Pacific Journal of Mathematics

SEMIGROUPS AND THEIR SUBSEMIGROUP LATTICES

TAKAYUKI TAMURA

Vol. 13, No. 2 April 1963

SEMIGROUPS AND THEIR SUBSEMIGROUP LATTICES

TAKAYUKI TAMURA

1. Introduction. Let S be a semigroup of order at least 2, and $L(S)$ be the system of all subsemigroups of S. Generally $L(S)$, including the empty subset, is a lattice with respect to inclusion. $L(S)$ is called the subsemigroup lattice of S. A semigroup S contains at least one nonempty subsemigroup besides S itself. In the previous paper [4], as the first step towards the investigation of the structure of S with a given type of $L(S)$, we determined all the Γ-semigroups,[1] namely, the semigroups S's in which $L(S)$'s are chains. In the present paper we shall define Γ^*-semigroups as generalization of Γ-semigroups and shall obtain all the types of Γ^*-semigroups except for infinite simple Γ^*-groups.

Since all the semigroups of order 2 are Γ^*-semigroups, we shall treat non-trivial Γ^*-semigroups, namely, those of order $\geqq 3$ in the discussion below. First, in § 2 we shall prove that Γ^*-semigroups of order $\geqq 3$ are unipotent, i.e., having a unique idempotent, and that they are periodic; and hence a Γ^*-semigroup is determined by a group and a Z-semigroup, i.e., a unipotent semigroup with zero. Accordingly, in § 3 we shall determine all the types of Γ^*-Z-semigroups which will have to be of order <5; in § 4 we shall treat solvable Γ^*-groups and prove that finite Γ^*-groups or non-simple Γ^*-groups are solvable; finally in § 5, unipotent Γ^*-semigroups which are neither groups nor Z-semigroups will be discussed. It is interesting that there are no infinite unipotent Γ^*-semigroups except groups.

For convenience, the results from the paper [4] are stated as follows:

LEMMA 1.1. *A semigroup is a Γ-semigroup if and only if it has one of the following types.*[2] *Except for* (1.3) *they are all cyclic semigroups, i.e., semigroups generated by an element d. We show defining relations below.*

(1.1) *Z-semigroups*:

(1.1.1) $\quad d^2 = d^3 \quad$ (*order* 2)

(1.1.2) $\quad d^3 = d^4 \quad$ (*order* 3)

Received February 21, 1962, and in revised form February 23, 1963. This paper was delivered in the meeting of the American Mathematical Society in Seattle in June, 1961; and the rapid report was published in [7].

[1] The author called them Γ-monoids in [4].

[2] As the trivial case, a semigroup of order 1 is also regarded as a Γ-semigroup. This remark will be needed for the definition of a Γ^*-semigroup.

(1.2) *Cyclic groups* $G(p^m)$ *of a prime power order*: $d = d^{p^m+1}$
(1.3) *Quasicyclic groups* [1]: $G(p^\infty)$, *i.e.*,

$$G(p^\infty) = \sum_{k=1}^{\infty} G(p^k)$$

where $Q(p) \subset G(p^2) \subset \cdots \subset G(p^k) \subset \cdots$, *p being a prime.*

(1.4) *Unipotent semigroups of order n, the kernel (the least ideal) of which is a group* $G(p^m)$:

(1.4.1)	*if* $p = 2$	$d^2 = d^{2^m+2}$	(*order* $n = 2^m + 1$)
(1.4.2)	*if* $p \neq 2$		
(1.4.2.1)		$d^2 = d^{p^m+2}$	(*order* $n = p^m + 1$)
(1.4.2.2)		$d^3 = d^{p^m+3}$	(*order* $n = p^m + 2$)

2. Preliminaries.

DEFINITION. A semigroup S is called a Γ^*-semigroup if every subsemigroup different from S is a Γ-semigroup.

S is a Γ^*-semigroup if and only if the subsemigroup lattice $L(S)$ is a lattice satisfying

(2.1) Any subset which cantains the greatest element 1 is a subsemilattice with respect to join, equivalently to

(2.1′) Let x, y be any elements of a lattice. Then

$$x \cup y = x \text{ or } y \text{ or } 1 .$$

Notation. If X and Y are subsets of S, $X \mid Y$ means either $X \subseteqq Y$ or $X \supseteqq Y$; $X \parallel Y$ means that X and Y are incomparable, that is, neither is contained in the other. $((X, Y, \cdots))$ denotes the subsemigroup generated by $X, Y, \cdots$. In particular, $((x))$ denotes the subsemigroup generated by an element x, $((x, y))$ the subsemigroup generated by elements x and y, while $\{x_1, x_2, \cdots\}$ is the set composed of $x_1, x_2, \cdots$.

S is a Γ^*-semigroup if and only if any two subsemigroups A and B satisfy the following condition: $A \parallel B$ implies $S = ((A, B))$. Of course a Γ-semigroup is a Γ^*-semigroup. Since the homomorphic image of a Γ-semigroup is also a Γ-semigroup, we get easily

LEMMA 2.1. *A homomorphic image of a* Γ^*-*semigroup is a* Γ^*-*semigroup.*

LEMMA 2.2. *A* Γ^*-*semigroup is periodic.*

Proof. Suppose there is an element x of infinite order. S contains an infinite cyclic subsemigroup $\{x^i; i = 1, 2, \cdots\}$. Hence we can consider a proper subsemigroup[3] T of S.

$$T = \{x^{2i}; i = 1, 2, \cdots\}$$

which contains two incomparable subsemigroups T_1 and T_2:

$$T_1 = \{x^{4i}; i = 1, 2, \cdots\}, \qquad T_2 = \{x^{6i}; i = 1, 2, \cdots\}.$$

This contradicts the assumption of S.

By Lemma 2.2, we have seen that a Γ^*-semigroup has at least one idempotent. However, we have

THEOREM 2.1. *A Γ^*-semigroup of order >2 is unipotent.*

Proof. Suppose that a Γ^*-semigroup S of order >2 contains at least two idempotents, say, e, f. First, since e is a right identity of Se, and f is a left identity of fS, we see easily that if $Se = fS$, then $e = f$. Second, we shall say that either both of Se and Sf or both of eS and fS are proper subsemigroups. Suppose either of Se and Sf is equal to S, say, $Se = S$. Then, by the above fact, $fS \subset S$, and so we have to show $eS \subset S$. Let us assume $Se = eS = S$. There is a proper subsemigroup $\{e, f\}$ of order 2 because $ef = fe = f$; but $\{e, f\}$ is not a Γ-semigroup since e and f are both idempotents. This is a contradiction. Therefore $eS \subset S$.

Next, assume that both eS and fS are proper subsemigroups of S. Since eS and fS are Γ-semigroups with left identities, they are groups by Lemma 1.1. We shall prove that $\{e, f\}$ is a proper subsemigroup which is not a Γ-semigroup, and then the contradiction will be derived. For proof, the idempotency of ef and fe is shown as follows:

$$(ef)(ef) = (efe)f = (ef)f = e(ff) = ef$$
$$(fe)(fe) = (fef)e = (fe)e = f(ee) = fe$$

because e and f are two-sided identities of the groups eS and fS respectively. Since $ef \in eS$ and $fe \in fS$, we have

$$ef = e, \qquad fe = f$$

whence $\{e, f\}$ is a subsemigroup. We can have the same result, when $Se \subset S$ and $Sf \subset S$. Thus the proof of the theorem has been completed.

LEMMA 2.3. *The index of an element a of a Γ^*-semigroup S cannot exceed* 3.

[3] By "a proper subsemigroup T of S" we mean "a subsemigroup T which is different from S."

Proof. Let a have index greater than 1. Then $((a)) - \{a\}$ is a Γ-semigroup, so $((a^2)) \mid ((a^3))$. Hence there is a positive integer n such that either

$$a^2 = a^{3n} \quad \text{or} \quad a^3 = a^{2n} .$$

This shows that a has index 2 or 3.

3. **Γ^*-Z-Semigroups.** In this section we shall determine the types of Γ^*-Z-semigroups, i.e., unipotent Γ^*-semigroup with zero 0.

Let S be a Γ^*-Z-semigroup with 0. Since S is periodic, every element of S is nilpotent, that is, some power of the element is 0. By Lemma 2.3,

$$x^3 = 0 \quad \text{for all} \quad x \in S .$$

LEMMA 3.1. *$x = xy$ implies $x = 0$; $x = yx$ implies $x = 0$.*

Proof. $x = xy = xy^2 = xy^3 = 0$; the proof of the second part is obtained in a similar way.

LEMMA 3.2. *If $x^2 = 0$, then $xy = yx = 0$ for all y.*

Proof. We may assume $x \neq 0$, let $y \neq 0$. If $((x)) \mid ((xy))$, $xy = 0$ because of Lemma 3.1. If $((x)) \parallel ((xy))$, then $S = ((x, xy))$ and so $y = xu$ for some u.

$$xy = x^2u = 0 .$$

The proof of $yx = 0$ is similar.

To determine the types of Γ^*-Z-semigroups, we consider the possible three cases:

Case I. $x^2 = 0$ for all $x \neq 0$.

Case II. There exists only one nonzero element x such that $x^3 = 0$, $x^2 \neq 0$.

Case III. There exist at least two nonzero elements x and y such that $x^3 = 0$, $x^2 \neq 0$, $y^3 = 0$, $y^2 \neq 0$.

THEOREM 3.1. *S is a non-trivial Γ^*-Z-semigroup if and only if S is isomorphic or anti-isomorphic to one of the following:*

Case I. *$S = \{0, a, b\}$ where $xy = 0$ for all $x, y \in S$.*

Case II. $S=\{0, a, a^2\}$ *where* $a^3=0$. *This is a* Γ*-semigroup which is isomorphic to* (1.1.2).

Case III. $S = \{0, a, b, c\}$ *where* $a^2 = b^2 = c$, $a^2x = xa^2 = 0$ *for all* $x \in S$.

Subcase III_1 $ab = ba = c$

Subcase III_2 $ab = c, ba = 0$

Subcase III_3 $ab = ba = 0$

Proof.

Case I. Let a and b be distinct nonzero elements of S. Since $((a)) \| ((b))$, $S = ((a, b))$. By Lemma 3.2, we have $ab = ba = 0$. Hence

$$S = ((a, b)) = \{0, a, b\} .$$

Case II. Let a be an element with index 3. Suppose that there is $b \in S - ((a))$. In the present case we know $b^2 = 0$. By Lemma 3.2, $ab = ba = 0$, whence $A = \{0, a^2, b\}$ is a subsemigroup which does not contain a, and hence A is a Γ-semigroup. On the other hand, since $b \neq a^2$, we have $((a^2)) \| ((b))$. It is impossible in a Γ^*-semigroup S. Therefore $S = ((a))$.

Case III. Let a and b be distinct nonzero elements, both of which have index 3. Since $(a^2)^2 = (b^2)^2 = 0$, Lemma 3.2 gives us

$$a^2b = ba^2 = b^2a = ab^2 = 0 \quad \text{and so} \quad a^2b^2 = b^2a^2 = 0 . \tag{3.1}$$

Using (3.1) and Lemma 3.2 repeatedly, since $(aba)^2 = aba^2ba = 0$, we have

$$(ab)^2 = (aba)b = 0 \tag{3.2}$$

and hence

$$aba = 0 . \tag{3.3}$$

Similarly we get

$$bab = 0 . \tag{3.3'}$$

Now we have two subsemigroups $T = ((a^2, b^2))$ and $U = ((ab, a^2))$:

$$T = ((a^2, b^2)) = \{0, a^2, b^2\} \not\ni a$$

where we see $a \neq b^2$, otherwise, $a = b^2$ would imply $a^2 = 0$; also

$$U = ((ab, a^2)) = \{0, ab, a^2\} \not\ni b .$$

Accordingly both T and U are Γ-semigroups and so

$$((a^2)) \mid ((b^2)) \quad \text{and} \quad ((ab)) \mid ((a^2)) \,.$$

The first implies (3.4); the second implies (3.5)

$$a^2 = b^2 \tag{3.4}$$

$$ab = a^2 \quad \text{or} \quad 0 \,. \tag{3.5}$$

Similarly we have

$$ba = a^2 \quad \text{or} \quad 0 \,. \tag{3.5$'$}$$

Clearly $((a)) \| ((b))$. By (3.1) through (3.5′),

$$S = ((a, b)) = \{0, a, b, a^2\}$$

which consists of exactly four elements. Thus we have obtained the three types for Case III. It is easy to show that the systems thus obtained are Γ^*-Z-semigroups.

4. **Γ^*-groups.** By Lemma 2.2, a group G is a Γ^*-semigroup if and only if it is a Γ^*-group, i.e, every proper gubgroup of G is a Γ-group. By Lemma 1.1, every Γ-group is of type $G(p^k)$, $k \leqq \infty$. In this chapter we determine all solvable Γ^*-groups. We also show that every finite Γ^*-group is solvable. The question whether infinite simple Γ^*-groups can exist remains open.

LEMMA 4.1. *Let G be a non-abelian solvable Γ^*-group which is not also a Γ-group. Then G contains a proper normal subgroup $N \neq 1$ and an element a not in N, such that*

$$N \| ((a)), \quad \textit{so that} \quad G = ((N, a)) \tag{4.1}$$

$$a^q \in N \quad \textit{for a prime number } q \,. \tag{4.2}$$

Proof. Since G is solvable, it contains a proper normal subgroup N such that G/N is abelian. $N \neq 1$ since G is not abelian. Since N is a proper subgroup of G, it is a Γ-group. Since G is not itself a Γ-group, there exist a and b in G such that $((a)) \| ((b))$, and then we have $G = ((a, b))$. If $N \| ((b))$, then (4.1) holds with b instead of a. To prove (4.1) suppose $N \| ((b))$. If $N \supseteqq ((b))$, then $N \not\supseteqq ((a))$, since N is a Γ-group; and $((a)) \| ((b))$, and $N \not\subseteqq ((a))$ since otherwise $((b)) \subseteqq N \subseteqq ((a))$. Hence $N \| ((a))$ in this case. If $N \subseteqq ((b))$, then, since G/N is abelian, $aba^{-1}b^{-1} \in N \subseteqq ((b))$, so $aba^{-1} \in ((b))b \subseteqq ((b))$. Since $G = ((a, b))$, we conclude that $N' = ((b))$ is a normal subgroup of G, and (4.1) holds with N' instead of N. Hence N and a exist such that (4.1) holds. Let k be the least positive integer such that $a^k \in N$, and let $k = k'q$ with q a prime. Let $a' = a^{k'}$. Then (4.1) and (4.2) hold with a' instead of a.

THEOREM 4.1. *Let G be a solvable Γ^*-group which is not a Γ-group. Then one of the following holds:*

(4.3) *G is a group of order pq, p and q primes excluding the cyclic group of order p^2.*

(4.4) *G is the quaternion group of order 8.*

Proof. First let us take the case G abelian. If G were directly indecomposable, it would be isomorphic with $G(p^k)$ for some $k \leqq \infty$ (cf. Theorem 10, p. 22, [2]), and so would be a Γ-group. Hence G is directly decomposable: $G = G_1 \times G_2$ where $G_1 \neq 1$, $G_2 \neq 1$. Let a_i be an element of G_i of prime order p_i ($i = 1, 2$). Then $((a_1)) \| (a_2))$, so $G = ((a_1, a_2)) = ((a_1)) \times ((a_2))$. Thus G has type (4.3).

Let G be non-abelian. By Lemma 4.1, G contains a proper normal subgroup $N \neq 1$, and an element a not in N such that $N \| ((a))$ and $a^q \in N$ for some prime q. Since N is a proper subgroup of G, it is isomorphic with $G(p^k)$ for some prime p and some $k \leqq \infty$. Hence a^q has prime power order p^n, say.

If $q \neq p$, then $a_1 = a^{p^n} \notin N$, and $a_1^q = 1$. If b is any element of N of order p, we have $((a_1)) \| ((b))$ and hence $G = ((a_1, b))$. Since $a_1 N a_1^{-1} \subseteqq N$, and every subgroup of N is characteristic, $a_1((b))a_1^{-1} \subseteqq ((b))$. Hence G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a_1))$ of order q.

We may now assume $q = p$. Since $N \nsubseteqq ((a))$, there exists b in N such that $b^p = a^p$. Let $c = a^p = b^p$. Since c commutes with a and b, and $G = ((a, b))$, c belongs to the center C of G. If $c = 1$, then, as in the above statements, G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a))$ of order p. Hence we may assume that the order of c is p^n with $n > 0$.

Since $((b))$ is invariant under a, we have $aba^{-1} = b^r$ for some positive integer $r > 1$. Then

$$c = b^p = ab^pa^{-1} = (aba^{-1})^p = b^{rp} = c^r ,$$

whence $r = 1 + sp^n$ for some integer s. Hence

$$aba^{-1} = b^r = bd \quad \text{or} \quad b^{-1}aba^{-1} = d \neq 1$$

where $d = b^{sp^n} = c^{sp^{n-1}}$ is an element of C of order p. As in the familiar way,

$$(ab^{-1})^p = d^{p(p-1)/2}a^pb^{-p} = d^{p(p-1)/2} .$$

If p is odd, we conclude that $(ab^{-1})^p = 1$. Let $a_1 = ab^{-1}$. Then $a_1^p = 1$ and this case is reduced to the previous case $c = 1$. We are left with the case $p = 2$. Setting $a_1 = ab^{-1}$, we have $a_1^2 = d$. Let b_1 be an element of N such that $b_1^2 = d$. Then $G = ((a_1, b_1))$, and $((b_1))$ is invariant under a_1. Since $b_1^4 = 1$, and G is not abelian, we must have

$$a_1 b_1 a_1^{-1} = b_1^3 .$$

Together with $a_1^4 = b_1^4 = 1$, this shows that G is the quaternion group of order 8. Thus this theorem has been proved.

THEOREM 4.2. *A finite Γ^*-group is solvable.*

Proof. For Γ-groups, the theorem is obvious. Let G be a finite Γ^*-group which is not a Γ-group. If G is of order p^m of a prime power, then this theorem holds, since G has a normal subgroup of order p^{m-1} by the familiar theorem of p-groups. So we may assume that the order of G has at least two distinct prime divisors.

First we shall prove that G has a proper normal subgroup. Let M be a Sylow subgroup of G, and consider the normalizer H of M. If $H = G$, then M is normal; if $M \subseteqq H \subset G$, then H is a Γ-group, a cyclic group. By Burnside's theorem ([8], p. 169), G has a proper normal subgroup N such that $G = NH$, $N \cap H = 1$.

Since N is a proper subgroup, it is a Γ-group, say, $G(p^{\alpha_1})$. Then, suppose the order of the factor group G/N is

$$p^{\alpha_2} q^{\beta} r^{\gamma} \cdots, \quad \alpha_2 \geqq 0, \beta \geqq 1, \quad \gamma \geqq 0, \cdots \tag{4.5}$$

which has a prime divisor $q \neq p$. Since G/N has a subgroup of order q, G has a proper subgroup of order $p^{\alpha_1} q$, which contains two incomparable subgroups, unless

$$\alpha_2 = 0, \beta = 1 . \tag{4.6}$$

Thus we have proved that the index of N is a prime q.

THEOREM 4.3. *A non-simple Γ^*-group is solvable.*

Proof. Let G be a non-simple Γ^*-group and N be a proper normal subgroup of G. We may assume that G/N contains a proper subgroup $\bar{H}$ of prime order p, since G/N is a Γ^*-group and so G/N is periodic. Consider a coset xN which is a generator of $\bar{H}$ and take an element $a \in xN$. Then $H = ((a))$ is a group of order p, and there is a subgroup K of G such that $K/N \cong \bar{H}$. Clearly $K = NH$. On the other hand, since $N \| H$, we have $G = ((N, H)) = NH = K$. Accordingly, $G/N \cong \bar{H}$, which is prime order. Thus the proof has been completed.

Consequently, (4.3) and (4.4) of Theorem 4.1 give us all the types of finite or non-simple Γ^*-groups which are not Γ-groups.

5. Unipotent Γ^*-semigroups.

1. In this chapter we shall discuss unipotent Γ^*-semigroups S's which are neither groups nor Z-semigroups. By Lemma 2.2 and Theorem 2.1 we see that a Γ^*-semigroup S of order >2 is a unipotent inversible semigroup. By "inversible" we mean "for any element a of S there is an element b such that $ab = e$ where e is a unique idempotent." According to [5], [6], a unipotent inversible semigroup which contains properly a group is determined by a group G (or kernel, i.e., least ideal), and a Z-semigroup D (the difference semigroup of S modulo G), and certain mapping of the bases of D into G: $a \to ea$.

First of all we shall prove that the kernel is finite.

LEMMA 5.1. *Let S be a unipotent inversible semigroup with the kernel G of type $G(p^k)$, k being infinite or finite, and let d be an element of S which is not in G such that ed generates $G(p^m)$, $m < k$, and $d^{l-1} \notin G(p^k)$, $d^l \in G(p^k)$. Then there is a subsemigroup H of order $p^{m+1} + l - 1$ of S which contains two incomparable subsemigroups: $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$.*

Proof. Let $a = ed$. As is easily seen (cf. [5]), we have

$$a = ed = de,\ d^i = a^i,\ i \geqq l \tag{5.1}$$

$$xd = dx = xa = ax \quad \text{for every} \quad x \in G\ . \tag{5.2}$$

Especially for $x \in G(p^{m+1})$, $xd = dx \in G(p^{m+1})$. Therefore the set union $H = G(p^{m+1}) \cup \{d^i; l - 1 \geqq i \geqq 1\}$ is a subsemigroup of S; and the two subsemigroups $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$ are incomparable, because $\{d^i; i \geqq l\} \subseteqq G(p^m)$.

THEOREM 5.1. *Let S be a unipotent inversible semigroup which is neither a group nor a Z-semigroup. If S is a Γ^*-semigroup, then S is finite.*

Proof. The proper subgroup G is a Γ-group $G(p^\infty)$ or $G(p^n)$, and the difference semigroup $D = (S: G)$ of S modulo G in Rees' sense [3] is a Γ^*-Z-semigroup of order $\leqq 4$ by theorems in § 3. There is an element z_1 outside G such that $z_1^2 \in G$, for example, we may take a nonzero annihilator as z_1 (cf. [6]); and let m be a positive integer such that ez_1 generates a subgroup $G(p^m)$. If S is infinite, then G is of the type $G(p^\infty)$ and so S has a proper subsemigroup of order $p^{m+1} + 1$, which contains two incomparable subsemigroups by Lemma 5.1. This contradicts the definition of Γ^*-semigroups of S. Thus the theorem has been proved.

Hereafter we shall determine the desired semigroups S in each case according as the order of D.

2. The case with D of order 2.

Let $G(p^n)$ denote the kernel of S, and let d be a unique element outside $G(p^n)$. Of course $d^2 \in G(p^n)$. $G(p^k)$ denotes the subgroup generated by $a = ed$. If $k = n$, then, by (5.1), we have

$$S = \{d^i; i \geqq 1\}, \qquad G(p^n) = \{d^i; i \geqq 2\}$$

that is, S is a Γ-semigroup of type (1.4.1) or (1.4.2.1).

If $k < n$, then by Lemma 5.1 there is a subsemigroup $H = G(p^{k+1}) \cup \{d\}$ of order $p^{k+1} + 1$ which contains two incomparable subsemigroups, so that $S = H$ and hence we have $k = n - 1$. In other words, a is a generator of $G(p^{n-1})$; this a determines S and there is a unique S to within isomorphism, independent of choice of generator a (cf. [6]). Conversely, a semigroup S thus obtained is easily seen to be a Γ^*-semigroup. In fact, by (5.1) we see that a proper subsemigroup incomparable to $G(p^n)$ is nothing but

$$G(p^{n-1}) \cup \{d\} = ((d)) .$$

3. The case with D of type Case I of order 3.

Let $S = G(p^n) \cup \{d_1, d_2\}$ where $d_1d_2, d_1^2, d_2^2, d_2d_1 \in G(p^n)$. S is determined by the two elements a_1, a_2, i.e.,

$$a_1 = ed_1, \qquad a_2 = ed_2$$

where a_1 and a_2 can be taken independently arbitrarily. The proper subsemigroups $G(p^n) \cup \{d_1\}$ and $G(p^n) \cup \{d_2\}$ are Γ-semigroups of type (1.4.1) or (1.4.2.1). We have already known that a_1 and a_1 are the generators of $G(p^n)$, and

$$G(p^n) \cup \{d_1\} = ((d_1)), \qquad G(p^n) \cup \{d_2\} = ((d_2)) .$$

We can easily prove that there are two possible distinct types

$$a_1 = a_2, \qquad a_1 \neq a_2$$

in all cases except for the case $p = 2$ and $n = 1$. They are immediately seen to be Γ^*-semigroups.

4. The case with D of type Case II of order 3.

Let d be a generator of D: $D = \{0, d, d^2\}, d^3 = 0$, and let $S =$

$G(p^n) \cup \{d, d^2\}$. We shall prove that $a = ed$ generates $G(p^n)$. Suppose that an element a generates $G(p^k)$, $k < n$. Then, since $ed^2 = (ed)^2$ and $(d^2)^2 \in G(p^n)$, ed^2 generates a subgroup $G(p^m)$, $m \leqq k$, and a subsemigroup $K = G(p^{m+1}) \cup \{d^2\}$ contains two incomparable subsemigroups by Lemma 5.1. K is a proper subsemigroup of S because

$$p^{m+1} + 1 < p^n + 2 .$$

This contradicts the assumption of Γ^*-semigroup of S. Hence it has been proved that $G(p^n)$ is generated by ed. Accordingly we get $G(p^n) = \{d^i; i \geqq 3\}$ by (5.1), whence S is generated by d. In the same way as the Case with D of order 2, we see that arditrary different generators of $G(p^n)$ give some isomorphic S's.

The remaining thing to do is to testify the subsemigroup lattice of such semigroups.

If $p \neq 2$, then ed^2 generates $G(p^n)$, and only a proper subsemigroup between S and $G(p^n)$ is

$$((d^2)) = G(p^n) \cup \{d^2\} \quad \text{by } (5.1)$$

and so S is a Γ-semigroup of type (1.4.2.2).

If $p = 2$, then ed^2 generates $G(2^{n-1})$ and so, by Lemma 5.1, we have a proper subsemigroup

$$G(2^n) \cup \{d^2\}$$

which contain, two incomparable $G(2^n)$ and $((d^2))$. Therefore, S is not a Γ^*-semigroup.

5. The case with D of order 4.

Let $S = G(p^n) \cup \{d_1, d_2, d_3\}$ where $d_1 = d_2^2 = d_3^2$. D has any one of the types of Case III with elements denoted by d_1, d_2, d_3 instead of a, b, c, respectively. Since the proper subsemigroups $G(p^n) \cup \{d_1, d_2\}$ and $G(p^n) \cup \{d_1, d_3\}$ are both Γ-semigroups of type (1.4.2.2), we have by (5.1)

$$G(p^n) \cup \{d_1, d_2\} = ((d_2)) , \qquad G(p^n) \cup \{d_1, d_3\} = ((d_3))$$

where $p \neq 2$, and $a_2 = ed_2$ and $a_3 = ed_3$ are both generators of $G(p^n)$. One the other hand, there are relations between a_2 and a_3 as follows: (We called these relations the primary equations for D in [6], § 3.)

$$a_2^2 = a_3^2 \qquad \text{in Case III}_3 ,$$
$$a_2 = a_3 \qquad \text{in Cases III}_1 \text{ and III}_2 .$$

We see easily that $a_2^2 = a_3^2$ in $G(p^n)$ implies $a_2 = a_3$ because $p \neq 2$. Thus for $G(p^n)$ and each D, there is a unique S to within isomorphism.

As far as the subsemigroups containing $G(p^n)$ are concerned, besides $((d_2))$ and $((d_3))$, there is $((d_1))$ and we have

$$((d_1)) = ((d_2)) \cap ((d_3))$$

because $p \neq 2$. Accordingly it can be seen that S is a Γ^*-semigroup. Thus we have

THEOREM 5.2. *When $G(p^n)$ is given, all the possible unipotent Γ^*-semigroups S whose kernel is $G(p^n)$ and which are not Γ-semigroups are determined as shown below. Let e be the unique idempotent of S, and let $D = (S: G(p^n))$. We remark $G(p^0) = 1$, $G(p^{-1}) =$ empty.*

(5.3.1) *In the case D of order 2, $S = G(p^n) \cup \{d\}$, $n \neq 0$, $ed \in G(p^{n-1}) - G(p^{n-2})$*

(5.3.2) *In the case D of order 3, D is of Case I and $S = G(p^n) \cup \{d_1, d_2\}$, $n \neq 0$*

(5.3.2.1) $$ed_1 = ed_2 \in G(p^n) - G(p^{n-1})$$

(5.3.2.2) $$p^n \neq 2,\ ed_1 \neq ed_2, \quad \text{and} \quad ed_1, ed_2 \in G(p^n) - G(p^{n-1})$$

(5.3.3) *In the case D of order 4, $S = G(p^n) \cup \{d_1, d_2, d_3\}$, $d_2^2 = d_3^2 = d_1$, $n \neq 0$, $p \neq 2$*

(5.3.3.1) *D of type Case* III_1

(5.3.3.2) *D of type Case* III_2

(5.3.3.3) *D of type Case* III_3

$$ed_2 = ed_3 \in G(p^n) - G(p^{n-1}) .$$

After all, under the given $G(p^n)$, if $p \neq 2$, then there are six types of S; if $p = 2$ and $n \neq 1$, then three types of S; if $p = 2$ and $n = 1$, then two types of S.

REFERENCES

1. L. Fuchs, *Abelian groups*, Publishing House of the Hungarian Acad. of Sci., Budapest, (1958).
2. I. Kaplansky, *Infinite abelian groups*, University of Michigan Press, (1954).
3. D. Rees, On semigroups, Proc. Cambridge Phil. Soc., **36** (1940), 387-400.
4. T. Tamura, *On a monoid whose submonoids form a chain*, Jour. of Gakugei, Tokushima Univ., **5** (1954), 8-16.
5. ———, *Note on unipotent inversible semigroups*, Kodai Math. Sem. Rep., **3** (1954), 93-95.
6. ———, *The theory of construction of finite semigroups III*, Osaka Math. Jour., **10** (1958), 191-204.
7. ———, *Note on Γ^*-semigroups*, Bull. Amer. Math. Soc., **68** (1962), 505-508.
8. H. Zassenhauss, *The theory of groups*, Chelsea Publ. Comp., New York (2nd edition), (1956).

UNIVERSITY OF CALIFORNIA, DAVIS

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 13, No. 2 April, 1963