then \( w \in M_{e}B \) and hence

\[
\lim_{\omega \to W} \int_{\beta}^{+} \log |w| | d\mu_{\omega} < \infty
\]

and

\[
\sum_{a \in W} g(z, a) < \infty \text{ on } W - \{a\}
\]

as well.

Indeed, the characteristic

\[
T(\Omega) = u_{\omega}(\zeta, w) = x_{\omega}(\zeta, w) + y_{\omega}(\zeta, w)
\]

\[
= \frac{1}{2\pi} \int_{\beta}^{+} \log |w| d\mu_{\omega} + \sum_{b \in W} g_{\omega}(\zeta, b)
\]

is \( O(1) \) if (69), (70) hold. Properties (71), (72) then follow from \( w \in M_{e}B \).

Another sufficient condition for \( w \in M_{e}B \) is, of course, that

\[
\int_{\beta}^{+} \log |w|^{-1} d\mu_{\omega} \text{ is bounded and } \Sigma g(\zeta, a) < \infty \text{ in } W - \{a\}.
\]

28. For “entire” functions in \( M_{e}B \) the conditions simplify. Let \( E_{e}B \) be the class of such functions, characterized by \( w(z) \neq \infty \) on \( W \).

**Theorem.** A necessary and sufficient condition for \( w \in E_{e}B \) on \( W \) is that

\[
\int_{\beta}^{+} \log |w| d\mu_{\omega} = O(1) .
\]
The proof is evident.

29. Consider the class $H$ of regular harmonic functions $h$ on $W$ and let $HP$ be the subclass of nonnegative functions. Set $h^+ = \max(0, h)$.

**Theorem.** A harmonic function $h$ on $W$ has a decomposition

\begin{equation}
    h = u_1 - u_2, \quad u_1, u_2 \in HP
\end{equation}

if and only if, for some $\zeta$,

\begin{equation}
    \int_{\beta_0} h^+ \, dp_0^\# = O(1),
\end{equation}

or, equivalently,

\begin{equation}
    \lim_{\Omega \to W} \int_{\beta_0} |h| \, dp_0^\# < \infty.
\end{equation}

**Proof.** The multiple-valued function $w = e^{h+i\zeta}$ is in $M_{c}$, and $w \not= 0, \infty$ on $W$. If (74) is given, then $\log |w| = u_1 - u_2$ and $w \in M_{c}B$. This implies

\[
    \lim_{\Omega \to W} \int_{\beta_0} |\log |w|| \, dp_0^\# = \lim_{\Omega \to W} \int_{\beta_0} |h| \, dp_0^\# < \infty
\]

and consequently $\int_{\beta_0} h^+ \, dp_0^\# = O(1)$. Conversely, suppose the latter condition holds,

\[
    \int_{\beta_0} \log |w| \, dp_0^\# = O(1).
\]

Then $w \in M_{c}B$ and

\[
    h = \log |w| = x(z, w) - x(z, w^{-1}),
\]

the $y$-terms vanishing because of the absence of zeros and poles of $w$.

It is known that functions $u$ harmonic in the interior $W$ of a compact bordered Riemann surface and with property (76) have a Poisson-Stieltjes representation (e.g., Rodin [6]). For further interesting results see Rao [5].

30. It is clear that theorems on $\log |w|$ can also be expressed directly in terms of $|w|$. Theorem 7, e.g., takes the following form:

**Theorem.** $w \in M_{c}B$ if and only if
(77) \[ |w| = \left| \frac{\eta(z, w)}{\eta(z, w^{-1})} \right|, \]
where \( \eta \in M_eB \) and \( |\eta| < 1 \) on \( W \).

Proof. Suppose \( w \in M_eB \), hence

(78) \[ \log |w| = u(z, w) - u(z, w^{-1}), \]
\( u \in LP \). Set

(79) \[ \eta(z, w) = \exp \left[ -u(z, w^{-1}) - iu(z, w^{-1})^* \right], \]
and (77) follows. Conversely, if (77) is given, then

(80) \[ \log |w| = \log |\eta(z, w)| - \log |\eta(z, w^{-1})| \]
is a difference of two functions in \( LP \), and we have \( w \in M_eB \).

31. The counterpart of Theorem 21 is as follows:

**Theorem.** \( w \in M_eB \) if and only if

(81) \[ |w| = \left| \frac{\varphi(z, w)\psi(z, w)}{\varphi(z, w^{-1})\psi(z, w^{-1})} \right|, \]
where \( \varphi, \psi \in M_eB \) and \( \varphi \neq 0 \) on \( W \), \( |\varphi| < 1 \), \( |\psi| < 1 \).

If \( w \in M_eB \), choose

(82) \[ \varphi(z, w) = \exp \left[ -x(z, w^{-1}) - ix(z, w^{-1})^* \right], \]
\[ \psi(z, w) = \exp \left[ -y(z, w^{-1}) - iy(z, w^{-1})^* \right], \]
and we have (81). Conversely, (81) gives \( \log |w| = u_1 - u_2 \) with \( u_1, u_2 \in LP \), hence \( w \in M_eB \).

32. We introduce the classes \( O_{MB} \) and \( O_{M_eB} \) of Riemann surfaces on which there are no nonconstant functions in \( MB \) and \( M_eB \) respectively. Similarly, let \( O_{EB} \) and \( O_{E_eB} \) be the subclasses determined by entire functions \( w(z) \neq \infty \) on \( W \) in \( MB \) and \( M_eB \). The problem here is to arrange these four classes in the general classification scheme of Riemann surfaces [1].

The inclusion relations

(83) \[ O_{M_eB} \subset O_{MB} \subset O_{EB}, \]
\[ O_{M_eB} \subset O_{E_eB} \subset O_{EB} \]
are immediately verified.
33. The smallest class in (83) is easily identified:

THEOREM. All functions in $M_eB$ on $W$ reduce to constants if and only if $W$ is parabolic,

(84) $$O_e = O_{M_eB}.$$ 

Proof. If $W \in O_e$, there is a Green's function $g(z, \zeta)$, and

(85) $$w = e^{-g-i\varphi} \in M_eB.$$ 

In fact, $g$ is bounded above in any $W - \Omega$, hence $m(\Omega, w) = O(1)$, and $N(\Omega, w) = 0$ gives $T(\Omega) = O(1)$. Conversely, if there is a non-constant $w \in M_eB$ on $W$, then $\log |w| = u_1 - u_2$ where at least one $u_i \in LP$ is nonconstant superharmonic. This means that $W \notin O_e$. The same proof gives $O_e = O_{eMB}$.

34. By the preceding theorem, every $M_e$-function on a parabolic $W$ has unbounded characteristic. Even more can be said of $M$-functions on the larger class $O_{MB}$ by comparing $T(\Omega)$ with $k_\alpha$ (no. 4):

THEOREM. On $W \in O_{MB}$, the characteristic $T(\Omega)$ of any $w \in M$ tends so rapidly to infinity that

(86) $$\lim_{W \to \Omega} \frac{T(\Omega)}{k_\alpha} \geq 1.$$ 

Proof. Let $w(\zeta) = a$. The counting function of $w$ for $a$ is, by definition,

$$N(\Omega, a) = \int_{-\infty}^{k_\alpha} (n(h, a) - n(-\infty, a))dh + n(-\infty, a)k_\alpha,$$

where $n(h, a)$ is the number of $a$-points of $w$ in the set $\mathcal{O}_h: p_\alpha \leq h \leq k_\alpha$. We obtain from the first fundamental theorem [7] that

(87) $$T(\Omega) + O(1) \geq N(\Omega, a) \geq n(-\infty, a)k_\alpha,$$

and (86) follows.

Thus (86) is obviously a property of every $w \in M, w \notin MB$, on every $W$.

35. We also observe:

THEOREM. A function $w \in M$ on $W \in O_{MB}$ cannot omit a set of values of positive capacity.
More accurately, the counting function $N(\Omega, a)$ of $w \in M$ on $O_{MB}$ is unbounded on any set $E$ of positive capacity. To see this we distribute mass $d\mu(a) > 0$ at $a \in E$, with $\int_{E} d\mu = 1$, and integrate Jensen’s formula

$$\log | w(\zeta) - a | = \frac{1}{2\pi} \int_{\beta_{\Omega}} \log | w - a | d\mu_{\Omega}^{*} + N(\Omega, \infty) - N(\Omega, a)$$

($w(\zeta) \neq \infty$) over $E$ with respect to $d\mu(a)$. We obtain Frostman’s formula on $W$:

$$N(\Omega, \infty) - \frac{1}{2\pi} \int_{\beta_{\Omega}} u(w) d\mu_{\Omega}^{*} = \int_{E} N(\Omega, a) d\mu(a) - u(w(\zeta)),$$

where $u(w) = \int_{E} \log | w - a | d\mu(a)$. For equilibrium distribution $d\mu$ it is known from the classical theory that $u(w) = -\log | w | + O(1)$, and a fortiori $\int_{\beta_{\Omega}} u(w) d\mu_{\Omega}^{*} = -2\pi m(\Omega, \infty) + O(1)$, where $O(1)$ depends on $E$ only. Substitution into (89) gives

$$T(\Omega) = \int_{E} N(\Omega, a) d\mu(a) + O(1).$$

This proves our assertion.

36. A comprehensive study of the role played by $O_{MB}$ in the classification theory of Riemann surfaces is contained in the doctoral dissertation of K. V. R. Rao [5].

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