SEMIGROUPS AND THEIR SUBSEMIGROUP LATTICES

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1. Introduction. Let $S$ be a semigroup of order at least 2, and $L(S)$ be the system of all subsemigroups of $S$. Generally $L(S)$, including the empty subset, is a lattice with respect to inclusion. $L(S)$ is called the subsemigroup lattice of $S$. A semigroup $S$ contains at least one nonempty subsemigroup besides $S$ itself. In the previous paper [4], as the first step towards the investigation of the structure of $S$ with a given type of $L(S)$, we determined all the $\Gamma'$-semigroups, namely, the semigroups $S$'s in which $L(S)$'s are chains. In the present paper we shall define $\Gamma^*$-semigroups as generalization of $\Gamma'$-semigroups and shall obtain all the types of $\Gamma^*$-semigroups except for infinite simple $\Gamma^*$-groups. Since all the semigroups of order 2 are $\Gamma^*$-semigroups, we shall treat non-trivial $\Gamma^*$-semigroups, namely, those of order $\geq 3$ in the discussion below. First, in §2 we shall prove that $\Gamma^*$-semigroups of order $\geq 3$ are unipotent, i.e., having a unique idempotent, and that they are periodic; and hence a $\Gamma^*$-semigroup is determined by a group and a $Z$-semigroup, i.e., a unipotent semigroup with zero. Accordingly, in §3 we shall determine all the types of $\Gamma^*$-$Z$-semigroups which will have to be of order $<5$; in §4 we shall treat solvable $\Gamma^*$-groups and prove that finite $\Gamma^*$-groups or non-simple $\Gamma^*$-groups are solvable; finally in §5, unipotent $\Gamma^*$-semigroups which are neither groups nor $Z$-semigroups will be discussed. It is interesting that there are no infinite unipotent $\Gamma^*$-semigroups except groups.

For convenience, the results from the paper [4] are stated as follows:

**Lemma 1.1.** A semigroup is a $\Gamma'$-semigroup if and only if it has one of the following types.\(^2\) Except for (1.3) they are all cyclic semigroups, i.e., semigroups generated by an element $d$. We show defining relations below.

(1.1) $Z$-semigroups:

\[
\begin{align*}
(1.1.1) & \quad d^2 = d^3 \quad (\text{order } 2) \\
(1.1.2) & \quad d^3 = d^4 \quad (\text{order } 3)
\end{align*}
\]

\(^1\) The author called them $\Gamma'$-monoids in [4].

\(^2\) As the trivial case, a semigroup of order 1 is also regarded as a $\Gamma'$-semigroup. This remark will be needed for the definition of a $\Gamma^*$-semigroup.
(1.2) Cyclic groups $G(p^n)$ of a prime power order: $d = d^{p^{m+1}}$

(1.3) Quasicyclic groups [1]: $G(p^n)$, i.e.,

$$G(p^n) = \sum_{k=1}^{\infty} G(p^k)$$

where $Q(p) \subset G(p^2) \subset \cdots \subset G(p^k) \subset \cdots$, $p$ being a prime.

(1.4) Unipotent semigroups of order $n$, the kernel (the least ideal) of which is a group $G(p^n)$:

(1.4.1) if $p = 2$

$$d^2 = d^{2^{m+2}}$$

(order $n = 2^m + 1$)

(1.4.2) if $p \neq 2$

(1.4.2.1) $d^2 = d^{p^{m+2}}$ (order $n = p^m + 1$)

(1.4.2.2) $d^3 = d^{p^{m+3}}$ (order $n = p^m + 2$)

2. Preliminaries.

**DEFINITION.** A semigroup $S$ is called a $\Gamma^*$-semigroup if every subsemigroup different from $S$ is a $\Gamma$-semigroup.

$S$ is a $\Gamma^*$-semigroup if and only if the subsemigroup lattice $L(S)$ is a lattice satisfying

(2.1) Any subset which contains the greatest element 1 is a subsemilattice with respect to join, equivalently to

(2.1') Let $x, y$ be any elements of a lattice. Then

$$x \cup y = x \text{ or } y \text{ or } 1.$$  

**Notation.** If $X$ and $Y$ are subsets of $S$, $X \upharpoonright Y$ means either $X \subseteq Y$ or $X \supseteq Y$; $X \parallel Y$ means that $X$ and $Y$ are incomparable, that is, neither is contained in the other. $((X, Y, \cdots))$ denotes the subsemigroup generated by $X, Y, \cdots$. In particular, $((x))$ denotes the subsemigroup generated by an element $x$, $((x, y))$ the subsemigroup generated by elements $x$ and $y$, while $\{x_1, x_2, \cdots\}$ is the set composed of $x_1, x_2, \cdots$.

$S$ is a $\Gamma^*$-semigroup if and only if any two subsemigroups $A$ and $B$ satisfy the following condition: $A \parallel B$ implies $S = ((A, B))$. Of course a $\Gamma$-semigroup is a $\Gamma^*$-semigroup. Since the homomorphic image of a $\Gamma$-semigroup is also a $\Gamma$-semigroup, we get easily

**LEMMA 2.1.** A homomorphic image of a $\Gamma^*$-semigroup is a $\Gamma^*$-semigroup.

**LEMMA 2.2.** A $\Gamma^*$-semigroup is periodic.

**Proof.** Suppose there is an element $x$ of infinite order. $S$ con-
Private note: By "a proper subsemigroup $T$ of $S$" we mean "a subsemigroup $T$ which is different from $S$."

contains an infinite cyclic subsemigroup $\{x^i; i = 1, 2, \cdots\}$. Hence we can consider a proper subsemigroup $T$ of $S$.

$$T = \{x^{2i}; i = 1, 2, \cdots\}$$

which contains two incomparable subsemigroups $T_1$ and $T_2$:

$$T_1 = \{x^{4i}; i = 1, 2, \cdots\}, \quad T_2 = \{x^{6i}; i = 1, 2, \cdots\}.$$  

This contradicts the assumption of $S$.

By Lemma 2.2, we have seen that a $I^*$-semigroup has at least one idempotent. However, we have

**THEOREM 2.1.** A $I^*$-semigroup of order $\geq 2$ is unipotent.

**Proof.** Suppose that a $I^*$-semigroup $S$ of order $\geq 2$ contains at least two idempotents, say, $e, f$. First, since $e$ is a right identity of $Se$, and $f$ is a left identity of $fS$, we see easily that if $Se = fS$, then $e = f$. Second, we shall say that either both of $Se$ and $Sf$ or both of $eS$ and $fS$ are proper subsemigroups. Suppose either of $Se$ and $Sf$ is equal to $S$, say, $Se = S$. Then, by the above fact, $fS \subseteq S$, and so we have to show $eS \subseteq S$. Let us assume $Se = eS = S$. There is a proper subsemigroup $\{e, f\}$ of order 2 because $ef = fe = f$; but $\{e, f\}$ is not a $I^*$-semigroup since $e$ and $f$ are both idempotents. This is a contradiction. Therefore $eS \subseteq S$.

Next, assume that both $eS$ and $fS$ are proper subsemigroups of $S$. Since $eS$ and $fS$ are $I^*$-semigroups with left identities, they are groups by Lemma 1.1. We shall prove that $\{e, f\}$ is a proper subsemigroup which is not a $I^*$-semigroup, and then the contradiction will be derived. For proof, the idempotency of $ef$ and $fe$ is shown as follows:

$$(ef)(ef) = (efe)f = (ef)f = e(ff) = ef$$

$$(fe)(fe) = (fefe) = (fe)e = f(ee) = fe$$

because $e$ and $f$ are two-sided identities of the groups $eS$ and $fS$ respectively. Since $ef \in eS$ and $fe \in fS$, we have

$$ef = e, \quad fe = f$$

whence $\{e, f\}$ is a subsemigroup. We can have the same result, when $Se \subseteq S$ and $Sf \subseteq S$. Thus the proof of the theorem has been completed.

**LEMMA 2.3.** The index of an element $a$ of a $I^*$-semigroup $S$ cannot exceed 3.
Proof. Let $a$ have index greater than 1. Then $((a)) \setminus \{a\}$ is a $I^*$-semigroup, so $((a^2)) | ((a^3))$. Hence there is a positive integer $n$ such that either

$$a^2 = a^{3n} \text{ or } a^3 = a^{3n}.$$ 

This shows that $a$ has index 2 or 3.

3. $I^*-Z$-Semigroups. In this section we shall determine the types of $I^*Z$-semigroups, i.e., unipotent $I^*$-semigroup with zero $0$.

Let $S$ be a $I^*Z$-semigroup with $0$. Since $S$ is periodic, every element of $S$ is nilpotent, that is, some power of the element is $0$. By Lemma 2.3,

$$x^3 = 0 \text{ for all } x \in S.$$ 

**Lemma 3.1.** $x = xy$ implies $x = 0$; $x = yx$ implies $x = 0$.

**Proof.** $x = xy = xy^2 = xy^3 = 0$; the proof of the second part is obtained in a similar way.

**Lemma 3.2.** If $x^3 = 0$, then $xy = yx = 0$ for all $y$.

**Proof.** We may assume $x \neq 0$, let $y \neq 0$. If $((x)) | ((xy))$, $xy = 0$ because of Lemma 3.1. If $((x)) || ((xy))$, then $S = ((x, xy))$ and so $y = xu$ for some $u$.

$$xy = x^3u = 0.$$ 

The proof of $yx = 0$ is similar.

To determine the types of $I^*Z$-semigroups, we consider the possible three cases:

**Case I.** $x^3 = 0$ for all $x \neq 0$.

**Case II.** There exists only one nonzero element $x$ such that $x^3 = 0$, $x^2 \neq 0$.

**Case III.** There exist at least two nonzero elements $x$ and $y$ such that $x^3 = 0$, $x^2 \neq 0$, $y^3 = 0$, $y^2 \neq 0$.

**Theorem 3.1.** $S$ is a non-trivial $I^*Z$-semigroup if and only if $S$ is isomorphic or anti-isomorphic to one of the following:

**Case I.** $S = \{0, a, b\}$ where $xy = 0$ for all $x, y \in S$. 
Case II. \( S = \{0, a, a^2\} \) where \( a^3 = 0 \). This is a \( I^-\)-semigroup which is isomorphic to (1.1.2).

Case III. \( S = \{0, a, b, c\} \) where \( a^3 = b^3 = c, a^2x = xa^2 = x = 0 \) for all \( x \in S \).

Subcase III_1 \( ab = ba = c \)
Subcase III_2 \( ab = c, ba = 0 \)
Subcase III_3 \( ab = ba = 0 \)

Proof.

Case I. Let \( a \) and \( b \) be distinct nonzero elements of \( S \). Since \((a) \parallel (b), S = ((a, b))\). By Lemma 3.2, we have \( ab = ba = 0 \). Hence

\[ S = ((a, b)) = \{0, a, b\} . \]

Case II. Let \( a \) be an element with index 3. Suppose that there is \( b \in S - ((a)) \). In the present case we know \( b^i = 0 \). By Lemma 3.2, \( ab = ba = 0 \), whence \( A = \{0, a^3, b\} \) is a subsemigroup which does not contain \( a \), and hence \( A \) is a \( I^-\)-semigroup. On the other hand, since \( b \neq a^2 \), we have \((a^2) \parallel (b))\). It is impossible in a \( I^*\)-semigroup \( S \). Therefore \( S = ((a)) \).

Case III. Let \( a \) and \( b \) be distinct nonzero elements, both of which have index 3. Since \((a^2)^2 = (b^2)^2 = 0 \), Lemma 3.2 gives us

\[ a^2b = ba^2 = b^2a = ab^2 = 0 \quad \text{and so} \quad a^2b^2 = b^2a^2 = 0 . \]

Using (3.1) and Lemma 3.2 repeatedly, since \((aba)^2 = ababa = 0 \), we have

\[ (ab)^2 = (aba)b = 0 \]

and hence

\[ aba = 0 . \]

Similarly we get

\[ bab = 0 . \]

Now we have two subsemigroups \( T = ((a^2, b^2)) \) and \( U = ((ab, a^2))\):

\[ T = ((a^2, b^2)) = \{0, a^2, b^2\} \ni a \]

where we see \( a \neq b^2 \), otherwise, \( a = b^2 \) would imply \( a^2 = 0 \); also

\[ U = ((ab, a^2)) = \{0, ab, a^2\} \ni b . \]
Accordingly both $T$ and $U$ are $\Gamma$-semigroups and so
$$((a^2)) | ((b^2)) \quad \text{and} \quad ((ab)) | ((a^2)).$$
The first implies (3.4); the second implies (3.5)

(3.4) \hspace{1cm} a^2 = b^2
(3.5) \hspace{1cm} ab = a^2 \quad \text{or} \quad 0.

Similarly we have

(3.5') \hspace{1cm} ba = a^2 \quad \text{or} \quad 0.

Clearly $((a)) || ((b))$. By (3.1) through (3.5'),
$$S = ((a, b)) = \{0, a, b, a^2\}$$
which consists of exactly four elements. Thus we have obtained the three types for Case III. It is easy to show that the systems thus obtained are $I^*\text{-Z}$-semigroups.

4. $I^*$-groups. By Lemma 2.2, a group $G$ is a $I^*$-semigroup if and only if it is a $I^*$-group, i.e., every proper subgroup of $G$ is a $\Gamma$-group. By Lemma 1.1, every $\Gamma$-group is of type $G(p^k), k \leq \infty$. In this chapter we determine all solvable $I^*$-groups. We also show that every finite $I^*$-group is solvable. The question whether infinite simple $I^*$-groups can exist remains open.

Lemma 4.1. Let $G$ be a non-abelian solvable $I^*$-group which is not also a $\Gamma$-group. Then $G$ contains a proper normal subgroup $N \neq 1$ and an element $a$ not in $N$, such that

(4.1) \hspace{1cm} N || ((a)), \quad \text{so that} \quad G = ((N, a))

(4.2) \hspace{1cm} a^q \in N \quad \text{for a prime number} \quad q.

Proof. Since $G$ is solvable, it contains a proper normal subgroup $N$ such that $G/N$ is abelian. $N \neq 1$ since $G$ is not abelian. Since $N$ is a proper subgroup of $G$, it is a $\Gamma$-group. Since $G$ is not itself a $I^*$-group, there exist $a$ and $b$ in $G$ such that $((a)) || ((b))$, and then we have $G = ((a, b))$. If $N || ((b))$, then (4.1) holds with $b$ instead of $a$. To prove (4.1) suppose $N \supseteq ((b))$. If $N \supseteq ((b))$, then $N \supseteq ((a))$, since $N$ is a $I^*$-group; and $((a)) || ((b))$, and $N \not\supseteq ((a))$ since otherwise $((b)) \subseteq N \subseteq ((a))$. Hence $N || ((a))$ in this case. If $N \subseteq ((b))$, then, since $G/N$ is abelian, $aba^{-1}b^{-1} \in N \subseteq ((b))$, so $aba^{-1} \in ((b))b \subseteq ((b))$. Since $G = ((a, b))$, we conclude that $N' = ((b))$ is a normal subgroup of $G$, and (4.1) holds with $N'$ instead of $N$. Hence $N$ and $a$ exist such that (4.1) holds. Let $k$ be the least positive integer such that $a^k \in N$,
and let $k = k'q$ with $q$ a prime. Let $a' = a^{k'}$. Then (4.1) and (4.2) hold with $a'$ instead of $a$.

**Theorem 4.1.** Let $G$ be a solvable $I'$-group which is not a $I$-group. Then one of the following holds:

(4.3) $G$ is a group of order $pq$, $p$ and $q$ primes excluding the cyclic group of order $p^5$.

(4.4) $G$ is the quaternion group of order 8.

**Proof.** First let us take the case $G$ abelian. If $G$ were directly indecomposable, it would be isomorphic with $G(p^k)$ for some $k \leq \infty$ (cf. Theorem 10, p. 22, [2]), and so would be a $I'$-group. Hence $G$ is directly decomposable: $G = G_1 \times G_2$ where $G_1 \neq 1$, $G_2 \neq 1$. Let $a_i$ be an element of $G$ of prime order $p_i$ ($i = 1, 2$). Then $((a_i)) \parallel ((a_i))$, so $G = ((a_i, a_2)) = ((a_1)) \times ((a_2))$. Thus $G$ has type (4.3).

Let $G$ be non-abelian. By Lemma 4.1, $G$ contains a proper normal subgroup $N \neq 1$, and an element $a$ not in $N$ such that $N \not\parallel ((a))$ and $a^r \in N$ for some prime $q$. Since $N$ is a proper subgroup of $G$, it is isomorphic with $G(p^k)$ for some prime $p$ and some $k \leq \infty$. Hence $a^r$ has prime power order $p^s$, say.

If $q \neq p$, then $a_i = a_i^{p^s} \in N$, and $a_i^s = 1$. If $b$ is any element of $N$ of order $p$, we have $((a_i)) \parallel ((b))$ and hence $G = ((a_i, b))$. Since $a_iNa_i^{-1} \subseteq N$, and every subgroup of $N$ is characteristic, $a_i((b))a_i^{-1} \subseteq ((b))$. Hence $G$ is an extension of the cyclic group $((b))$ of order $p$ by the cyclic group $((a_i))$ of order $q$.

We may now assume $q = p$. Since $N \not\subseteq ((a))$, there exists $b$ in $N$ such that $b^p = a^r$. Let $e = a^p = b^p$. Since $e$ commutes with $a$ and $b$, and $G = ((a, b))$, $e$ belongs to the center $C$ of $G$. If $e = 1$, then, as in the above statements, $G$ is an extension of the cyclic group $((b))$ of order $p$ by the cyclic group $((a))$ of order $p$. Hence we may assume that the order of $e$ is $p^n$ with $n > 0$.

Since $((b))$ is invariant under $a$, we have $aba^{-1} = b^r$ for some positive integer $r > 1$. Then

$$c = b^p = abpa^{-1} = (aba^{-1})^p = b^{rp} = c^r,$$

whence $r = 1 + sp^n$ for some integer $s$. Hence

$$aba^{-1} = b^r = bd \text{ or } b^{-1}aba^{-1} = d \neq 1$$

where $d = b^{rp^n} = e^{p^{n-1}}$ is an element of $C$ of order $p$. As in the familiar way,

$$\left(ab^{-1}\right)^p = d^{p(\frac{p-1}{2})^2}a^q b^p = d^{p(\frac{p-1}{2})^2}.$$
If $p$ is odd, we conclude that $(ab^{-1})^p = 1$. Let $a_1 = ab^{-1}$. Then $a_1^p = 1$ and this case is reduced to the previous case $c = 1$. We are left with the case $p = 2$. Setting $a_1 = ab^{-1}$, we have $a_1^2 = d$. Let $b_1$ be an element of $N$ such that $b_1^2 = d$. Then $G = ((a_1, b_1))$, and $((b_1))$ is invariant under $a_1$. Since $b_1^2 = 1$, and $G$ is not abelian, we must have

$$a_1b_1a_1^{-1} = b_1^2.$$

Together with $a_1^2 = b_1^2 = 1$, this shows that $G$ is the quaternion group of order 8. Thus this theorem has been proved.

**Theorem 4.2.** A finite $I^*$-group is solvable.

**Proof.** For $I$-groups, the theorem is obvious. Let $G$ be a finite $I^*$-group which is not a $I$-group. If $G$ is of order $p^n$ of a prime power, then this theorem holds, since $G$ has a normal subgroup of order $p^{n-1}$ by the familiar theorem of $p$-groups. So we may assume that the order of $G$ has at least two distinct prime divisors.

First we shall prove that $G$ has a proper normal subgroup. Let $M$ be a Sylow subgroup of $G$, and consider the normalizer $H$ of $M$. If $H = G$, then $M$ is normal; if $M \subseteq H \subset G$, then $H$ is a $I$-group, a cyclic group. By Burnside's theorem ([8], p. 169), $G$ has a proper normal subgroup $N$ such that $G = NH$, $N \cap H = 1$.

Since $N$ is a proper subgroup, it is a $I$-group, say, $G(p^n)$. Then, suppose the order of the factor group $G/N$ is

$$p^\alpha q^\beta r^\gamma \cdots, \quad \alpha \geq 0, \beta \geq 1, \gamma \geq 0, \cdots$$

which has a prime divisor $q \neq p$. Since $G/N$ has a subgroup of order $q$, $G$ has a proper subgroup of order $p^\alpha q$, which contains two incomparable subgroups, unless

$$\alpha = 0, \beta = 1.$$

Thus we have proved that the index of $N$ is a prime $q$.

**Theorem 4.3.** A non-simple $I^*$-group is solvable.

**Proof.** Let $G$ be a non-simple $I^*$-group and $N$ be a proper normal subgroup of $G$. We may assume that $G/N$ contains a proper subgroup $\tilde{H}$ of prime order $p$, since $G/N$ is a $I^*$-group and so $G/N$ is periodic. Consider a coset $xN$ which is a generator of $\tilde{H}$ and take an element $a \in xN$. Then $H = ((a))$ is a group of order $p$, and there is a subgroup $K$ of $G$ such that $K/N \cong \tilde{H}$. Clearly $K = NH$. On the other hand, since $N \lVert H$, we have $G = ((N, H)) = NH = K$. Accordingly, $G/N \cong \tilde{H}$, which is prime order. Thus the proof has been completed.
Consequently, (4.3) and (4.4) of Theorem 4.1 give us all the types of finite or non-simple $\Gamma^*$-groups which are not $\Gamma$-groups.

5. Unipotent $\Gamma^*$-semigroups.

1. In this chapter we shall discuss unipotent $\Gamma^*$-semigroups $S$'s which are neither groups nor $Z$-semigroups. By Lemma 2.2 and Theorem 2.1 we see that a $\Gamma^*$-semigroup $S$ of order $>2$ is a unipotent invertible semigroup. By “invertible” we mean “for any element $a$ of $S$ there is an element $b$ such that $ab = e$ where $e$ is a unique idempotent.” According to [5], [6], a unipotent invertible semigroup which contains properly a group is determined by a group $G$ (or kernel, i.e., least ideal), and a $Z$-semigroup $D$ (the difference semigroup of $S$ modulo $G$), and certain mapping of the bases of $D$ into $G$: $a \rightarrow ea$.

First of all we shall prove that the kernel is finite.

**Lemma 5.1.** Let $S$ be a unipotent invertible semigroup with the kernel $G$ of type $G(p^k)$, $k$ being infinite or finite, and let $d$ be an element of $S$ which is not in $G$ such that $ed$ generates $G(p^m)$, $m < k$, and $d^{i-1} \in G(p^i)$, $d^i \in G(p^i)$. Then there is a subsemigroup $H$ of order $p^{m+1} + 1$ of $S$ which contains two incomparable subsemigroups: $G(p^m)$ and $\{d\^{i}; \ i \geq 1\}$.

**Proof.** Let $a = ed$. As is easily seen (cf. [5]), we have

\begin{align}
5.1 & \quad a = ed = de, \quad d^i = a^i, \quad i \geq l \\
5.2 & \quad xd = dx = xa = ax \quad \text{for every} \quad x \in G.
\end{align}

Especially for $x \in G(p^{m+1})$, $xd = dx \in G(p^{m+1})$. Therefore the set union $H = G(p^{m+1}) \cup \{d^i; \ l - 1 \geq i \geq 1\}$ is a subsemigroup of $S$; and the two subsemigroups $G(p^{m+1})$ and $\{d^i; \ i \geq 1\}$ are incomparable, because $\{d^i; \ i \geq l\} \subseteq G(p^n)$.

**Theorem 5.1.** Let $S$ be a unipotent invertible semigroup which is neither a group nor a $Z$-semigroup. If $S$ is a $\Gamma^*$-semigroup, then $S$ is finite.

**Proof.** The proper subgroup $G$ is a $\Gamma$-group $G(p^n)$ or $G(p^o)$, and the difference semigroup $D = (S; G)$ of $S$ modulo $G$ in Rees’ sense [3] is a $\Gamma^*$-$Z$-semigroup of order $\leq 4$ by theorems in §3. There is an element $z_i$ outside $G$ such that $z^i_i \in G$, for example, we may take a nonzero annihilator as $z_i$ (cf. [6]); and let $m$ be a positive integer such that $ez_i$ generates a subgroup $G(p^m)$. If $S$ is infinite, then $G$ is of the type $G(p^n)$ and so $S$ has a proper subsemigroup of order $p^{m+1} + 1$,
which contains two incomparable subsemigroups by Lemma 5.1. This
contradicts the definition of \( \Gamma^\ast \)-semigroups of \( S \). Thus the theorem
has been proved.

Hereafter we shall determine the desired semigroups \( S \) in each
case according as the order of \( D \).

2. The case with \( D \) of order 2.

Let \( G(p^n) \) denote the kernel of \( S \), and let \( d \) be a unique element
outside \( G(p^n) \). Of course \( d^2 \in G(p^n) \). \( G(p^k) \) denotes the subgroup generated by \( a = ed \). If \( k = n \), then, by (5.1), we have

\[
S = \{d^i; i \geq 1\}, \quad G(p^n) = \{d^i; i \geq 2\}
\]

that is, \( S \) is a \( \Gamma^\ast \)-semigroup of type (1.4.1) or (1.4.2.1).

If \( k < n \), then by Lemma 5.1 there is a subsemigroup \( H = G(p^{k+1}) \cup \{d\} \) of order \( p^{k+1} + 1 \) which contains two incomparable subsemigroups, so that \( S = H \) and hence we have \( k = n - 1 \). In other
words, \( a \) is a generator of \( G(p^{n-1}) \); this \( a \) determines \( S \) and there is
a unique \( S \) to within isomorphism, independent of choice of generator
\( a \) (cf. [6]). Conversely, a semigroup \( S \) thus obtained is easily seen
to be a \( \Gamma^\ast \)-semigroup. In fact, by (5.1) we see that a proper subsemigroup incomparable to \( G(p^n) \) is nothing but

\[
G(p^{n-1}) \cup \{d\} = ((d)).
\]

3. The case with \( D \) of type Case I of order 3.

Let \( S = G(p^n) \cup \{d_1, d_2\} \) where \( d_1, d_2, d_1^2, d_2^2 \in G(p^n) \). \( S \) is
determined by the two elements \( a_1, a_2 \), i.e.,

\[
a_1 = ed_1, \quad a_2 = ed_2
\]

where \( a_1 \) and \( a_2 \) can be taken independently arbitrarily. The proper
subsemigroups \( G(p^n) \cup \{d_1\} \) and \( G(p^n) \cup \{d_2\} \) are \( \Gamma^\ast \)-semigroups of type
(1.4.1) or (1.4.2.1). We have already known that \( a_1 \) and \( a_2 \) are the
generators of \( G(p^n) \), and

\[
G(p^n) \cup \{d_1\} = ((d_1)), \quad G(p^n) \cup \{d_2\} = ((d_2)).
\]

We can easily prove that there are two possible distinct types

\[
a_1 = a_2, \quad a_1 \neq a_2
\]
in all cases except for the case \( p = 2 \) and \( n = 1 \). They are immediately
seen to be \( \Gamma^\ast \)-semigroups.

4. The case with \( D \) of type Case II of order 3.

Let \( d \) be a generator of \( D: D = \{0, d, d^2\}, d^3 = 0 \), and let \( S =

We shall prove that $a = ed$ generates $G(p^n)$. Suppose that an element $a$ generates $G(p^k), k < n$. Then, since $ed^2 = (ed)^2$ and $(d^2)^2 = G(p^n)$, $ed^2$ generates a subgroup $G(p^m), m \leq k$, and a subsemigroup $K = G(p^{m+1}) \cup \{d^2\}$ contains two incomparable subsemigroups by Lemma 5.1. $K$ is a proper subsemigroup of $S$ because

$$p^{m+1} + 1 < p^n + 2.$$ 

This contradicts the assumption of $I^*$-semigroup of $S$. Hence it has been proved that $G(p^n)$ is generated by $ed$. Accordingly we get $G(p^n) = \{d^i; i \geq 3\}$ by (5.1), whence $S$ is generated by $d$. In the same way as the Case with $D$ of order 2, we see that arbitrary different generators of $G(p^n)$ give some isomorphic $S$'s.

The remaining thing to do is to testify the subsemigroup lattice of such semigroups.

If $p \neq 2$, then $ed^2$ generates $G(p^n)$, and only a proper subsemigroup between $S$ and $G(p^n)$ is

$$(d^2) = G(p^n) \cup \{d^2\} \text{ by (5.1)}$$

and so $S$ is a $I^*$-semigroup of type (1.4.2.2).

If $p = 2$, then $ed^2$ generates $G(2^n)$ and so, by Lemma 5.1, we have a proper subsemigroup

$$G(2^n) \cup \{d^2\}$$

which contains two incomparable $G(2^n)$ and $((d^2))$. Therefore, $S$ is not a $I^*$-semigroup.

5. The case with $D$ of order 4.

Let $S = G(p^n) \cup \{d_1, d_2, d_3\}$ where $d_1 = d_2 = d_3$. $D$ has any one of the types of Case III with elements denoted by $d_1, d_2, d_3$ instead of $a, b, c$, respectively. Since the proper subsemigroups $G(p^n) \cup \{d_1, d_2\}$ and $G(p^n) \cup \{d_1, d_3\}$ are both $I^*$-semigroups of type (1.4.2.2), we have by (5.1)

$$G(p^n) \cup \{d_1, d_2\} = ((d_3)), \quad G(p^n) \cup \{d_1, d_3\} = ((d_2))$$

where $p \neq 2$, and $a_2 = ed_2$ and $a_3 = ed_3$ are both generators of $G(p^n)$. One the other hand, there are relations between $a_2$ and $a_3$ as follows: (We called these relations the primary equations for $D$ in [6], § 3.)

$$a_2^2 = a_3^2 \text{ in Case III}_3,$$

$$a_2 = a_3 \text{ in Case III}_1 \text{ and III}_2.$$ 

We see easily that $a_2^2 = a_3^2$ in $G(p^n)$ implies $a_2 = a_3$ because $p \neq 2$. Thus for $G(p^n)$ and each $D$, there is a unique $S$ to within isomorphism.
As far as the subsemigroups containing $G(p^n)$ are concerned, besides $((d_3))$ and $((d_4))$, there is $((d_5))$ and we have

$$((d_5)) = ((d_3)) \cap ((d_4))$$

because $p \neq 2$. Accordingly it can be seen that $S$ is a $I^*\text{-semigroup}$. Thus we have

**THEOREM** 5.2. When $G(p^n)$ is given, all the possible unipotent $I^*\text{-semigroups}$ $S$ whose kernel is $G(p^n)$ and which are not $I\text{-semigroups}$ are determined as shown below. Let $e$ be the unique idempotent of $S$, and let $D = (S: G(p^n))$. We remark $G(p^n) = 1$, $G(p^{-1}) = \emptyset$.

(5.3.1) In the case $D$ of order 2, $S = G(p^n) \cup \{d\}$, $n \neq 0$, $ed \in G(p^{n-2}) - G(p^{n-1})$

(5.3.2) In the case $D$ of order 3, $D$ is of Case I and $S = G(p^n) \cup \{d_1, d_2\}$, $n \neq 0$

(5.3.2.1) $ed_1 = ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.2.2) $p^n \neq 2, ed_1 \neq ed_2$, and $ed_1, ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.3) In the case $D$ of order 4, $S = G(p^n) \cup \{d_1, d_2, d_3\}$, $d_1^2 = d_2^2 = d_3 = d_1, n \neq 0$, $p \neq 2$

(5.3.3.1) $D$ of type Case III$_1$

(5.3.3.2) $D$ of type Case III$_2$ $ed_2 = ed_3 \in G(p^n) - G(p^{n-1})$.

(5.3.3.3) $D$ of type Case III$_3$

After all, under the given $G(p^n)$, if $p \neq 2$, then there are six types of $S$; if $p = 2$ and $n \neq 1$, then three types of $S$; if $p = 2$ and $n = 1$, then two types of $S$.

REFERENCES


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