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SOLVABILITY OF GROUPS OF ODD ORDER

WALTER FEIT AND JOHN G. THOMPSON

CHAPTER I

1. Introduction

The purpose of this paper is to prove the following result:

THEOREM. *All finite groups of odd order are solvable.*

Some consequences of this theorem and a discussion of the proof may be found in [11].

The paper contains six chapters, the first three being of a general nature. The first section in each of Chapters IV and V summarizes the results proved in that chapter. These results provide the starting point of the succeeding chapter. Other than this, there is no cross reference between Chapters IV, V and VI. The methods used in Chapter IV are purely group theoretical. The work in Chapter V relies heavily on the theory of group characters. Chapter VI consists primarily of a study of generators and relations of a special sort.

2. Notation and Definitions

Most of the following lengthy notation is familiar. Some comes from a less familiar set of notes of P. Hall [20], while some has arisen from the present paper. In general, groups and subsets of groups are denoted by German capitals, while group elements are denoted by ordinary capitals. Other sets of various kinds are denoted by English script capitals. All groups considered in this paper are finite, except when explicitly stated otherwise.

Ordinary lower case letters denote numbers or sometimes elements of sets other than subsets of the group under consideration. Greek letters usually denote complex valued functions on groups. However,

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σ and τ are reserved for field automorphisms, permutations or other mappings, and ε is used with or without subscripts to denote a root of unity. Bold faced letters are used to denote operators on subsets of groups.

The rational numbers are denoted by $\mathscr{Q}$, while $\mathscr{Q}_n$ denotes the field of nth roots of unity over $\mathscr{Q}$.

Set theoretic union is denoted by $\cup$. If $\mathfrak{A}$ and $\mathfrak{B}$ are sets, $\mathfrak{A} - \mathfrak{B}$ denotes the elements of $\mathfrak{A}$ which are not in $\mathfrak{B}$. $\mathfrak{A} \subset \mathfrak{B}$ means that $\mathfrak{A}$ is a proper subset of $\mathfrak{B}$.

$\langle \cdots \mid \cdots \rangle$	the group generated by $\cdots$ such that $\cdots$. $\langle 1 \rangle$ will be identified with 1.
$\{\cdots \mid \cdots\}$	the set of $\cdots$ such that $\cdots$.
$gp\langle \cdots \mid \cdots \rangle$	the group defined by the generators $\cdots$ with the relations $\cdots$.
$\lvert \mathfrak{X} \rvert$	the number of elements in the set $\mathfrak{X}$.
$\mathfrak{X}^{\#}$	the set of non identity elements in the set $\mathfrak{X}$.
π	a set of primes. If $\pi = \{p\}$, we customarily identify π with p.
π'	the complementary set of primes.
π-number	a non zero integer all of whose prime factors are in π.
n_π	the largest π-number dividing the non zero integer n.
π-group	a group $\mathfrak{X}$ with $\lvert \mathfrak{X} \rvert = \lvert \mathfrak{X} \rvert_\pi$.
π-element	a group element X such that $\langle X \rangle$ is a π-group.
S_π-subgroup of $\mathfrak{X}$	a subgroup $\mathfrak{S}$ of $\mathfrak{X}$ with $\lvert \mathfrak{S} \rvert = \lvert \mathfrak{X} \rvert_\pi$.
S-subgroup of $\mathfrak{X}$	a S_π-subgroup of $\mathfrak{X}$ for suitable π.
Hall subgroup of $\mathfrak{X}$	a S-subgroup of $\mathfrak{X}$.
$\mathfrak{K} \lhd \mathfrak{X}$	$\mathfrak{K}$ is a normal subgroup of $\mathfrak{X}$.
$\mathfrak{K}$ char $\mathfrak{X}$	$\mathfrak{K}$ is a characteristic subgroup of $\mathfrak{X}$.
$f(\mathfrak{X} \bmod \mathfrak{N})$	the inverse image in $\mathfrak{X}$ of $f(\mathfrak{X}/\mathfrak{N})$. Here $\mathfrak{N} \lhd \mathfrak{X}$, and f is a function from groups to subgroups.
$\boldsymbol{O}_\pi(\mathfrak{X})$	the maximal normal π-subgroup of $\mathfrak{X}$.
$\boldsymbol{O}_{\pi_1,\cdots,\pi_n}(\mathfrak{X})$	$\boldsymbol{O}_{\pi_n}(\mathfrak{X} \bmod \boldsymbol{O}_{\pi_1,\cdots,\pi_{n-1}}(\mathfrak{X}))$.
π-closed group	we say that $\mathfrak{X}$ is π-closed if and only if $\mathfrak{X}$ has a normal S_π-subgroup.
$\boldsymbol{F}(\mathfrak{X})$	the Fitting subgroup of $\mathfrak{X}$, the maximal normal nilpotent subgroup of $\mathfrak{X}$.
$\boldsymbol{D}(\mathfrak{X})$	the Frattini subgroup of $\mathfrak{X}$, the intersection of all maximal subgroups of $\mathfrak{X}$.
$\boldsymbol{Z}_n(\mathfrak{X})$	the nth term in the ascending central series of $\mathfrak{X}$, defined inductively by: $\boldsymbol{Z}_0(\mathfrak{X}) = 1$, $\boldsymbol{Z}_1(\mathfrak{X}) =$

	$Z(\mathfrak{X})$ = center of $\mathfrak{X}$, $Z_{n+1}(\mathfrak{X}) = Z(\mathfrak{X} \bmod Z_n(\mathfrak{X}))$.
$O^{\pi}(\mathfrak{X})$	the smallest normal subgroup $\mathfrak{Y}$ of $\mathfrak{X}$ such that $\mathfrak{X}/\mathfrak{Y}$ is a π-group.
$[X, Y]$	$X^{-1}Y^{-1}XY = X^{-1}X^{Y}$.
$[X_1, \cdots, X_n]$	$[[X_1, \cdots, X_{n-1}], X_n]$, $n \geqq 3$.
$[\mathfrak{A}, \mathfrak{B}]$	$\langle [A, B] \mid A \in \mathfrak{A}, B \in \mathfrak{B} \rangle$, $\mathfrak{A}$ and $\mathfrak{B}$ being subsets of a group.
$[\mathfrak{A}_1, \cdots, \mathfrak{A}_n]$	$[[\mathfrak{A}_1, \cdots, \mathfrak{A}_{n-1}], \mathfrak{A}_n]$, $n \geqq 3$.
$\mathfrak{X}^{\mathfrak{Y}}$	$\langle X^{Y} \mid X \in \mathfrak{X}, Y \in \mathfrak{Y} \rangle$. If $\mathfrak{X} \subseteqq \mathfrak{Y}$, $\mathfrak{X}^{\mathfrak{Y}}$ is called the normal closure of $\mathfrak{X}$ in $\mathfrak{Y}$.
$\mathfrak{X}'$	$[\mathfrak{X}, \mathfrak{X}]$, the commutator subgroup of $\mathfrak{X}$.
$C_n(\mathfrak{X})$	the nth term of the descending central series of $\mathfrak{X}$, defined inductively by: $C_1(\mathfrak{X}) = \mathfrak{X}$, $C_{n+1}(\mathfrak{X}) = [C_n(\mathfrak{X}), \mathfrak{X}]$.
$\Omega_n(\mathfrak{X})$	the subgroup of the p-group $\mathfrak{X}$ generated by the elements of order at most p^n.
$\mho^n(\mathfrak{X})$	the subgroup of the p-group $\mathfrak{X}$ generated by the p^nth powers of elements of $\mathfrak{X}$.
$m(\mathfrak{X})$	the minimal number of generators of $\mathfrak{X}$.
$m_p(\mathfrak{X})$	$m(\mathfrak{P})$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{X}$.
$\mathrm{cl}(\mathfrak{X})$	the class of nilpotency of the nilpotent group $\mathfrak{X}$, that is, the smallest integer n such that $\mathfrak{X} = Z_n(\mathfrak{X})$.
$C_{\mathfrak{B}}(\mathfrak{A})$	the largest subset of $\mathfrak{B}$ commuting elementwise with $\mathfrak{A}$, $\mathfrak{A}$ and $\mathfrak{B}$ being subsets of a group $\mathfrak{X}$. In case there is no danger of confusion, we set $C(\mathfrak{A}) = C_{\mathfrak{X}}(\mathfrak{A})$.
$N_{\mathfrak{B}}(\mathfrak{A})$	the largest subset of $\mathfrak{B}$ which normalizes $\mathfrak{A}$, $\mathfrak{A}$ and $\mathfrak{B}$ being subsets of a group $\mathfrak{X}$. In case there is no danger of confusion, we set $N(\mathfrak{A}) = N_{\mathfrak{X}}(\mathfrak{A})$.
$\ker(\mathfrak{X} \xrightarrow{\alpha} \mathfrak{Y})$	the kernel of the homomorphism α of the group $\mathfrak{X}$ into the group $\mathfrak{Y}$. α will often be suppressed.
$\mathrm{ccl}_{\mathfrak{X}}(\mathfrak{A})$	$\{\mathfrak{A}^{X} \mid X \in \mathfrak{X}\}$, $\mathfrak{A}$ being a subset of $\mathfrak{X}$.
$V(\mathrm{ccl}_{\mathfrak{X}}(\mathfrak{A}); \mathfrak{B})$	$\langle \mathfrak{A}^{X} \mid X \in \mathfrak{X}, \mathfrak{A}^{X} \subseteqq \mathfrak{B} \rangle$, the weak closure of $\mathrm{ccl}_{\mathfrak{X}}(\mathfrak{A})$ in $\mathfrak{B}$ with respect to the group $\mathfrak{X}$. Here $\mathfrak{A}$ and $\mathfrak{B}$ are subgroups of $\mathfrak{X}$. If $\mathfrak{A} = V(\mathrm{ccl}_{\mathfrak{X}}(\mathfrak{A}); \mathfrak{B})$, we say that $\mathfrak{A}$ is weakly closed in $\mathfrak{B}$ with respect to $\mathfrak{X}$.
$\pi(\mathfrak{X})$	the set of primes which divide $\lvert \mathfrak{X} \rvert$.
J_n	the n by n matrix with 1 in positions (i, i) and $(j, j+1)$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n-1$, zero elsewhere.

$SL(2, p)$	the group of 2 by 2 matrices of determinant one with coefficients in $GF(p)$, the field of p elements.
special p-group	an elementary abelian p-group, or a non abelian p-group whose center, commutator subgroup and Frattini subgroup coincide and are elementary.
extra special p-group	a non abelian special p-group whose center is of order p.
self centralizing subgroup of $\mathfrak{X}$	a subgroup $\mathfrak{A}$ of $\mathfrak{X}$ such that $\mathfrak{A} = C(\mathfrak{A})$. Notice that self centralizing subgroups are abelian.
self normalizing subgroup of $\mathfrak{X}$	a subgroup $\mathfrak{A}$ of $\mathfrak{X}$ such that $\mathfrak{A} = N(\mathfrak{A})$.
$\mathscr{SCN}(\mathfrak{X})$	the set of self centralizing normal subgroups of $\mathfrak{X}$.
$\mathscr{SCN}_m(\mathfrak{X})$	$\{\mathfrak{A} \mid \mathfrak{A} \in \mathscr{SCN}(\mathfrak{X}), m(\mathfrak{A}) \geqq m\}$.
$И_{\mathfrak{X}}(\mathfrak{A})$	the set of subgroups of $\mathfrak{X}$ which $\mathfrak{A}$ normalizes and which intersect $\mathfrak{A}$ in the identity only. In case there is no danger of confusion, we set $И_{\mathfrak{X}}(\mathfrak{A}) = И(\mathfrak{A})$. If $И(\mathfrak{A})$ contains only the identity subgroup, we say that $И(\mathfrak{A})$ is trivial.
$И_{\mathfrak{X}}(\mathfrak{A}; \pi)$	the π-subgroups in $И(\mathfrak{A})$.
section	if $\mathfrak{H}$ and $\mathfrak{K}$ are subgroups of the group $\mathfrak{X}$, and $\mathfrak{H} \lhd \mathfrak{K}$, then $\mathfrak{K}/\mathfrak{H}$ is called a section.
factor	if $\mathfrak{H}$ and $\mathfrak{K}$ are normal subgroups of $\mathfrak{X}$ and $\mathfrak{H} \subseteqq \mathfrak{K}$, then $\mathfrak{K}/\mathfrak{H}$ is called a factor of $\mathfrak{X}$.
chief factor	if $\mathfrak{K}/\mathfrak{H}$ is a factor of $\mathfrak{X}$ and a minimal normal subgroup of $\mathfrak{X}/\mathfrak{H}$, it is called a chief factor of $\mathfrak{X}$.

If $\mathfrak{H}/\mathfrak{K}$ and $\mathfrak{L}/\mathfrak{M}$ are sections of $\mathfrak{X}$, and if each coset of $\mathfrak{K}$ in $\mathfrak{H}$ has a non empty intersection with precisely one coset of $\mathfrak{M}$ in $\mathfrak{L}$ and each coset of $\mathfrak{M}$ in $\mathfrak{L}$ has a non empty intersection with precisely one coset of $\mathfrak{K}$ in $\mathfrak{H}$, then $\mathfrak{H}/\mathfrak{K}$ and $\mathfrak{L}/\mathfrak{M}$ are *incident sections*.

If $\mathfrak{H}/\mathfrak{K}$ is a section of $\mathfrak{X}$ and $\mathfrak{L}$ is a subgroup of $\mathfrak{X}$ which contains at least one element from each coset of $\mathfrak{K}$ in $\mathfrak{H}$, we say that $\mathfrak{L}$ *covers* $\mathfrak{H}/\mathfrak{K}$. We say that $\mathfrak{L}$ *dominates* the subgroup $\mathfrak{K}$ provided $\mathfrak{L}$ covers the section $N_{\mathfrak{X}}(\mathfrak{K})/C_{\mathfrak{X}}(\mathfrak{K})$. The idea to consider such objects stems from [17].

If $\mathfrak{F} = \mathfrak{H}/\mathfrak{K}$ is a factor of $\mathfrak{X}$, we let $C_{\mathfrak{X}}(\mathfrak{F})$ denote the kernel of the homomorphism of $\mathfrak{X}$ into Aut $\mathfrak{F}$ induced by conjugation. Similarly, we say that X in $\mathfrak{X}$ centralizes $\mathfrak{F}$ (or acts trivially on $\mathfrak{F}$) provided $X \in C(\mathfrak{F})$.

We say that $\mathfrak{X}$ has a *Sylow series* if $\mathfrak{X}$ possesses a unique $S_{p_1,\ldots,p_i}$-subgroup for each $i = 1, \cdots, n$, where $\pi(\mathfrak{X}) = \{p_1, \cdots, p_n\}$. The ordered

n-tuple $(p_1, \cdots, p_n)$ is called the *complexion* of the series [18].

A set of pairwise permutable Sylow subgroups of $\mathfrak{X}$, one for each prime dividing $|\mathfrak{X}|$, is called a *Sylow system* for $\mathfrak{X}$. This definition differs only superficially from that given in [16].

P. Hall [18] introduced and studied the following propositions:

E_π — $\mathfrak{X}$ contains at least one S_π-subgroup.

C_π — $\mathfrak{X}$ satisfies E_π, and any two S_π-subgroups of $\mathfrak{X}$ are conjugate in $\mathfrak{X}$.

D_π — $\mathfrak{X}$ satisfies C_π, and any π-subgroup of $\mathfrak{X}$ is contained in a S_π-subgroup of $\mathfrak{X}$.

E_π^* — $\mathfrak{X}$ contains a nilpotent S_π-subgroup.

In [19], P. Hall studied the *stability group* $\mathfrak{A}$ of the chain $\mathscr{C}: \mathfrak{X} = \mathfrak{X}_0 \supseteqq \mathfrak{X}_1 \supseteqq \cdots \supseteqq \mathfrak{X}_n = 1$, that is, the group of all automorphisms α of $\mathfrak{X}$ such that $(\mathfrak{X}_i X)^\alpha = \mathfrak{X}_i X$ for all X in $\mathfrak{X}_{i-1}$ and each $i = 1, \cdots, n$. If $\mathfrak{B}$ and $\mathfrak{X}$ are subgroups of a larger group, and if $\mathfrak{B}$ normalizes $\mathfrak{X}$, we say that $\mathfrak{B}$ *stabilizes* $\mathscr{C}$ provided $\mathfrak{B}/C_{\mathfrak{B}}(\mathfrak{X})$ is a subgroup of the stability group of $\mathscr{C}$.

By a *character* of $\mathfrak{X}$ we always mean a complex character of $\mathfrak{X}$ unless this is precluded by the context. A *linear character* is a character of degree one. An *integral linear combination* of characters is a linear combination of characters whose coefficients are rational integers. Such an integral linear combination is called a *generalized character*. If $\mathscr{S}$ is a collection of generalized characters of a group, let $\mathscr{I}(\mathscr{S})(\mathscr{C}(\mathscr{S}))$ be respectively the set of all integral (complex) linear combinations of elements in $\mathscr{S}$. Let $\mathscr{I}_0(\mathscr{S})$, $\mathscr{C}_0(\mathscr{S})$ be the subsets of $\mathscr{I}(\mathscr{S})$, $\mathscr{C}(\mathscr{S})$ respectively consisting of all elements α with $\alpha(1) = 0$.

If α and β are complex valued class functions on $\mathfrak{X}$, then the inner product and weight are denoted by

$$(\alpha, \beta)_{\mathfrak{X}} = \frac{1}{|\mathfrak{X}|} \sum_{X \in \mathfrak{X}} \alpha(X)\overline{\beta(X)}\,,$$

$$\|\alpha\|_{\mathfrak{X}}^2 = (\alpha, \alpha)_{\mathfrak{X}}\,.$$

The subscript $\mathfrak{X}$ is dropped in cases where it is clear from the context which group is involved.

The principal character of $\mathfrak{X}$ is denoted by $1_{\mathfrak{X}}$; the character of the regular representation of $\mathfrak{X}$ is denoted by $\rho_{\mathfrak{X}}$. If α is a complex valued class function of a subgroup $\mathfrak{H}$ of $\mathfrak{X}$, then α^* denotes the class function of $\mathfrak{X}$ induced by α.

The *kernel of a character* is the kernel of the representation with the given character.

A generalized character is *n-rational* if the field of its values is linearly disjoint from $\mathscr{Q}_n$.

A subset $\mathfrak{A}$ of the group $\mathfrak{X}$ is said to be a *trivial intersection set in* $\mathfrak{X}$, or a *T.I. set in* $\mathfrak{X}$ if and only if for every X in $\mathfrak{X}$, either

$$X^{-1}\mathfrak{A}X \cap \mathfrak{A} \subseteq \{1\}$$

or

$$X^{-1}\mathfrak{A}X = \mathfrak{A} .$$

If $\mathfrak{H}$ is a normal subgroup of the group $\mathfrak{X}$ and θ is a character of $\mathfrak{H}$, $\mathfrak{J}(\theta)$ denotes the *inertial group* of θ, that is

$$\mathfrak{J}(\theta) = \{X \mid X \in \mathfrak{X}, \theta(X^{-1}HX) = \theta(H) \quad \text{for all} \quad H \in \mathfrak{H}\} .$$

Clearly, $\mathfrak{H} \subseteq \mathfrak{J}(\theta)$ for all characters θ of $\mathfrak{H}$.

A group $\mathfrak{X}$ is a *Frobenius group* with *Frobenius kernel* $\mathfrak{H}$ if and only if $\mathfrak{H}$ is a proper normal subgroup of $\mathfrak{X}$ which contains the centralizer of every element in $\mathfrak{H}^{\#}$. It is well known (see 3.16) that the Frobenius kernel $\mathfrak{H}$ of $\mathfrak{X}$ is also characterized by the conditions

1. $\mathfrak{H} \lhd \mathfrak{X}, 1 \subset \mathfrak{H} \subset \mathfrak{X}$.
2. $\mathfrak{J}(\theta) = \mathfrak{H}$ for every non principal irreducible character θ of $\mathfrak{H}$.

We say that $\mathfrak{X}$ is of *Frobenius type* if and only if the following conditions are satisfied:

(i) If $\mathfrak{H}$ is the maximal normal nilpotent S-subgroup of $\mathfrak{X}$, then $1 \subset \mathfrak{H} \subset \mathfrak{X}$.

(ii) If $\mathfrak{E}$ is a complement for $\mathfrak{H}$ in $\mathfrak{X}$, then $\mathfrak{E}$ contains a normal abelian subgroup $\mathfrak{A}$ such that $\mathfrak{J}(\theta) \cap \mathfrak{E} \subseteq \mathfrak{A}$ for every non principal irreducible character θ of $\mathfrak{H}$.

(iii) $\mathfrak{E}$ contains a subgroup $\mathfrak{E}_0$ of the same exponent as $\mathfrak{E}$ such that $\mathfrak{E}_0\mathfrak{H}$ is a Frobenius group with Frobenius kernel $\mathfrak{H}$.

In case $\mathfrak{X}$ is of Frobenius type, the maximal normal nilpotent S-subgroup of $\mathfrak{X}$ will be called the *Frobenius kernel* of $\mathfrak{X}$.

A group $\mathfrak{S}$ is a *three step group* if and only if

(i) $\mathfrak{S} = \mathfrak{S}'\mathfrak{Q}^*$, where $\mathfrak{Q}^*$ is a cyclic S-subgroup of $\mathfrak{S}, \mathfrak{Q}^* \neq 1$, and $\mathfrak{S}' \cap \mathfrak{Q}^* = 1$.

(ii) $\mathfrak{S}$ contains a non cyclic normal S-subgroup $\mathfrak{H}$ such that $\mathfrak{S}'' \subseteq \mathfrak{H}C(\mathfrak{H}) \subseteq \mathfrak{S}'$, $\mathfrak{H}C(\mathfrak{H})$ is nilpotent and $\mathfrak{H}$ is the maximal normal nilpotent S-subgroup of $\mathfrak{S}$.

(iii) $\mathfrak{H}$ contains a cyclic subgroup $\mathfrak{H}^* \neq 1$ such that for Q in $\mathfrak{Q}^{*\#}$, $C_{\mathfrak{S}'}(Q) = \mathfrak{H}^*$.

3. Quoted Results

For convenience we single out various published results which are of use.

3.1. ([19] *Lemma* 1, *Three subgroups lemma*). *If* $\mathfrak{H}, \mathfrak{K}, \mathfrak{L}$ *are subgroups of the group* $\mathfrak{X}$ *and*

$$[\mathfrak{H}, \mathfrak{K}, \mathfrak{L}] = [\mathfrak{K}, \mathfrak{L}, \mathfrak{H}] = 1, \quad \textit{then} \ [\mathfrak{L}, \mathfrak{H}, \mathfrak{K}] = 1 .$$

3.2. [20] $F(\mathfrak{X}) = \cap C_{\mathfrak{X}}(\mathfrak{D})$, *the intersection being taken over all chief factors* $\mathfrak{D}$ *of the group* $\mathfrak{X}$.

3.3. [20] *If* $\mathfrak{X}$ *is solvable, then* $C(F(\mathfrak{X})) = Z(F(\mathfrak{X}))$.

3.4. *Let* p *be an odd prime and* $\mathfrak{X}$ *a* p*-group. If every normal abelian subgroup of* $\mathfrak{X}$ *is cyclic, then* $\mathfrak{X}$ *is cyclic. If every normal abelian subgroup of* $\mathfrak{X}$ *is generated by two elements, then* $\mathfrak{X}$ *is isomorphic to one of the following groups*:

(i) *a central product of a cyclic group and the non abelian group of order* p^3 *and exponent* p.

(ii) *a metacyclic group.*

(iii) $gp\langle A, B \mid [B, A] = C, [C, A] = B^{rp^{n-1}}, C^p = [B, C] = A^p = B^{p^n} = 1, n > 1, (r, p) = 1\rangle$.

(iv) *a* 3*-group.*

A proof of this result, together with a complete determination of the relevant 3-groups, can be found in the interesting papers [1] and [2].

3.5. [20] *If* $\mathfrak{X}$ *is a non abelian* p*-group,* p *is odd, and if every characteristic abelian subgroup of* $\mathfrak{X}$ *is cyclic, then* $\mathfrak{X}$ *is a central product of a cyclic group and an extra special group of exponent* p.

3.6. ([22] *Hilfssatz* 1.5). *If* σ *is a* p'*-automorphism of the* p*-group* $\mathfrak{X}$, p *is odd, and* σ *acts trivially on* $\Omega_1(\mathfrak{X})$, *then* $\sigma = 1$.

3.7. [20] *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are subgroups of a larger group, then* $[\mathfrak{A}, \mathfrak{B}] \lhd \langle\mathfrak{A}, \mathfrak{B}\rangle$.

3.8. *If the* S_p*-subgroup* $\mathfrak{P}$ *of the group* $\mathfrak{X}$ *is metacyclic, and if* p *is odd, then* $\mathfrak{P} \cap O^p(\mathfrak{X})$ *is abelian.*

This result is a consequence of ([23] Satz 1.5) and the well known fact that subgroups of metacyclic groups are metacyclic.

3.9. [28] *If* $\mathfrak{A}$ *is a normal abelian subgroup of the nilpotent group* $\mathfrak{X}$ *and* $\mathfrak{A}$ *is not a proper subgroup of any normal abelian subgroup of* $\mathfrak{X}$, *then* $\mathfrak{A}$ *is self centralizing.*

3.10. *If* $\mathfrak{P}$ *is a* S_p*-subgroup of the group* $\mathfrak{X}$, *and* $\mathfrak{A} \in \mathscr{SCN}(\mathfrak{P})$, *then* $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$ *where* $\mathfrak{D}$ *is a* p'*-group.* The proof of Lemma 5.7 in [27] is valid for all finite groups, and yields the preceding statement.

3.11. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be subgroups of a group* $\mathfrak{X}$, *where* $\mathfrak{A}$ *is a* p*-group and* $\mathfrak{B}$ *is a* p'*-group normalized by* $\mathfrak{A}$. *Suppose* $\mathfrak{A}_1$ *is a subgroup of* $\mathfrak{A}$ *which does not centralize* $\mathfrak{B}$. *If* $\mathfrak{B}_1$ *is a subgroup of* $\mathfrak{B}$ *of least order subject to being normalized by* $\mathfrak{A}$ *and not centralized by* $\mathfrak{A}_1$, *then* $\mathfrak{B}_1$ *is a special* q*-group for some prime* q, $\mathfrak{A}_1$ *acts trivially on* $D(\mathfrak{B}_1)$ *and* $\mathfrak{A}$ *acts irreducibly on* $\mathfrak{B}_1/D(\mathfrak{B}_1)$. This statement is a paraphrase of Theorem C of Hall and Higman [21].

3.12. ([3] *Lemma* 1). *Let* A *be a nonsingular matrix and let* σ *be a permutation of the elements of* A. *Suppose that* $\sigma(A)$ *can be derived from* A *by permuting the columns of* A *and* $\sigma(A)$ *can also be derived from* A *by permuting the rows of* A. *Then the number of rows left fixed by* σ *is equal to the number of columns left fixed by* σ.

The next two results follow from applying 3.12 to the character table of a group $\mathfrak{X}$.

3.13 (*Burnside*). *A group of odd order has no non principal real valued irreducible characters.*

3.14. *If* σ *is an automorphism of the group* $\mathfrak{X}$ *then the number of irreducible characters fixed by* σ *is equal to the number of conjugate classes fixed by* σ.

3.15. ([8] *Lemma* 2.1). *Let* $\mathfrak{P}$ *be a* p*-group for some prime* p *and let* θ *be an irreducible character of* $\mathfrak{P}$ *with* $\theta(1) > 1$. *Then* $\Sigma\theta_i(1)^2 \equiv 0 \pmod{\theta(1)^2}$, *where the summation ranges over all irreducible characters* θ_i *of* $\mathfrak{P}$ *with* $\theta_i(1) < \theta(1)$.

Let $\mathfrak{L}$ *be a Frobenius group with Frobenius kernel* $\mathfrak{H}$. *Then*

3.16. (i). ([7], [26]). $\mathfrak{H}$ *is a nilpotent* S*-subgroup of* $\mathfrak{L}$ *and* $\mathfrak{L}$ $\mathfrak{H}\mathfrak{E}$ *for some subgroup* $\mathfrak{E}$ *of* $\mathfrak{L}$ *with* $\mathfrak{H} \cap \mathfrak{E} = 1$.

3.16. (ii). ([4] *p.* 334). *If* p, q *are primes then every subgroup of* $\mathfrak{E}$ *of order* pq *is cyclic. If* $p \neq 2$ *then a* S_p*-subgroup of* $\mathfrak{E}$ *is cyclic.*

3.16. (iii). ([7] *Lemma* 2.1 *or* [10] *Lemma* 2.1). *A non principal irreducible character of* $\mathfrak{H}$ *induces an irreducible character of* $\mathfrak{L}$. *Furthermore every irreducible character of* $\mathfrak{L}$ *which does not have* $\mathfrak{H}$ *in its kernel is induced by a character of* $\mathfrak{H}$. *Thus in particular any complex representation of* $\mathfrak{L}$, *which does not have* $\mathfrak{H}$ *in its kernel,* *contains the regular representation of $\mathfrak{E}$ as a constituent when restricted to $\mathfrak{E}$.*

We will often use the fact that the last sentence of 3.16 (iii) is valid if "complex representation of $\mathfrak{L}$" is replaced by "representation of $\mathfrak{L}$ over a field of characteristic prime to $|\mathfrak{L}|$".

4. Elementary Results

LEMMA 4.1. *Let $\mathfrak{X}$ be a group with center $\mathfrak{Z}$ and let λ be an irreducible character of $\mathfrak{X}$. Then $\lambda(1)^2 \leqq |\mathfrak{X}:\mathfrak{Z}|$.*

Proof. For $Z \in \mathfrak{Z}$, $|\lambda(Z)| = \lambda(1)$. Therefore

$$|\mathfrak{X}| \geqq \Sigma_{\mathfrak{Z}} |\lambda(Z)|^2 = |\mathfrak{Z}| \lambda(1)^2$$

LEMMA 4.2. *Let α be a generalized character of the group $\mathfrak{X}$. Suppose that R, X are commuting elements of $\mathfrak{X}$ and the order of R is a power of a prime r. Let $\mathscr{F}$ be an algebraic number field which contains the $|\mathfrak{X}|$th roots of unity and let $\mathfrak{r}$ be a prime ideal in the ring of integers of $\mathscr{F}$ which divides r. Then*

$$\alpha(RX) \equiv \alpha(X) \pmod{\mathfrak{r}} .$$

Proof. It is clearly sufficient to prove the result for a generalized character, and thus for every irreducible character, of the abelian group $\langle R, X \rangle$. If α is an irreducible character of $\langle R, X\rangle$ then $\alpha(RX) = \alpha(R)\alpha(X)$ and $\alpha(R) \equiv 1 \pmod{\mathfrak{r}}$. This implies the required congruence.

LEMMA 4.3. *Let $\mathfrak{H}$ be a normal subgroup of the group $\mathfrak{X}$ and let λ be an irreducible character of $\mathfrak{X}$ which does not contain $\mathfrak{H}$ in its kernel. If $X \in \mathfrak{X}$ and $C(X) \cap \mathfrak{H} = \langle 1 \rangle$, then $\lambda(X) = 0$.*

Proof. Let $\mu_1, \mu_2, \cdots$ be all the irreducible characters of $\mathfrak{X}/\mathfrak{H} = \bar{\mathfrak{X}}$. Let $\lambda_1, \lambda_2, \cdots$ be all the remaining irreducible characters of $\mathfrak{X}$. If $C(X) \cap \mathfrak{H} = \langle 1 \rangle$, then $C(X)$ is mapped isomorphically into $C(\bar{X})$ where $\bar{X}$ is the image of X in $\bar{\mathfrak{X}}$. Consequently

$$\Sigma_i |\mu_i(X)|^2 = |C(\bar{X})| \geqq |C(X)| = \Sigma_i |\mu_i(X)|^2 + \Sigma_i |\lambda_i(X)|^2 .$$

This yields the required result.

Lemma 4.3 is of fundamental importance in this paper.

LEMMA 4.4. *Let $\mathfrak{H}$ be a normal subgroup of the group $\mathfrak{X}$. Assume that if θ is any nonprincipal irreducible character of $\mathfrak{H}$ then θ^* is* *a sum of irreducible characters of $\mathfrak{X}$, all of which have the same degree and occur with the same multiplicity in θ^*. For any integer d let ξ_d be the sum of all the irreducible characters of $\mathfrak{X}$ of degree d which do not have $\mathfrak{H}$ in their kernel. Then $\xi_d = a\gamma^*$, where a is a rational number and γ is a generalized character of $\mathfrak{H}$.*

Proof. Let $\theta_1^*, \theta_2^*, \cdots$ be all the distinct characters of $\mathfrak{X}$ which are induced by non principal irreducible characters of $\mathfrak{H}$ and which are sums of irreducible characters of $\mathfrak{X}$ of degree d. Suppose that $\theta_i^* = a_i \Sigma_j \lambda_{ij}$, where λ_{ij} is an irreducible character of $\mathfrak{X}$ for all values of j. It is easily seen that $\theta_1^*, \theta_2^*, \cdots$ form a set of pairwise orthogonal characters. Hence $\xi_d = \Sigma_i (1/a_i)\theta_i^*$. This proves the lemma.

If $\mathfrak{H}$ is a normal subgroup of the group $\mathfrak{X}$, $X \in \mathfrak{X}$, and φ is a character of $\mathfrak{H}$, then φ^X is defined by $\varphi^X(H) = \varphi(X^{-1}HX)$, $H \in \mathfrak{H}$.

LEMMA 4.5. *Let $\mathfrak{H}$ be a normal subgroup of the group $\mathfrak{X}$ and let θ be an irreducible character of $\mathfrak{H}$. Suppose $\mathfrak{X}$ contains a normal subgroup $\mathfrak{X}_0$ such that $\mathfrak{J}(\theta) \subseteq \mathfrak{X}_0$ and such that $\mathfrak{X}_0/\mathfrak{H}$ is abelian. Then θ^* is a sum of irreducible characters of $\mathfrak{X}$ which have the same degree and occur with the same multiplicity in θ^*. This common degree is a multiple of $|\mathfrak{X} : \mathfrak{J}(\theta)|$. If furthermore $\mathfrak{H}$ is a S-subgroup of $\mathfrak{X}_0$, then θ^* is a sum of $|\mathfrak{J}(\theta) : \mathfrak{H}|$ distinct irreducible characters of degree $|\mathfrak{X} : \mathfrak{J}(\theta)|\theta(1)$.*

Proof. Let θ_1 be the character of $\mathfrak{J}(\theta) = \mathfrak{J}$ induced by θ. Let λ be an irreducible constituent of θ_1 and let $\mu_1, \mu_2, \cdots, \mu_m$ be all the irreducible characters of $\mathfrak{J}/\mathfrak{H}$. Choose the notation so that $\lambda\mu_i = \lambda$ if and only if $1 \leqq i \leqq n$. Since $\theta_{1|\mathfrak{H}} = |\mathfrak{J} : \mathfrak{H}|\theta$, we get that $\lambda_{|\mathfrak{H}} = a\theta$ for some integer a. Thus,

$$\sum_{j=1}^{m} \lambda\mu_j = a\theta_1 . \tag{4.1}$$

Hence, every irreducible constituent of $a\theta_1$ is of the form $\lambda\mu_j$, so all irreducible constituents of θ_1 have the same degree. The characters $\mu_1, \mu_2, \cdots$ form a group $\mathfrak{M}$ which permutes the irreducible constituents of $a\theta_1$ transitively by multiplication. Hence for every value of j there are exactly n values of i such that $\lambda\mu_j\mu_i = \lambda\mu_j$. If now $\lambda_1, \lambda_2, \cdots$, are the distinct irreducible characters which are constituents of $a\theta_1$, then (4.1) implies that $a\theta_1 = n\Sigma\lambda_i$.

Suppose $\mathfrak{A}$ is a complement to $\mathfrak{H}$ in $\mathfrak{J}$, $\mathfrak{H}$ being a S-subgroup of $\mathfrak{J}$. We must show that θ_1 is a sum of $|\mathfrak{A}|$ distinct irreducible characters of $\mathfrak{J}$. For any subgroups $\mathfrak{R}_1, \mathfrak{R}$ of $\mathfrak{J}$ with $\mathfrak{H} \subseteq \mathfrak{R}_1 \subseteq \mathfrak{R}$, and any character φ of $\mathfrak{R}_1$, let $\varphi^{\mathfrak{R}}$ denote the character of $\mathfrak{R}$ induced by φ.

Suppose $\mathfrak{R}$ has the property that $\theta^{\mathfrak{R}}$ is a sum of $|\mathfrak{R} : \mathfrak{H}|$ distinct irreducible characters of $\mathfrak{K}$, where $\mathfrak{H} \subseteq \mathfrak{K} \subseteq \mathfrak{J}$. Let $\mathfrak{M}_{\mathfrak{K}}$ be the multiplicative group of linear characters of $\mathfrak{K}$ which have $\mathfrak{H}$ in their kernel, and let $\lambda_{\mathfrak{K}}$ be an irreducible constituent of $\theta^{\mathfrak{K}}$. Then $\lambda_{\mathfrak{K}}(1) = \theta(1)$ is prime to $|\mathfrak{A} \cap \mathfrak{K}|$, and it follows from Lemma 4.2 that $\lambda_{\mathfrak{K}}$ does not vanish on any element of $\mathfrak{A} \cap \mathfrak{K}$ of prime power order. This in turn implies that

$$\theta^{\mathfrak{K}} = \sum_{\mu \in \mathfrak{M}_{\mathfrak{K}}} \lambda_{\mathfrak{K}} \mu .$$

If $\mathfrak{K} = \mathfrak{J}$, we are done. Otherwise, let $\mathfrak{L}$ contain $\mathfrak{K}$ as a subgroup of prime index. It suffices to show that $\lambda_{\mathfrak{K}}^{\mathfrak{L}}$ is reducible, or equivalently, that $\lambda_{\mathfrak{K}}^{L} = \lambda_{\mathfrak{K}}$ for every L in $\mathfrak{L}$. This is immediate, since $(\theta^{\mathfrak{K}})^L = \theta^{\mathfrak{K}}$, so that $\lambda_{\mathfrak{K}}^{L} = \lambda_{\mathfrak{K}}\mu$ for some μ in $\mathfrak{M}_{\mathfrak{K}}$. Since $\mathfrak{A}$ is abelian, it follows that $\mu = 1$, as required.

To complete the proof of the lemma (now that the necessary properties of $\mathfrak{J}$ have been established), it suffices to show that if

$$\theta_1 = b\Sigma_i \lambda_i ,$$

where the λ_i are distinct irreducible characters of $\mathfrak{J}$, then each $\lambda_i^{\mathfrak{X}_0}$ is irreducible, and $\lambda_i^{\mathfrak{X}_0} \neq \lambda_j^{\mathfrak{X}_0}$ for $\lambda_i \neq \lambda_j$. For if this is proved, the normality of $\mathfrak{X}_0$ in $\mathfrak{X}$ implies the lemma. The definition of $\mathfrak{J}$ implies that $\lambda_{i|\mathfrak{J}}^{\mathfrak{X}_0}$ is a sum of $|\mathfrak{X}_0 : \mathfrak{J}|$ distinct irreducible characters of $\mathfrak{J}$. Furthermore, λ_i is the only irreducible constituent of $\lambda_{i|\mathfrak{J}}^{\mathfrak{X}_0}$ whose restriction to $\mathfrak{H}$ is not orthogonal to θ. Thus, if $\lambda_i^{\mathfrak{X}_0} = \lambda_j^{\mathfrak{X}_0}$, then $\lambda_i = \lambda_j$. Since $\lambda_i^{\mathfrak{X}_0}$ vanishes outside $\mathfrak{J}$, a simple computation yields that $\|\lambda_i^{\mathfrak{X}_0}\|^2 = 1$. Therefore $\lambda_i^{\mathfrak{X}_0}$ is irreducible. The proof is complete.

LEMMA 4.6. *Let p be an odd prime and let $\mathfrak{P}$ be a normal S_p-subgroup of the group $\mathfrak{P}\mathfrak{H}\mathfrak{E}$. Assume that $\mathfrak{H}\mathfrak{E}$ is a Frobenius group with Frobenius kernel $\mathfrak{H}$, $\mathfrak{H}\mathfrak{E}$ is a p'-group and $\mathfrak{H} \cap \mathfrak{E} = 1$.*

(i) *If $C_{\mathfrak{P}}(\mathfrak{E}) = 1$, then $\mathfrak{H} \subseteq C(\mathfrak{P})$.*

(ii) *If $C_{\mathfrak{P}}(E)$ is cyclic for all elements $E \in \mathfrak{E}^{\#}$, then $|\mathfrak{E}|$ is a prime or $\mathfrak{H} \subseteq C(\mathfrak{P})$.*

(iii) *If $1 \neq C_{\mathfrak{P}}(\mathfrak{H}) \subseteq C_{\mathfrak{P}}(\mathfrak{E})$, then either $\mathfrak{P}$ is cyclic or $C_{\mathfrak{P}}(\mathfrak{E})$ is not cyclic.*

Proof. $\mathfrak{H}\mathfrak{E}$ is represented on $\mathfrak{P}/D(\mathfrak{P})$. Suppose that $\mathfrak{H} \not\subseteq C(\mathfrak{P})$. By 3.16 (iii) $\mathfrak{E}$ has a fixed point on $\mathfrak{P}/D(\mathfrak{P})$, and thus on $\mathfrak{P}$. This proves (i). If $|\mathfrak{E}|$ is not a prime, let $1 \subset \mathfrak{E}_0 \subset \mathfrak{E}$. Then 3.16 (iii) implies that $\mathfrak{E}_0$ has a non-cyclic fixed point set on $\mathfrak{P}/D(\mathfrak{P})$, and thus on $\mathfrak{P}$. This proves (ii).

As for (iii), let k be the largest integer such that $\mathfrak{H}$ has a non trivial fixed point on $Z_k(\mathfrak{P})/Z_{k-1}(\mathfrak{P})$. It follows that $\mathfrak{H}$ has a non trivial fixed point on $Z_k(\mathfrak{P})/D(Z_k(\mathfrak{P}))$. If $Z_k(\mathfrak{P})$ is not cyclic then since $\mathfrak{H}\mathfrak{E}$ is completely reducible on $Z_k(\mathfrak{P})/D(Z_k(\mathfrak{P}))$ (i) implies (iii) by 3.16 (iii). Suppose that $Z_k(\mathfrak{P})$ is cyclic. If $k \geqq 2$, then by [10] Lemma 1.4, $\mathfrak{P}$ is cyclic. Since $Z_2(\mathfrak{P})$ is of class 1 or 2, $\Omega_1(Z_2(\mathfrak{P}))$ is of exponent p. As $Z_2(\mathfrak{P})$ is not cyclic neither is $\Omega_1(Z_2(\mathfrak{P}))$. Thus it may be assumed that $\mathfrak{P} = \Omega_1(Z_2(\mathfrak{P}))$ is non cyclic of exponent p and class at most 2. If $\mathfrak{P}$ is abelian then (iii) follows from (i). If $\mathfrak{P}$ is of class 2 then by (i) $\mathfrak{E}$ has a fixed point on $\mathfrak{P}/\mathfrak{P}'$ and on $\mathfrak{P}'$. As $\mathfrak{P}$ has exponent p this implies that $C_{\mathfrak{P}}(\mathfrak{E})$ is not cyclic as required.

5. Numerical Results

In this section we state some elementary number theoretical results and some inequalities. The inequalities can all be proved by the methods of elementary calculus and their proof is left to the reader.

LEMMA 5.1. *If p, q are primes and*

$$p \equiv 1 \pmod q , \qquad q^3 \equiv 1 \pmod p$$

then $p = 1 + q + q^2$.

Proof. Let $p = 1 + nq$. Since $p > q$, $q \not\equiv 1 \pmod p$. Hence

$$1 + q + q^2 = mp .$$

Reading $(\mathrm{mod}\ q)$ yields $m = 1 + rq$. Therefore

$$1 + q + q^2 = 1 + (r + n)q + rnq^2 .$$

If $r \neq 0$ then the right hand side of the previous equation is strictly larger than the left hand side. Thus $r = 0$ as required.

The first statement of the following lemma is proved in [5]. The second can be proved in a similar manner.

LEMMA 5.2. *Let p, q be odd primes and let $n \geqq 1$.*
(i) *If q^m divides $(p^n - 1)(p^{n-1} - 1) \cdots (p - 1)$ then $q^m < p^n$.*
(ii) *If q^m divides $(p^{2n} - 1)(p^{2(n-1)} - 1) \cdots (p^2 - 1)$ then $q^m < p^{2n-1}$.*

If $x \geqq 5$, then

$$3^{x-2} > x^2 , \tag{5.1}$$

$$5^{x-1} > 80x , \tag{5.2}$$

$$3^x x > 20(2x^2 + 1) . \tag{5.3}$$

If $x \geqq 7$, then

$$3^{x-2} > 2x^2 , \tag{5.4}$$

(5.5) $$3^x - 3 > 28x^2\,,$$

(5.6) $$7^x > 4x^2 \cdot 3^x + 1\,.$$

(5.7) $$5^x > 4x^2 3^x + 1 \quad \text{for } x \geqq 13\,.$$

(5.8) $$(x^y - 1) - (x - 1)y - \frac{(x-1)^3}{4} > 0 \quad \text{for } x, y \geqq 3\,.$$

(5.9) $$x^y - 1 > 4y^2 \quad \text{for } x \geqq 3,\ y \geqq 5, \text{ or } x \geqq 5, y \geqq 3\,.$$

(5.10) $$x^{y-2} > y^2 \quad \text{for } x \geqq 3,\ y \geqq 5 \text{ or } x \geqq 10,\ y \geqq 3\,.$$

(5.11) $$\frac{y^x - 1}{y - 1} > \frac{x^y - 1}{x - 1} \quad \text{for } x > y \geqq 3\,.$$

(5.12) $$y^2\frac{(y^{x-1} - 1)}{y - 1} > x^2\frac{(x^{y-1} - 1)}{x - 1} \quad \text{for } x > y \geqq 3\,.$$

CHAPTER II

6. Preliminary Lemmas of Lie Type

Hypothesis 6.1.

(i) *p is a prime, $\mathfrak{P}$ is a normal S_p-subgroup of $\mathfrak{P}\mathfrak{U}$, and $\mathfrak{U}$ is a non identity cyclic p'-group.*

(ii) $C_{\mathfrak{U}}(\mathfrak{P}) = 1$.

(iii) *$\mathfrak{P}'$ is elementary abelian and $\mathfrak{P}' \subseteq \boldsymbol{Z}(\mathfrak{P})$.*

(iv) *$|\mathfrak{P}\mathfrak{U}|$ is odd.*

Let $\mathfrak{U} = \langle U \rangle$, $|\mathfrak{U}| = u$, and $|\mathfrak{P} : \boldsymbol{D}(\mathfrak{P})| = p^n$. Let $\mathscr{L}$ be the Lie ring associated to $\mathfrak{P}$ ([12] p. 328). Then $\mathscr{L} = \mathscr{L}_1^* \oplus \mathscr{L}_2$ where $\mathscr{L}_1^*$ and $\mathscr{L}_2$ correspond to $\mathfrak{P}/\mathfrak{P}'$ and $\mathfrak{P}'$ respectively. Let $\mathscr{L}_1 = \mathscr{L}_1^*/p\mathscr{L}_1^*$. For $i = 1, 2$, let U_i be the linear transformation induced by U on $\mathscr{L}_i$.

LEMMA 6.1. *Assume that Hypothesis* 6.1 *is satisfied. Let $\varepsilon_1, \cdots, \varepsilon_n$ be the characteristic roots of U_1. Then the characteristic roots of U_2 are found among the elements $\varepsilon_i\varepsilon_j$ with $1 \leq i < j \leq n$.*

Proof. Suppose the field is extended so as to include $\varepsilon_1, \cdots, \varepsilon_n$. Since $\mathfrak{U}$ is a p'-group, it is possible to find a basis $x_1, \cdots, x_n$ of $\mathscr{L}_1$ such that $x_i U_1 = \varepsilon_i x_i$, $1 \leq i \leq n$. Therefore, $x_i U_1 \cdot x_j U_1 = \varepsilon_i \varepsilon_j x_i \cdot x_j$. As U induces an automorphism of $\mathscr{L}$, this yields that

$$(x_i \cdot x_j) U_2 = x_i U_1 \cdot x_j U_1 = \varepsilon_i \varepsilon_j x_i \cdot x_j \ .$$

Since the vectors $x_i \cdot x_j$ with $i < j$ span $\mathscr{L}_2$, the lemma follows.

By using a method which differs from that used below, M. Hall proved a variant of Lemma 6.2. We are indebted to him for showing us his proof.

LEMMA 6.2. *Assume that Hypothesis* 6.1 *is satisfied, and that U_1 acts irreducibly on $\mathscr{L}_1$. Assume further that $n = q$ is an odd prime and that U_1 and U_2 have the same characteristic polynomial. Then $q > 3$ and*

$$u < 3^{q/2}$$

Proof. Let ε^{p^i} be the characteristic roots of U_1, $0 \leq i < n$. By Lemma 6.1 there exist integers i, j, k such that $\varepsilon^{p^i}\varepsilon^{p^j} = \varepsilon^{p^k}$. Raising this equation to a suitable power yields the existence of integers a and b with $0 \leq a < b < q$ such that $\varepsilon^{p^a + p^b - 1} = 1$. By Hypothesis 6.1 (ii), the preceding equality implies $p^a + p^b - 1 \equiv 0 (\text{mod } u)$. Since U_1 acts irreducibly, we also have $p^q - 1 \equiv 0 (\text{mod } u)$. Since $\mathfrak{U}$ is a p'-group,

$ab \neq 0$. Consequently,

$$\begin{aligned} &p^a + p^b - 1 \equiv 0 \pmod{u}\,, \\ &p^q - 1 \equiv 0 \pmod{u},\ 0 < a < b < q\,. \end{aligned} \tag{6.1}$$

Let d be the resultant of the polynomials $f = x^a + x^b - 1$ and $g = x^q - 1$. Since q is a prime, the two polynomials are relatively prime, so d is a nonzero integer. Also, by a basic property of resultants,

$$d = hf + kg \tag{6.2}$$

for suitable integral polynomials h and k.

Let ε_q be a primitive qth root of unity over $\mathscr{Q}$, so that we also have

$$\begin{aligned} d^2 &= \prod_{i=0}^{q-1} (\varepsilon_q^{ia} + \varepsilon_q^{ib} - 1) \prod_{i=0}^{q-1} (\varepsilon_q^{-ia} + \varepsilon_q^{-ib} - 1) \\ &= \prod_{i=0}^{q-1} \{3 + \varepsilon_q^{i(a-b)} + \varepsilon_q^{i(b-a)} - \varepsilon_q^{ia} - \varepsilon_q^{ib} - \varepsilon_q^{-ib} - \varepsilon_q^{-ia}\}\,. \end{aligned} \tag{6.3}$$

For $q = 3$, this yields that $d^2 = (3 - 1 + 1 + 1)^2 = 4^2$, so that $d = \pm 4$. Since u is odd (6.1) and (6.2) imply that $u = 1$. This is not the case, so $q > 3$.

Each term on the right hand side of (6.3) is non negative. As the geometric mean of non negative numbers is at most the arithmetic mean, (6.3) implies that

$$d^{2/q} \leqq \frac{1}{q} \sum_{i=0}^{q-1} \{3 + \varepsilon_q^{i(a-b)} + \varepsilon_q^{i(b-a)} - \varepsilon_q^{ia} - \varepsilon_q^{-ia} - \varepsilon_q^{ib} - \varepsilon_q^{-ib}\}\,.$$

The algebraic trace of a primitive qth root of unity is -1, hence

$$d^{2/q} \leqq 3\,.$$

Now (6.1) and (6.2) imply that

$$u \leqq |d| \leqq 3^{q/2}\,.$$

Since $3^{q/2}$ is irrational, equality cannot hold.

LEMMA 6.3. *If $\mathfrak{P}$ is a p-group and $\mathfrak{P}' = D(\mathfrak{P})$, then $C_n(\mathfrak{P})/C_{n+1}(\mathfrak{P})$ is elementary abelian for all n.*

Proof. The assertion follows from the congruence

$$[A_1, \cdots, A_n]^p \equiv [A_1, \cdots, A_{n-1}, A_n^p] \pmod{C_{n+1}(\mathfrak{P})}\,,$$

valid for all $A_1, \cdots, A_n$ in $\mathfrak{P}$.

LEMMA 6.4. *Suppose that σ is a fixed point free p'-automorphism* *of the p-group* $\mathfrak{P}$, $\mathfrak{P}' = D(\mathfrak{P})$ *and* $A^\sigma \equiv A^x \pmod{\mathfrak{P}'}$ *for some integer* x *independent of* A. *Then* $\mathfrak{P}$ *is of exponent* p.

Proof. Let $A^\sigma = A^x \cdot A^\phi$ so that A^ϕ is in $\mathfrak{P}'$ for all A in $\mathfrak{P}$. Then

$$\begin{aligned}[A_1, \cdots, A_n]^\sigma &= [A_1^\sigma, \cdots, A_n^\sigma] = [A_1^x \cdot A_1^\phi, \cdots, A_n^x \cdot A_n^\phi] \\ &\equiv [A_1^x, \cdots, A_n^x] \equiv [A_1, \cdots, A_n]^{x^n} \pmod{C_{n+1}(\mathfrak{P}))} .\end{aligned}$$

Since σ is regular on $\mathfrak{P}$, σ is also regular on each C_n/C_{n+1}. As the order of σ divides $p-1$ the above congruences now imply that $\mathrm{cl}(\mathfrak{P}) \leqq p-1$ and so $\mathfrak{P}$ is a regular p-group. If $\mho^1(\mathfrak{P}) \neq 1$, then the mapping $A \longrightarrow A^p$ induces a non zero linear map of $\mathfrak{P}/D(\mathfrak{P})$ to $C_n(\mathfrak{P})/C_{n+1}(\mathfrak{P})$ for suitable n. Namely, choose n so that $\mho^1(\mathfrak{P}) \subseteqq C_n(\mathfrak{P})$ but $\mho^1(\mathfrak{P}) \nsubseteqq C_{n+1}(\mathfrak{P})$, and use the regularity of $\mathfrak{P}$ to guarantee linearity. Notice that $n \geqq 2$, since by hypothesis $\mho^1(\mathfrak{P}) \subseteqq \mathfrak{P}'$. We find that $x \equiv x^n \pmod{p}$, and so $x^{n-1} \equiv 1 \pmod{p}$ and σ has a fixed point on C_{n-1}/C_n, contrary to assumption. Hence, $\mho^1(\mathfrak{P}) = 1$.

7. Preliminary Lemmas of Hall-Higman Type

Theorem B of Hall and Higman [21] is used frequently and will be referred to as (B).

LEMMA 7.1. *If* $\mathfrak{X}$ *is a p-solvable linear group of odd order over a field of characteristic p, then* $O_p(\mathfrak{X})$ *contains every element whose minimal polynomial is* $(x-1)^2$.

Proof. Let $\mathscr{V}$ be the space on which $\mathfrak{X}$ acts. The hypotheses of the lemma, together with (B), guarantee that either $O_p(\mathfrak{X}) \neq 1$ or $\mathfrak{X}$ contains no element whose minimal polynomial is $(x-1)^2$.

Let X be an element of $\mathfrak{X}$ with minimal polynomial $(x-1)^2$. Then $O_p(\mathfrak{X}) \neq 1$, and the subspace $\mathscr{V}_0$ which is elementwise fixed by $O_p(\mathfrak{X})$ is proper and is $\mathfrak{X}$-invariant. Since $O_p(\mathfrak{X})$ is a p-group, $\mathscr{V}_0 \neq 0$. Let

$$\mathfrak{K}_0 = \ker(\mathfrak{X} \longrightarrow \mathrm{Aut}\ \mathscr{V}_0) , \qquad \mathfrak{K}_1 = \ker(\mathfrak{X} \longrightarrow \mathrm{Aut}(\mathscr{V}/\mathscr{V}_0)) .$$

By induction on $\dim \mathscr{V}$, $X \in O_p(\mathfrak{X} \bmod \mathfrak{K}_i)$, $i = 0, 1$. Since

$$O_p(\mathfrak{X} \bmod \mathfrak{K}_0) \cap O_p(\mathfrak{X} \bmod \mathfrak{K}_1)$$

is a p-group, the lemma follows.

LEMMA 7.2. *Let* $\mathfrak{X}$ *be a p-solvable group of odd order, and* $\mathfrak{A}$ *a p-subgroup of* $\mathfrak{X}$. *Any one of the following conditions guarantees that* $\mathfrak{A} \subseteqq O_{p',p}(\mathfrak{X})$:

1. $\mathfrak{A}$ *is abelian and* $|\mathfrak{X}: N(\mathfrak{A})|$ *is prime to* p.
2. $p \geqq 5$ *and* $[\mathfrak{P}, \mathfrak{A}, \mathfrak{A}, \mathfrak{A}, \mathfrak{A}] = 1$ *for some* S_p*-subgroup* $\mathfrak{P}$ *of* $\mathfrak{X}$.
3. $[\mathfrak{P}, \mathfrak{A}, \mathfrak{A}] = 1$ *for some* S_p*-subgroup* $\mathfrak{P}$ *of* $\mathfrak{X}$.
4. $\mathfrak{A}$ *acts trivially on the factor* $O_{p',p,p'}(\mathfrak{X})/O_{p',p}(\mathfrak{X})$.

Proof. Conditions 1, 2, or 3 imply that each element of $\mathfrak{A}$ has a minimal polynomial dividing $(x-1)^{p-1}$ on $O_{p',p}(\mathfrak{X})/\mathfrak{D}$, where $\mathfrak{D} = D(O_{p',p}(\mathfrak{X}) \bmod O_{p'}(\mathfrak{X}))$. Thus (B) and the oddness of $|\mathfrak{X}|$ yield 1, 2, and 3. Lemma 1.2.3 of [21] implies 4.

LEMMA 7.3. *If* $\mathfrak{X}$ *is* p*-solvable, and* $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{X}$, *then* $\mathsf{M}(\mathfrak{P})$ *is a lattice whose maximal element is* $O_{p'}(\mathfrak{X})$.

Proof. Since $O_{p'}(\mathfrak{X}) \lhd \mathfrak{X}$ and $\mathfrak{P} \cap O_{p'}(\mathfrak{X}) = 1$, $O_{p'}(\mathfrak{X})$ is in $\mathsf{M}(\mathfrak{P})$. Thus it suffices to show that if $\mathfrak{H} \in \mathsf{M}(\mathfrak{P})$, then $\mathfrak{H} \subseteq O_{p'}(\mathfrak{X})$. Since $\mathfrak{P}\mathfrak{H}$ is a group of order $|\mathfrak{P}|\cdot|\mathfrak{H}|$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{X}$, $\mathfrak{H}$ is a p'-group, as is $\mathfrak{H}O_{p'}(\mathfrak{X})$. In proving the lemma, we can therefore assume that $O_{p'}(\mathfrak{X}) = 1$, and try to show that $\mathfrak{H} = 1$. In this case, $\mathfrak{H}$ is faithfully represented as automorphisms of $O_p(\mathfrak{X})$, by Lemma 1.2.3 of [21]. Since $O_p(\mathfrak{X}) \subseteq \mathfrak{P}$, we see that $[\mathfrak{H}, O_p(\mathfrak{X})] \subseteq \mathfrak{H} \cap \mathfrak{P}$, and $\mathfrak{H} = 1$ follows.

LEMMA 7.4. *Suppose* $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{X}$ *and* $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}(\mathfrak{P})$. *Then* $\mathsf{M}(\mathfrak{A})$ *contains only* p'*-groups. If in addition,* $\mathfrak{X}$ *is* p*-solvable, then* $\mathsf{M}(\mathfrak{A})$ *is a lattice whose maximal element is* $O_{p'}(\mathfrak{X})$.

Proof. Suppose $\mathfrak{A}$ normalizes $\mathfrak{H}$ and $\mathfrak{A} \cap \mathfrak{H} = \langle 1 \rangle$. Let $\mathfrak{A}^*$ be a S_p-subgroup of $\mathfrak{A}\mathfrak{H}$ containing $\mathfrak{A}$. By Sylow's theorem, $\mathfrak{P}_1 = \mathfrak{A}^* \cap \mathfrak{H}$ is a S_p-subgroup of $\mathfrak{H}$. It is clearly normalized by $\mathfrak{A}$, and $\mathfrak{A} \cap \mathfrak{P}_1 = \langle 1 \rangle$. If $\mathfrak{P}_1 \neq \langle 1 \rangle$, a basic property of p-groups implies that $\mathfrak{A}$ centralizes some non identity element of $\mathfrak{P}_1$, contrary to 3.10. Thus, $\mathfrak{P}_1 = \langle 1 \rangle$ and $\mathfrak{H}$ is a p'-group. Hence we can assume that $\mathfrak{X}$ is p-solvable and that $O_{p'}(\mathfrak{X}) = \langle 1 \rangle$ and try to show that $\mathfrak{H} = \langle 1 \rangle$.

Let $\mathfrak{X}_1 = O_p(\mathfrak{X})\mathfrak{H}\mathfrak{A}$. Then $O_p(\mathfrak{X})\mathfrak{A}$ is a S_p-subgroup of $\mathfrak{X}_1$, and $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}(O_p(\mathfrak{X})\mathfrak{A})$. If $\mathfrak{X}_1 \subset \mathfrak{X}$, then by induction $\mathfrak{H} \subseteq O_{p'}(\mathfrak{X}_1)$ and so $[O_p(\mathfrak{X}), \mathfrak{H}] \subseteq O_p(\mathfrak{X}) \cap O_{p'}(\mathfrak{X}_1) = 1$ and $\mathfrak{H} = 1$. We can suppose that $\mathfrak{X}_1 = \mathfrak{X}$.

If $\mathfrak{A}$ centralizes $\mathfrak{H}$, then clearly $\mathfrak{A} \lhd \mathfrak{X}$, and so $\ker(\mathfrak{X} \longrightarrow \operatorname{Aut}\mathfrak{A}) = \mathfrak{A} \times \mathfrak{H}_1$, by 3.10 where $\mathfrak{H} \subseteq \mathfrak{H}_1$. Hence, $\mathfrak{H}_1$ char $\mathfrak{A} \times \mathfrak{H}_1 \lhd \mathfrak{X}$, and $\mathfrak{H}_1 \lhd \mathfrak{X}$, so that $\mathfrak{H}_1 = 1$. We suppose that $\mathfrak{A}$ does not centralize $\mathfrak{H}$, and that $\mathfrak{H}$ is an elementary q-group on which $\mathfrak{A}$ acts irreducibly. Let $\mathfrak{B} = O_p(\mathfrak{X})/D(O_p(\mathfrak{X})) = \mathfrak{B}_1 \times \mathfrak{B}_2$, where $\mathfrak{B}_1 = C_{\mathfrak{B}}(\mathfrak{H})$ and $\mathfrak{B}_2 = [\mathfrak{B}, \mathfrak{H}]$. Let $V \in \mathfrak{B}_2$ and $X \in V$, so that $[X, \mathfrak{A}] \subseteq \mathfrak{A}$. Hence, $[X, \mathfrak{A}]$ maps into $\mathfrak{B}_1$, since $[[X, \mathfrak{A}], \mathfrak{H}] \subseteq \mathfrak{H} \cap O_p(\mathfrak{X}) = 1$. But $\mathfrak{B}_2$ is $\mathfrak{X}$-invariant, so $[X, \mathfrak{A}]$ maps into $\mathfrak{B}_1 \cap \mathfrak{B}_2 = 1$. Thus, $\mathfrak{A} \subseteq \ker(\mathfrak{X} \longrightarrow \operatorname{Aut}\mathfrak{B}_2)$, and so $[\mathfrak{A}, \mathfrak{H}]$

centralizes $\mathfrak{V}_2$. As $\mathfrak{A}$ acts irreducibly on $\mathfrak{H}$, we have $\mathfrak{H} = [\mathfrak{H}, \mathfrak{A}]$, so $\mathfrak{V}_2 = 1$. Thus, $\mathfrak{H}$ centralizes $\mathfrak{V}$ and so centralizes $O_p(\mathfrak{X})$, so $\mathfrak{H} = 1$, as required.

LEMMA 7.5. *Suppose $\mathfrak{H}$ and $\mathfrak{H}_1$ are $S_{p,q}$-subgroups of the solvable group $\mathfrak{S}$. If $\mathfrak{B} \subseteq O_p(\mathfrak{H}_1) \cap \mathfrak{H}$, then $\mathfrak{B} \subseteq O_p(\mathfrak{H})$.*

Proof. We proceed by induction on $|\mathfrak{S}|$. We can suppose that $\mathfrak{S}$ has no non identity normal subgroup of order prime to pq. Suppose that $\mathfrak{S}$ possesses a non identity normal p-subgroup $\mathfrak{J}$. Then

$$\mathfrak{J} \subseteq O_p(\mathfrak{H}) \cap O_p(\mathfrak{H}_1) .$$

Let $\bar{\mathfrak{S}} = \mathfrak{S}/\mathfrak{J}$, $\bar{\mathfrak{B}} = \mathfrak{B}\mathfrak{J}/\mathfrak{J}$, $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{J}$, $\bar{\mathfrak{H}}_1 = \mathfrak{H}_1/\mathfrak{J}$. By induction, $\bar{\mathfrak{B}} \subseteq O_p(\bar{\mathfrak{H}})$, so $\mathfrak{B} \subseteq O_p(\mathfrak{H} \bmod \mathfrak{J}) = O_p(\mathfrak{H})$, and we are done. Hence, we can assume that $O_p(\mathfrak{S}) = \langle 1 \rangle$. In this case, $F(\mathfrak{S})$ is a q-group, and $F(\mathfrak{S}) \subseteq \mathfrak{H}_1$. By hypothesis, $\mathfrak{B} \subseteq O_p(\mathfrak{H}_1)$, and so $\mathfrak{B}$ centralizes $F(\mathfrak{S})$. By 3.3, we see that $\mathfrak{B} = \langle 1 \rangle$, so $\mathfrak{B} \subseteq O_p(\mathfrak{H})$ as desired.

The next two lemmas deal with a S_p-subgroup $\mathfrak{P}$ of the p-solvable group $\mathfrak{X}$ and with the set

$\mathscr{S} = \{\mathfrak{H} \mid$ 1. $\mathfrak{H}$ is a subgroup of $\mathfrak{X}$.
2. $\mathfrak{P} \subseteq \mathfrak{H}$.
3. The p-length of $\mathfrak{H}$ is at most two .
4. $|\mathfrak{H}|$ is not divisible by three distinct primes .$\}$

LEMMA 7.6. $\mathfrak{X} = \langle \mathfrak{H} \mid \mathfrak{H} \in \mathscr{S} \rangle$.

Proof. Let $\mathfrak{X}_1 = \langle \mathfrak{H} \mid \mathfrak{H} \in \mathscr{S} \rangle$. It suffices to show that $|\mathfrak{X}_1|_q = |\mathfrak{X}|_q$ for every prime q. This is clear if $q = p$, so suppose $q \neq p$. Since $\mathfrak{X}$ is p-solvable, $\mathfrak{X}$ satisfies $E_{p\,q}$, so we can suppose that $\mathfrak{X}$ is a p, q-group. By induction, we can suppose that $\mathfrak{X}_1$ contains every proper subgroup of $\mathfrak{X}$ which contains $\mathfrak{P}$. Since $\mathfrak{P}O_q(\mathfrak{X}) \in \mathscr{S}$, we see that $O_q(\mathfrak{X}) \subseteq \mathfrak{X}_1$. If $N(\mathfrak{P} \cap O_{q\,p}(\mathfrak{X})) \subset \mathfrak{X}$, then $N(\mathfrak{P} \cap O_p(\mathfrak{X})) \subseteq \mathfrak{X}_1$. Since $\mathfrak{X} = O_q(\mathfrak{X}) \cdot N(\mathfrak{P} \cap O_{q\,p}(\mathfrak{X}))$, we have $\mathfrak{X} = \mathfrak{X}_1$. Thus, we can assume that $O_p(\mathfrak{X}) = \mathfrak{P} \cap O_{q\,p}(\mathfrak{X})$. Since $\mathfrak{P}O_{p,q}(\mathfrak{X}) \in \mathscr{S}$, we see that $O_{p,q}(\mathfrak{X}) \subseteq \mathfrak{X}_1$. If $\mathfrak{P}O_{p\,q}(\mathfrak{X}) = \mathfrak{X}$, we are done, so suppose not. Then $N(\mathfrak{P} \cap O_{p,q\,p}(\mathfrak{X})) \subset \mathfrak{X}$, so that $\mathfrak{X}_1$ contains $N(\mathfrak{P} \cap O_{p,q\,p}(\mathfrak{X}))O_{p\,q}(\mathfrak{X}) = \mathfrak{X}$, as required.

LEMMA 7.7. *Suppose $\mathfrak{M}, \mathfrak{N}$ are subgroups of $\mathfrak{X}$ which contain $\mathfrak{P}$ such that $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{M})(\mathfrak{H} \cap \mathfrak{N})$ for all $\mathfrak{H}$ in $\mathscr{S}$. Then $\mathfrak{X} = \mathfrak{M}\mathfrak{N}$.*

Proof. It suffices to show that $|\mathfrak{M}\mathfrak{N}|_q \geqq |\mathfrak{X}|_q$ for every prime q. This is clear if $q = p$, so suppose $q \neq p$. Let $\mathfrak{Q}_1$ be a S_q-subgroup of

$\mathfrak{M} \cap \mathfrak{N}$ permutable with $\mathfrak{P}$, which exists by $E_{p,q}$ in $\mathfrak{M} \cap \mathfrak{N}$. Since $\mathfrak{X}$ satisfies $D_{p,q}$, there is a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{X}$ which contains $\mathfrak{Q}_1$ and is permutable with $\mathfrak{P}$, Set $\mathfrak{R} = \mathfrak{P}\mathfrak{Q}$. We next show that

$$\mathfrak{R} = (\mathfrak{R} \cap \mathfrak{M})(\mathfrak{R} \cap \mathfrak{N}) .$$

If $\mathfrak{R} \in \mathscr{S}$, this is the case by hypothesis, so we can suppose the p-length of $\mathfrak{R}$ is at least 3. Let $\mathfrak{P}_1 = \mathfrak{P} \cap O_{p,q,p}(\mathfrak{R})$, and $\mathfrak{L} = N_{\mathfrak{R}}(\mathfrak{P}_1)$. Then $\mathfrak{L}$ is a proper subgroup of $\mathfrak{R}$ so by induction on $|\mathfrak{X}|$, we have $\mathfrak{L} = (\mathfrak{L} \cap \mathfrak{M})(\mathfrak{L} \cap \mathfrak{N})$. Let $\mathfrak{K} = \mathfrak{P} \cdot O_{p,q,p}(\mathfrak{R}) = \mathfrak{P}O_{p,q}(\mathfrak{R})$. Since $\mathfrak{K}$ is in $\mathscr{S}$, we have $\mathfrak{K} = (\mathfrak{K} \cap \mathfrak{M})(\mathfrak{K} \cap \mathfrak{N})$. Furthermore, by Sylow's theorem, $\mathfrak{R} = \mathfrak{K}\mathfrak{L}$. Let $R \in \mathfrak{R}$. Then $R = KL$ with $K \in \mathfrak{K}$, $L \in \mathfrak{L}$. Then $K = PK_1$, with P in $\mathfrak{P}$, K_1 in $O_{p,q}(\mathfrak{R})$. Also, $L = MN$, M in $\mathfrak{L} \cap \mathfrak{M}$, N in $\mathfrak{L} \cap \mathfrak{N}$, and so $R = KL = PK_1MN = PMK_1^MN$. Since $K_1^M \in O_{p,q}(\mathfrak{R})$, we have $K_1^M = M_1N_1$ with M_1 in $\mathfrak{M} \cap \mathfrak{K}$, N_1 in $\mathfrak{N} \cap \mathfrak{K}$. Hence, $R = PMM_1 \cdot N_1N$ with PMM_1 in $\mathfrak{M} \cap \mathfrak{R}$, N_1N in $\mathfrak{N} \cap \mathfrak{R}$.

Since $\mathfrak{R} = (\mathfrak{R} \cap \mathfrak{M})(\mathfrak{R} \cap \mathfrak{N})$, we have

$$|\mathfrak{X}|_q = |\mathfrak{R}|_q = \frac{|\mathfrak{R} \cap \mathfrak{M}|_q \cdot |\mathfrak{R} \cap \mathfrak{N}|_q}{|\mathfrak{R} \cap \mathfrak{M} \cap \mathfrak{N}|_q} .$$

By construction, $|\mathfrak{R} \cap \mathfrak{M} \cap \mathfrak{N}|_q = |\mathfrak{M} \cap \mathfrak{N}|_q$. Furthermore, $|\mathfrak{R} \cap \mathfrak{M}|_q \leqq |\mathfrak{M}|_q$ and $|\mathfrak{R} \cap \mathfrak{N}|_q \leqq |\mathfrak{N}|_q$, so

$$|\mathfrak{M}\mathfrak{N}|_q = \frac{|\mathfrak{M}|_q|\mathfrak{N}|_q}{|\mathfrak{M} \cap \mathfrak{N}|_q} \geqq \frac{|\mathfrak{R} \cap \mathfrak{M}|_q \cdot |\mathfrak{R} \cap \mathfrak{N}|_q}{|\mathfrak{R} \cap \mathfrak{M} \cap \mathfrak{N}|_q} = |\mathfrak{X}|_q ,$$

completing the proof.

LEMMA 7.8. *Let $\mathfrak{X}$ be a finite group and $\mathfrak{H}$ a p'-subgroup of $\mathfrak{X}$ which is normalized by the p-subgroup $\mathfrak{A}$ of $\mathfrak{X}$. Set $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{H})$. Suppose $\mathfrak{L}$ is a p-solvable subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{H}$ and $\mathfrak{H} \not\subseteq O_{p'}(\mathfrak{L})$. Then there is a p-solvable subgroup $\mathfrak{K}$ of $\mathfrak{A}C_{\mathfrak{X}}(\mathfrak{A}_1)$ which contains $\mathfrak{A}\mathfrak{H}$ and $\mathfrak{H} \not\subseteq O_{p'}(\mathfrak{K})$.*

Proof. Let $\mathfrak{F} = O_{p',p}(\mathfrak{L})/O_{p'}(\mathfrak{L})$. Then $\mathfrak{H}$ does not centralize $\mathfrak{F}$. Let $\mathfrak{B}$ be a subgroup of $\mathfrak{F}$ which is minimal with respect to being $\mathfrak{A}\mathfrak{H}$-invariant and not centralized by $\mathfrak{H}$. Then $\mathfrak{B} = [\mathfrak{B}, \mathfrak{H}]$, and $[\mathfrak{B}, \mathfrak{A}_1] \subseteq D(\mathfrak{B})$, while $[D(\mathfrak{B}), \mathfrak{H}] = 1$. Hence, $[\mathfrak{B}, \mathfrak{A}_1, \mathfrak{H}] = [\mathfrak{A}_1, \mathfrak{H}, \mathfrak{B}] = 1$, and so $[\mathfrak{H}, \mathfrak{B}, \mathfrak{A}_1] = 1$. Since $[\mathfrak{H}, \mathfrak{B}] = \mathfrak{B}$, $\mathfrak{A}_1$ centralizes $\mathfrak{B}$. Since $\mathfrak{B}$ is a subgroup of $\mathfrak{F}$, we have $\mathfrak{B} = \mathfrak{L}_0/O_{p'}(\mathfrak{L})$ for suitable $\mathfrak{L}_0$. As $O_{p'}(\mathfrak{L})$ is a p'-group and $\mathfrak{B}$ is a p-group, we can find an $\mathfrak{A}$-invariant p-subgroup $\mathfrak{P}_0$ of $\mathfrak{L}_0$ incident with $\mathfrak{B}$. Hence, $\mathfrak{A}_1$ centralizes $\mathfrak{P}_0$. Set

$$\mathfrak{K} = \langle \mathfrak{A}, \mathfrak{P}_0, \mathfrak{H} \rangle \subseteq \mathfrak{L} .$$

As $\mathfrak{L}$ is p-solvable so is $\mathfrak{K}$. If $\mathfrak{H} \subseteq O_{p'}(\mathfrak{K})$, then

$$[\mathfrak{P}_0, \mathfrak{H}] \subseteq \mathfrak{L}_0 \cap O_{p'}(\mathfrak{K}) \subseteq O_{p'}(\mathfrak{L})$$

and $\mathfrak{H}$ centralizes $\mathfrak{B}$, contrary to construction. Thus, $\mathfrak{H} \not\subseteq O_{p'}(\mathfrak{K})$, as required.

LEMMA 7.9. *Let $\mathfrak{H}$ be a p-solvable subgroup of the finite group $\mathfrak{X}$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{H}$. Assume that one of the following conditions holds:*

(a) *$|\mathfrak{X}|$ is odd.*

(b) *$p \geqq 5$.*

(c) *$p = 3$ and a S_2-subgroup of $\mathfrak{H}$ is abelian.*

Let $\mathfrak{P}_0 = O_{p',p}(\mathfrak{H}) \cap \mathfrak{P}$ and let $\mathfrak{P}^*$ be a p-subgroup of $\mathfrak{X}$ containing $\mathfrak{P}$. If $\mathfrak{P}$ is a S_p-subgroup of $N_{\mathfrak{X}}(\mathfrak{P}_0)$, then $\mathfrak{P}_0$ contains every element of $\mathscr{SCN}(\mathfrak{P}^*)$.

Proof. Let $\mathfrak{A} \in \mathscr{SCN}(\mathfrak{P}^*)$. By (B) and (a), (b), (c), it follows that $\mathfrak{A} \cap \mathfrak{P} = \mathfrak{A} \cap \mathfrak{P}_0 = \mathfrak{A}_1$, say. If $\mathfrak{A}_1 \subset \mathfrak{A}$, then there is a $\mathfrak{P}_0$-invariant subgroup $\mathfrak{B}$ such that $\mathfrak{A}_1 \subset \mathfrak{B} \subseteq \mathfrak{A}$, $|\mathfrak{B} : \mathfrak{A}_1| = p$. Hence, $[\mathfrak{P}_0, \mathfrak{B}] \subseteq \mathfrak{A}_1 \subseteq \mathfrak{P}_0$, so $\mathfrak{B} \subseteq N_{\mathfrak{X}}(\mathfrak{P}_0) \cap \mathfrak{P}^*$. Hence, $\langle \mathfrak{B}, \mathfrak{P} \rangle$ is a p-subgroup of $N_{\mathfrak{X}}(\mathfrak{P}_0)$, so $\mathfrak{B} \subseteq \mathfrak{P}$. Hence, $\mathfrak{B} \subseteq \mathfrak{A} \cap \mathfrak{P} = \mathfrak{A}_1$, which is not the case, so $\mathfrak{A} = \mathfrak{A}_1$, as required.

8. Miscellaneous Preliminary Lemmas

LEMMA 8.1. *If $\mathfrak{X}$ is a π-group, and $\mathscr{C}$ is a chain $\mathfrak{X} = \mathfrak{X}_0 \supseteqq \mathfrak{X}_1 \supseteqq \cdots \supseteqq \mathfrak{X}_n = 1$, then the stability group $\mathfrak{A}$ of $\mathscr{C}$ is a π-group.*

Proof. We proceed by induction on n. Let $A \in \mathfrak{A}$. By induction, there is a π-number m such that $B = A^m$ centralizes $\mathfrak{X}_1$. Let $X \in \mathfrak{X}$; then $X^B = XY$ with Y in $\mathfrak{X}_1$, and by induction, $X^{B^r} = XY^r$. It follows that $B^{|\mathfrak{X}_1|} = 1$.

LEMMA 8.2. *If $\mathfrak{P}$ is a p-group, then $\mathfrak{P}$ possesses a characteristic subgroup $\mathfrak{C}$ such that*

(i) cl$(\mathfrak{C}) \leqq 2$, *and $\mathfrak{C}/Z(\mathfrak{C})$ is elementary.*

(ii) ker (Aut $\mathfrak{P} \xrightarrow{\text{res}}$ Aut $\mathfrak{C}$) *is a p-group.* (res *is the homomorphism induced by restricting A in* Aut $\mathfrak{P}$ *to* $\mathfrak{C}$.)

(iii) $[\mathfrak{P}, \mathfrak{C}] \subseteq Z(\mathfrak{C})$ *and* $C(\mathfrak{C}) = Z(\mathfrak{C})$.

Proof. Suppose $\mathfrak{C}$ can be found to satisfy (i) and (iii). Let $\mathfrak{K} =$ ker res. In commutator notation, $[\mathfrak{K}, \mathfrak{C}] = 1$, and so $[\mathfrak{K}, \mathfrak{C}, \mathfrak{P}] = 1$. Since $[\mathfrak{C}, \mathfrak{P}] \subseteq \mathfrak{C}$, we also have $[\mathfrak{C}, \mathfrak{P}, \mathfrak{K}] = 1$ and 3.1 implies $[\mathfrak{P}, \mathfrak{K}, \mathfrak{C}] = 1$, so that $[\mathfrak{P}, \mathfrak{K}] \subseteq Z(\mathfrak{C})$. Thus, $\mathfrak{K}$ stabilizes the chain $\mathfrak{P} \supseteqq \mathfrak{C} \supseteqq 1$ so is a p-group by Lemma 8.1.

If now some element of $\mathscr{SCN}(\mathfrak{P})$ is characteristic in $\mathfrak{P}$, then (i) and (iii) are satisfied and we are done. Otherwise, let $\mathfrak{A}$ be a maximal characteristic abelian subgroup of $\mathfrak{P}$, and let $\mathfrak{C}$ be the group generated by all subgroups $\mathfrak{D}$ of $\mathfrak{P}$ such that $\mathfrak{A} \subset \mathfrak{D}$, $|\mathfrak{D} : \mathfrak{A}| = p$, $\mathfrak{D} \subseteq \boldsymbol{Z}(\mathfrak{P} \bmod \mathfrak{A})$, $\mathfrak{D} \subseteq C(\mathfrak{A})$. By construction, $\mathfrak{A} \subseteq \boldsymbol{Z}(\mathfrak{C})$, and $\mathfrak{C}$ is seen to be characteristic. The maximal nature of $\mathfrak{A}$ implies that $\mathfrak{A} = \boldsymbol{Z}(\mathfrak{C})$. Also by construction $[\mathfrak{P}, \mathfrak{C}] \subseteq \mathfrak{A} = \boldsymbol{Z}(\mathfrak{C})$, so in particular, $[\mathfrak{C}, \mathfrak{C}] \subseteq \boldsymbol{Z}(\mathfrak{C})$ and $\mathrm{cl}(\mathfrak{C}) \leqq 2$. By construction, $\mathfrak{C}/\boldsymbol{Z}(\mathfrak{C})$ is elementary.

We next show that $C(\mathfrak{C}) = \boldsymbol{Z}(\mathfrak{C})$. This statement is of course equivalent to the statement that $C(\mathfrak{C}) \subseteq \mathfrak{C}$. Suppose by way of contradiction that $C(\mathfrak{C}) \not\subseteq \mathfrak{C}$. Let $\mathfrak{E}$ be a subgroup of $C(\mathfrak{C})$ of minimal order subject to (a) $\mathfrak{E} \lhd \mathfrak{P}$, and (b) $\mathfrak{E} \not\subseteq \mathfrak{C}$. Since $C(\mathfrak{C})$ satisfies (a) and (b), $\mathfrak{E}$ exists. By the minimality of $\mathfrak{E}$, we see that $[\mathfrak{P}, \mathfrak{E}] \subseteq \mathfrak{C}$ and $D(\mathfrak{E}) \subseteq \mathfrak{C}$. Since $\mathfrak{E}$ centralizes $\mathfrak{C}$, so do $[\mathfrak{P}, \mathfrak{E}]$ and $D(\mathfrak{E})$, so we have $[\mathfrak{P}, \mathfrak{E}] \subseteq \mathfrak{A}$ and $D(\mathfrak{E}) \subseteq \mathfrak{A}$. The minimal nature of $\mathfrak{E}$ guarantees that $\mathfrak{E}/\mathfrak{E} \cap \mathfrak{C}$ is of order p. Since $\mathfrak{E} \cap \mathfrak{C} = \mathfrak{E} \cap \mathfrak{A}$, $\mathfrak{E}/\mathfrak{E} \cap \mathfrak{A}$ is of order p, so $\mathfrak{E}\mathfrak{A}/\mathfrak{A}$ is of order p. By construction of $\mathfrak{C}$, we find $\mathfrak{E}\mathfrak{A} \subseteq \mathfrak{C}$, so $\mathfrak{E} \subseteq \mathfrak{C}$, in conflict with (b). Hence, $C(\mathfrak{C}) = \boldsymbol{Z}(\mathfrak{C})$, and (i) and (iii) are proved.

LEMMA 8.3. *Let $\mathfrak{X}$ be a p-group, p odd, and among all elements of $\mathscr{SCN}(\mathfrak{X})$, choose $\mathfrak{A}$ to maximize $m(\mathfrak{A})$. Then $\Omega_1(C(\Omega_1(\mathfrak{A}))) = \Omega_1(\mathfrak{A})$.*

REMARK. The oddness of p is required, as the dihedral group of order 16 shows.

Proof. We must show that whenever an element of $\mathfrak{X}$ of order p centralizes $\Omega_1(\mathfrak{A})$, then the element lies in $\Omega_1(\mathfrak{A})$.

If $X \in C(\Omega_1(\mathfrak{A}))$ and $X^p = 1$, let $\mathfrak{B}(X) = \mathfrak{B}_1 = \langle \Omega_1(\mathfrak{A}), X \rangle$, and let $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}_n = \langle \mathfrak{A}, X \rangle$ be an ascending chain of subgroups, each of index p in its successor. We wish to show that $\mathfrak{B}_1 \lhd \mathfrak{B}_n$. Suppose $\mathfrak{B}_1 \lhd \mathfrak{B}_m$ for some $m \leqq n - 1$. Then $\mathfrak{B}_m$ is generated by its normal abelian subgroups $\mathfrak{B}_1$ and $\mathfrak{B}_m \cap \mathfrak{A}$, so $\mathfrak{B}_m$ is of class at most two, so is regular. Let $Z \in \mathfrak{B}_m$, Z of order p. Then $Z = X^k A$, A in $\mathfrak{A}$, k an integer. Since $\mathfrak{B}_m$ is regular, $X^{-k}Z$ is of order 1 or p. Hence, $A \in \Omega_1(\mathfrak{A})$, and $Z \in \mathfrak{B}_1$. Hence, $\mathfrak{B}_1 = \Omega_1(\mathfrak{B}_m) \operatorname{char} \mathfrak{B}_m \lhd \mathfrak{B}_{m+1}$, and $\mathfrak{B}_1 \lhd \mathfrak{B}_n$ follows. In particular, X stabilizes the chain $\mathfrak{A} \supseteq \Omega_1(\mathfrak{A}) \supseteq \langle 1 \rangle$.

It follows that if $\mathfrak{D} = \Omega_1(C(\Omega_1(\mathfrak{A})))$, then $\mathfrak{D}'$ centralizes $\mathfrak{A}$. Since $\mathfrak{A} \in \mathscr{SCN}(\mathfrak{X})$, $\mathfrak{D}' \subseteq \mathfrak{A}$. We next show that $\mathfrak{D}$ is of exponent p. Since $[\mathfrak{D}, \mathfrak{D}] \subseteq \mathfrak{A}$, we see that $[\mathfrak{D}, \mathfrak{D}, \mathfrak{D}] \subseteq \Omega_1(\mathfrak{A})$, and so

$$[\mathfrak{D}, \mathfrak{D}, \mathfrak{D}, \mathfrak{D}] = 1\,,$$

and $\mathrm{cl}(\mathfrak{D}) \leqq 3$. If $p \geqq 5$, then $\mathfrak{D}$ is regular, and being generated by elements of order p, is of exponent p. It remains to treat the case $p = 3$, and we must show that the elements of $\mathfrak{D}$ of order at most 3 form a subgroup. Suppose false, and that $\langle X, Y\rangle$ is of minimal order subject to $X^3 = Y^3 = 1$, $(XY)^3 \neq 1$, X and Y being elements of $\mathfrak{D}$. Since $\langle Y, Y^X\rangle \subset \langle X, Y\rangle$, $[Y, X] = Y^{-1}$. $X^{-1}YX$ is of order three. Hence, $[X, Y]$ is in $\Omega_1(\mathfrak{A})$, and so $[Y, X]$ is centralized by both X and Y. It follows that $(XY)^3 = X^3Y^3[Y, X]^3 = 1$, so $\mathfrak{D}$ is of exponent p in all cases.

If $\Omega_1(\mathfrak{A}) \subset \mathfrak{D}$, let $\mathfrak{E} \lhd \mathfrak{X}$, $\mathfrak{E} \subseteq \mathfrak{D}$, $|\mathfrak{E} : \Omega_1(\mathfrak{A})| = p$. Since $\Omega_1(\mathfrak{A}) \subseteq \boldsymbol{Z}(\mathfrak{E})$, $\mathfrak{E}$ is abelian. But $m(\mathfrak{E}) = m(\mathfrak{A}) + 1 > m(\mathfrak{A})$, in conflict with the maximal nature of $\mathfrak{A}$, since $\mathfrak{E}$ is contained in some element of $\mathscr{SCN}(\mathfrak{X})$ by 3.9.

LEMMA 8.4. *Suppose p is an odd prime and $\mathfrak{X}$ is a p-group.*

(i) *If $\mathscr{SCN}_3(\mathfrak{X})$ is empty, then every abelian subgroup of $\mathfrak{X}$ is generated by two elements.*

(ii) *If $\mathscr{SCN}_3(\mathfrak{X})$ is empty and A is an automorphism of $\mathfrak{X}$ of prime order q, $p \neq q$, then q divides $p^2 - 1$.*

Proof. (i) Suppose $\mathfrak{A}$ is chosen in accordance with Lemma 8.3. Suppose also that $\mathfrak{X}$ contains an elementary subgroup $\mathfrak{E}$ of order p^3. Let $\mathfrak{E}_1 = C_{\mathfrak{E}}(\Omega_1(\mathfrak{A}))$, so that $\mathfrak{E}_1$ is of order p^2 at least. But by Lemma 8.3, $\mathfrak{E}_1 \subseteq \Omega_1(\mathfrak{A})$, a group of order at most p^2, and so $\mathfrak{E}_1 = \Omega_1(\mathfrak{A})$. But now Lemma 8.3 is violated since $\mathfrak{E}$ centralizes $\mathfrak{E}_1$.

(ii) Among the A-invariant subgroups of $\mathfrak{X}$ on which A acts non trivially, let $\mathfrak{H}$ be minimal. By 3.11, $\mathfrak{H}$ is a special p-group. Since p is odd, $\mathfrak{H}$ is regular, so 3.6 implies that $\mathfrak{H}$ is of exponent p. By the first part of this lemma, $\mathfrak{H}$ contains no elementary subgroup of order p^3. It follows readily that $m(\mathfrak{H}) \leq 2$, and (ii) follows from the well known fact that q divides $|\text{Aut } \mathfrak{H}/\boldsymbol{D}(\mathfrak{H})|$.

LEMMA 8.5. *If $\mathfrak{X}$ is a group of odd order, p is the smallest prime in $\pi(\mathfrak{X})$, and if in addition $\mathfrak{X}$ contains no elementary subgroup of order p^3, then $\mathfrak{X}$ has a normal p-complement.*

Proof. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{X}$. By hypothesis, if $\mathfrak{H}$ is a subgroup of $\mathfrak{P}$, then $\mathscr{SCN}_3(\mathfrak{H})$ is empty. Application of Lemma 8.4 (ii) shows that $N_{\mathfrak{X}}(\mathfrak{H})/C_{\mathfrak{X}}(\mathfrak{H})$ is a p-group for every subgroup $\mathfrak{H}$ of $\mathfrak{P}$. We apply Theorem 14. 4. 7 in [12] to complete the proof.

Application of Lemma 8.5 to a simple group $\mathfrak{G}$ of odd order implies that if p is the smallest prime in $\pi(\mathfrak{G})$, then $\mathfrak{G}$ contains an elementary subgroup of order p^3. In particular, if $3 \in \pi(\mathfrak{G})$, then $\mathfrak{G}$ contains an elementary subgroup of order 27.

LEMMA 8.6. *Let $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3$ be subgroups of a group $\mathfrak{X}$ and suppose that for every permutation σ of $\{1, 2, 3\}$,*

$$\mathfrak{N}_{\sigma(1)} \subseteq \mathfrak{N}_{\sigma(2)}\mathfrak{N}_{\sigma(3)}$$

Then $\mathfrak{N}_1\mathfrak{N}_2$ is a subgroup of $\mathfrak{X}$.

Proof. $\mathfrak{N}_2\mathfrak{N}_1 \subseteq (\mathfrak{N}_1\mathfrak{N}_3)(\mathfrak{N}_3\mathfrak{N}_2) \subseteq \mathfrak{N}_1\mathfrak{N}_3\mathfrak{N}_2 \subseteq \mathfrak{N}_1(\mathfrak{N}_1\mathfrak{N}_2)\mathfrak{N}_2 \subseteq \mathfrak{N}_1\mathfrak{N}_2$, as required.

LEMMA 8.7. *If $\mathfrak{A}$ is a p'-group of automorphisms of the p-group $\mathfrak{P}$, if $\mathfrak{A}$ has no fixed points on $\mathfrak{P}/\boldsymbol{D}(\mathfrak{P})$, and $\mathfrak{A}$ acts trivially on $\boldsymbol{D}(\mathfrak{P})$, then $\boldsymbol{D}(\mathfrak{P}) \subseteq \boldsymbol{Z}(\mathfrak{P})$.*

Proof. In commutator notation, we are assuming $[\mathfrak{P}, \mathfrak{A}] = \mathfrak{P}$, and $[\mathfrak{A}, \boldsymbol{D}(\mathfrak{P})] = 1$. Hence, $[\mathfrak{A}, \boldsymbol{D}(\mathfrak{P}), \mathfrak{P}] = 1$. Since $[\boldsymbol{D}(\mathfrak{P}), \mathfrak{P}] \subseteq \boldsymbol{D}(\mathfrak{P})$, we also have $[\boldsymbol{D}(\mathfrak{P}), \mathfrak{P}, \mathfrak{A}] = 1$. By the three subgroups lemma, we have $[\mathfrak{P}, \mathfrak{A}, \boldsymbol{D}(\mathfrak{P})] = 1$. Since $[\mathfrak{P}, \mathfrak{A}] = \mathfrak{P}$, the lemma follows.

LEMMA 8.8. *Suppose $\mathfrak{Q}$ is a q-group, q is odd, A is an automorphism of $\mathfrak{Q}$ of prime order p, $p \equiv 1 \pmod q$, and $\mathfrak{Q}$ contains a subgroup $\mathfrak{Q}_0$ of index q such that $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q}_0)$ is empty. Then $p = 1 + q + q^2$ and $\mathfrak{Q}$ is elementary of order q^3.*

Proof. Since $p \equiv 1 \pmod q$ and q is odd, p does not divide $q^2 - 1$. Since $\boldsymbol{D}(\mathfrak{Q}) \subseteq \mathfrak{Q}_0$, Lemma 8.4 (ii) implies that A acts trivially on $\boldsymbol{D}(\mathfrak{Q})$.

Suppose that A has a non trivial fixed point on $\mathfrak{Q}/\boldsymbol{D}(\mathfrak{Q})$. We can then find an A-invariant subgroup $\mathfrak{M}$ of index q in $\mathfrak{Q}$ such that A acts trivially on $\mathfrak{Q}/\mathfrak{M}$. In this case, A does not act trivially on $\mathfrak{M}$, and so $\mathfrak{M} \neq \mathfrak{Q}_0$, and $\mathfrak{M} \cap \mathfrak{Q}_0$ is of index q in $\mathfrak{M}$. By induction, $p = 1 + q + q^2$ and $\mathfrak{M}$ is elementary of order q^3. Since A acts trivially on $\mathfrak{Q}/\mathfrak{M}$, it follows that $\mathfrak{Q}$ is abelian of order q^4 If $\mathfrak{Q}$ were elementary, $\mathfrak{Q}_0$ would not exist. But if $\mathfrak{Q}$ were not elementary, then A would have a fixed point on $\Omega_1(\mathfrak{Q}) = \mathfrak{M}$, which is not possible. Hence A has no fixed points on $\mathfrak{Q}/\boldsymbol{D}(\mathfrak{Q})$, so by Lemma 8.7, $\boldsymbol{D}(\mathfrak{Q}) \subseteq \boldsymbol{Z}(\mathfrak{Q})$.

Next, suppose that A does not act irreducibly on $\mathfrak{Q}/\boldsymbol{D}(\mathfrak{Q})$. Let $\mathfrak{N}/\boldsymbol{D}(\mathfrak{Q})$ be an irreducible constituent of A on $\mathfrak{Q}/\boldsymbol{D}(\mathfrak{Q})$. By induction, $\mathfrak{N}$ is of order q^3, and $p = 1 + q + q^2$. Since $\boldsymbol{D}(\mathfrak{Q}) \subset \mathfrak{N}$, $\boldsymbol{D}(\mathfrak{Q})$ is a proper A-invariant subgroup of $\mathfrak{N}$. The only possibility is $\boldsymbol{D}(\mathfrak{Q}) = 1$, and $|\mathfrak{Q}| = q^3$ follows from the existence of $\mathfrak{Q}_0$.

If $|\mathfrak{Q}| = q^3$, then $p = 1 + q + q^2$ follows from Lemma 5.1. Thus, we can suppose that $|\mathfrak{Q}| > q^3$, and that A acts irreducibly on $\mathfrak{Q}/\boldsymbol{D}(\mathfrak{Q})$, and try to derive a contradiction. We see that $\mathfrak{Q}$ must be non abelian. This implies that $\boldsymbol{D}(\mathfrak{Q}) = \boldsymbol{Z}(\mathfrak{Q})$. Let $|\mathfrak{Q} : \boldsymbol{D}(\mathfrak{Q})| = q^n$. Since

$p \equiv 1 \pmod q$, and $q^n \equiv 1 \pmod p$, $n \geqq 3$. Since $D(\mathfrak{Q}) = Z(\mathfrak{Q})$, n is even, $\mathfrak{Q}/Z(\mathfrak{Q})$ possessing a non singular skew-symmetric inner product over integers mod q which admits A. Namely, let $\mathfrak{E}$ be a subgroup of order q contained in $\mathfrak{Q}'$ and let $\mathfrak{E}_1$ be a complement for $\mathfrak{E}$ in $\mathfrak{Q}'$. This complement exists since $\mathfrak{Q}'$ is elementary. Then $Z(\mathfrak{P} \bmod \mathfrak{E}_1)$ is A-invariant, proper, and contains $D(\mathfrak{Q})$. Since A acts irreducibly on $\mathfrak{Q}/D(\mathfrak{Q})$, we must have $D(\mathfrak{Q}) = Z(\mathfrak{Q} \bmod \mathfrak{E}_1)$, so a non singular skew-symmetric inner product is available. Now $\mathfrak{Q}$ is regular, since $\mathrm{cl}(\mathfrak{Q}) = 2$, and q is odd, so $|\Omega_1(\mathfrak{Q})| = |\mathfrak{Q} : \mho^1(\mathfrak{Q})|$, by [14]. Since $\mathrm{cl}(\mathfrak{Q}) = 2$, $\Omega_1(\mathfrak{Q})$ is of exponent q. Since

$$|\mathfrak{Q} : \mho^1(\mathfrak{Q})| \geqq |\mathfrak{Q} : D(\mathfrak{Q})| \geqq q^4 ,$$

we see that $|\Omega_1(\mathfrak{Q})| \geqq q^4$. Since $\mathfrak{Q}_0$ exists, $\Omega_1(\mathfrak{Q})$ is non abelian, of order exactly q^4, since otherwise $\mathfrak{Q}_0 \cap \Omega_1(\mathfrak{Q})$ would contain an elementary subgroup of order q^3. It follows readily that A centralizes $\Omega_1(\mathfrak{Q})$, and so centralizes $\mathfrak{Q}$, by 3.6. This is the desired contradiction.

LEMMA 8.9. *If $\mathfrak{P}$ is a p-group, if $\mathscr{SCN}_3(\mathfrak{P})$ is non empty and $\mathfrak{A}$ is a normal abelian subgroup of $\mathfrak{P}$ of type (p, p), then $\mathfrak{A}$ is contained in some element of $\mathscr{SCN}_3(\mathfrak{P})$.*

Proof. Let $\mathfrak{E}$ be a normal elementary subgroup of $\mathfrak{P}$ of order p^3, and let $\mathfrak{E}_1 = C_{\mathfrak{E}}(\mathfrak{A})$. Then $\mathfrak{E}_1 \lhd \mathfrak{P}$, and $\langle \mathfrak{A}, \mathfrak{E}_1 \rangle = \mathfrak{F}$ is abelian. If $|\mathfrak{F}| = p^2$, then $\mathfrak{A} = \mathfrak{E}_1 = \mathfrak{F} \subset \mathfrak{E}$, and we are done, since $\mathfrak{E}$ is contained in an element of $\mathscr{SCN}_3(\mathfrak{P})$. If $|\mathfrak{F}| \geqq p^3$, then again we are done, since $\mathfrak{F}$ is contained in an element of $\mathscr{SCN}_3(\mathfrak{P})$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are groups, we say that $\mathfrak{Y}$ *is involved in* $\mathfrak{X}$ provided some section of $\mathfrak{X}$ is isomorphic to $\mathfrak{Y}$ [18].

LEMMA 8.10. *Let $\mathfrak{P}$ be a S_p-subgroup of the group $\mathfrak{X}$. Suppose that $Z(\mathfrak{P})$ is cyclic and that for each subgroup $\mathfrak{A}$ in $\mathfrak{P}$ of order p which does not lie in $Z(\mathfrak{P})$, there is an element $X = X(\mathfrak{A})$ of $\mathfrak{P}$ which normalizes but does not centralize $\langle \mathfrak{A}, \Omega_1(Z(\mathfrak{P}))\rangle$. Then either $SL(2, p)$ is involved in $\mathfrak{X}$ or $\Omega_1(Z(\mathfrak{P}))$ is weakly closed in $\mathfrak{P}$.*

Proof. Let $\mathfrak{D} = \Omega_1(Z(\mathfrak{P}))$. Suppose $\mathfrak{E} = \mathfrak{D}^G$ is a conjugate of $\mathfrak{D}$ contained in $\mathfrak{P}$, but that $\mathfrak{E} \neq \mathfrak{D}$. Let $\mathfrak{D} = \langle D \rangle$, $\mathfrak{E} = \langle E \rangle$. By hypothesis, we can find an element $X = X(\mathfrak{E})$ in $\mathfrak{P}$ such that X normalizes $\langle E, D \rangle = \mathfrak{F}$, and with respect to the basis (E, D) has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Enlarge $\mathfrak{F}$ to a S_p-subgroup $\mathfrak{P}^*$ of $C_{\mathfrak{X}}(\mathfrak{E})$. Since $\mathfrak{E} = \mathfrak{D}^G$, $\mathfrak{P}^G \subseteq C_{\mathfrak{X}}(\mathfrak{E})$, so $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{X}$, and $\mathfrak{E} \subseteq Z(\mathfrak{P}^*)$. Since $Z(\mathfrak{P}^*)$ is cyclic by hypothesis, we have $\mathfrak{E} = \Omega_1(Z(\mathfrak{P}^*))$. By hypothesis, there is an element $Y = Y(\mathfrak{D})$ in $\mathfrak{P}^*$ which normalizes $\mathfrak{F}$ and with respect to the basis (E, D) has the matrix $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. Now $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ generate $SL(2, p)$ [6, Sections 262 and 263], so $SL(2, p)$ is involved in $N_{\mathfrak{X}}(\mathfrak{F})$, as desired.

LEMMA 8.11. *If $\mathfrak{A}$ is a p-subgroup and $\mathfrak{B}$ is a q-subgroup of $\mathfrak{X}$, $p \neq q$, and $\mathfrak{A}$ normalizes $\mathfrak{B}$ then* $[\mathfrak{B}, \mathfrak{A}] = [\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]$.

Proof. By 3.7, $[\mathfrak{A}, \mathfrak{B}] \lhd \mathfrak{A}\mathfrak{B}$. Since $\mathfrak{A}\mathfrak{B}/[\mathfrak{A}, \mathfrak{B}]$ is nilpotent, we can suppose that $[\mathfrak{A}, \mathfrak{B}]$ is elementary. With this reduction, $[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}] \lhd \mathfrak{A}\mathfrak{B}$, and we can assume that $[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}] = 1$. In this case, $\mathfrak{A}$ stabilizes the chain $\mathfrak{B} \supseteq [\mathfrak{B}, \mathfrak{A}] \supseteq 1$, so $[\mathfrak{B}, \mathfrak{A}] = 1$ follows from Lemma 8.1 and $p \neq q$.

LEMMA 8.12. *Let p be an odd prime, and $\mathfrak{E}$ an elementary subgroup of the p-group $\mathfrak{P}$. Suppose A is a p'-automorphism of $\mathfrak{P}$ which centralizes $\Omega_1(C_{\mathfrak{P}}(\mathfrak{E}))$. Then* $A = 1$.

Proof. Since $\mathfrak{E} \subseteq \Omega_1(C_{\mathfrak{P}}(\mathfrak{E}))$, A centralizes $\mathfrak{E}$. Since $\mathfrak{E}$ is A-invariant, so is $C_{\mathfrak{P}}(\mathfrak{E})$. By 3.6 A centralizes $C_{\mathfrak{P}}(\mathfrak{E})$, so if $\mathfrak{E} \subseteq Z(\mathfrak{P})$, we are done.

If $C_{\mathfrak{P}}(\mathfrak{E}) \subset \mathfrak{P}$, then $C_{\mathfrak{P}}(\mathfrak{E})D(\mathfrak{P}) \subset \mathfrak{P}$, and by induction A centralizes $D(\mathfrak{P})$. Now $[\mathfrak{P}, \mathfrak{E}] \subseteq D(\mathfrak{P})$ and so $[\mathfrak{P}, \mathfrak{E}, \langle A\rangle] = 1$. Also, $[\mathfrak{E}, \langle A\rangle] = 1$, so that $[\mathfrak{E}, \langle A\rangle, \mathfrak{P}] = 1$. By the three subgroups lemma, we have $[\langle A\rangle, \mathfrak{P}, \mathfrak{E}] = 1$, so that $[\mathfrak{P}, \langle A\rangle] \subseteq C_{\mathfrak{P}}(\mathfrak{E})$, and A stabilizes the chain $\mathfrak{P} \supseteq C_{\mathfrak{P}}(\mathfrak{E}) \supset 1$. It follows from Lemma 8.1 that $A = 1$.

LEMMA 8.13. *Suppose $\mathfrak{P}$ is a S_p-subgroup of the solvable group $\mathfrak{S}$, $\mathscr{SCN}_3(\mathfrak{P})$ is empty and $\mathfrak{S}$ is of odd order. Then $\mathfrak{S}'$ centralizes every chief p-factor of $\mathfrak{S}$.*

Proof. We assume without loss of generality that $O_{p'}(\mathfrak{S}) = 1$. We first show that $\mathfrak{P} \lhd \mathfrak{S}$. Let $\mathfrak{H} = O_p(\mathfrak{S})$, and let $\mathfrak{C}$ be a subgroup of $\mathfrak{H}$ chosen in accordance with Lemma 8.2. Let $\mathfrak{W} = \Omega_1(\mathfrak{C})$. Since p is odd and $\mathrm{cl}(\mathfrak{C}) \leq 2$, $\mathfrak{W}$ is of exponent p.

Since $O_{p'}(\mathfrak{S}) = 1$, Lemma 8.2 implies that $\ker(\mathfrak{S} \longrightarrow \operatorname{Aut}\mathfrak{C})$ is a p-group. By 3.6, it now follows that $\ker(\mathfrak{S} \xrightarrow{\alpha} \operatorname{Aut}\mathfrak{W})$ is a p-group. Since $\mathfrak{P}$ has no elementary subgroup of order p^3, neither does $\mathfrak{W}$, and so $|\mathfrak{W} : D(\mathfrak{W})| \leq p^2$. Hence no p-element of $\mathfrak{S}$ has a minimal polynomial $(x-1)^p$ on $\mathfrak{W}/D(\mathfrak{W})$. Now (B) implies that $\mathfrak{P}/\ker\alpha \lhd \mathfrak{S}/\ker\alpha$. and so $\mathfrak{P} \lhd \mathfrak{S}$, since $\ker\alpha \subseteq \mathfrak{P}$.

Since $\mathfrak{P} \lhd \mathfrak{S}$, the lemma is equivalent to the assertion that if $\mathfrak{L}$ is a $S_{p'}$-subgroup of $\mathfrak{S}$, then $\mathfrak{L}' = 1$. If $\mathfrak{L}' \neq 1$, we can suppose that $\mathfrak{L}'$ centralizes every proper subgroup of $\mathfrak{P}$ which is normal in $\mathfrak{S}$. Since $\mathfrak{L}$ is completely reducible on $\mathfrak{P}/D(\mathfrak{P})$, we can suppose that $[\mathfrak{P}, \mathfrak{L}'] = \mathfrak{P}$

and $[D(\mathfrak{P}), \mathfrak{L}'] = 1$. By Lemma 8.7 we have $D(\mathfrak{P}) \subseteq Z(\mathfrak{P})$ and so $\Omega_1(\mathfrak{P}) = \mathfrak{K}$ is of exponent p and class at most 2. Since $\mathfrak{P}$ has no elementary subgroup of order p^3, neither does $\mathfrak{K}$. If $\mathfrak{K}$ is of order p, $\mathfrak{L}'$ centralizes $\mathfrak{K}$ and so centralizes $\mathfrak{P}$ by 3.6, thus $\mathfrak{L}' = 1$. Otherwise, $|\mathfrak{K} : D(\mathfrak{K})| = p^2$ and $\mathfrak{L}$ is faithfully represented as automorphisms of $\mathfrak{K}/D(\mathfrak{K})$. Since $|\mathfrak{L}|$ is odd, $\mathfrak{L}' = 1$.

LEMMA 8.14. *If $\mathfrak{S}$ is a solvable group of odd order, and $\mathscr{SCN}_3(\mathfrak{P})$ is empty for every S_p-subgroup $\mathfrak{P}$ of $\mathfrak{S}$ and every prime p, then $\mathfrak{S}'$ is nilpotent.*

Proof. By the preceding lemma, $\mathfrak{S}'$ centralizes every chief factor of $\mathfrak{S}$. By 3.2, $\mathfrak{S}' \subseteq F(\mathfrak{S})$, a nilpotent group.

LEMMA 8.15. *Let $\mathfrak{S}$ be a solvable group of odd order and suppose that $\mathfrak{S}$ does not contain an elementary subgroup of order p^3 for any prime p. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{S}$ and let $\mathfrak{C}$ be any characteristic subgroup of $\mathfrak{P}$. Then $\mathfrak{C} \cap \mathfrak{P}' \lhd \mathfrak{S}$.*

Proof. We can suppose that $\mathfrak{C} \subseteq \mathfrak{P}'$, since $\mathfrak{C} \cap \mathfrak{P}'$ char $\mathfrak{P}$. By Lemma 8.14 $F(\mathfrak{S})$ normalizes $\mathfrak{C}$. Since $F(\mathfrak{S})\mathfrak{P} \lhd \mathfrak{S}$, we have $\mathfrak{S} = F(\mathfrak{S})N(\mathfrak{P})$. The lemma follows.

The next two lemmas involve a non abelian p-group $\mathfrak{P}$ with the following properties:

(1) p is odd.

(2) $\mathfrak{P}$ contains a subgroup $\mathfrak{P}_0$ of order p such that

$$C(\mathfrak{P}_0) = \mathfrak{P}_0 \quad \mathfrak{P}_1 ,$$

where $\mathfrak{P}_1$ is cyclic.
Also, $\mathfrak{A}$ is a p'-group of automorphisms of $\mathfrak{P}$ of odd order.

LEMMA 8.16. *With the preceding notation,*

(i) *$\mathfrak{A}$ is abelian.*

(ii) *No element of $\mathfrak{A}^\#$ centralizes $\Omega_1(C(\mathfrak{P}_0))$.*

(iii) *If $\mathfrak{A}$ is cyclic, then either $|\mathfrak{A}|$ divides $p \quad 1$ or $\mathscr{SCN}_3(\mathfrak{P})$ is empty.*

Proof. (ii) is an immediate consequence of Lemma 8.12.

Let $\mathfrak{B}$ be a subgroup of $\mathfrak{P}$ chosen in accordance with Lemma 8.2, and let $\mathfrak{W} = \Omega_1(\mathfrak{B})$ so that $\mathfrak{A}$ is faithfully represented on $\mathfrak{W}$. If $\mathfrak{P}_0 \not\subseteq \mathfrak{W}$, then $\mathfrak{P}_0\mathfrak{W}$ is of maximal class, so that with $\mathfrak{W}_0 = \mathfrak{W}$, $\mathfrak{W}_{i+1} = [\mathfrak{W}_i, \mathfrak{P}]$, we have $|\mathfrak{W}_i : \mathfrak{W}_{i+1}| = p$, $i = 0, 1, \cdots, n-1$, $|\mathfrak{W}| = p^n$, and both (i) and (iii) follow. If $\mathfrak{P}_0 \subseteq \mathfrak{W}$, then $m(\mathfrak{W}) = 2$. Since $[\mathfrak{W}, \mathfrak{P}] \subseteq Z(\mathfrak{W})$, it follows that $\langle \mathfrak{P}_0, \boldsymbol{Z}(\mathfrak{W})\rangle \lhd \mathfrak{P}$. By Lemma 8.9, $\mathscr{SCN}_3(\mathfrak{P})$ is empty. The lemma follows readily from 3.4.

LEMMA 8.17. *In the preceding notation, assume in addition that* $|\mathfrak{A}| = q$ *is a prime, that* q *does not divide* $p-1$, *that* $\mathfrak{P} = [\mathfrak{P}, \mathfrak{A}]$ *and that* $C_{\mathfrak{P}}(\mathfrak{A})$ *is cyclic. Then* $|\mathfrak{P}| = p^3$.

Proof. Since $q \nmid p-1$, $\mathfrak{A}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$, and so $\boldsymbol{Z}(\mathfrak{P}) \subseteq \mathfrak{P}'$. Since $C_{\mathfrak{P}}(\mathfrak{A})$ is cyclic, $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$ is not of type (p, p). Hence, $\mathfrak{P}_0 \subseteq \Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$. Since every automorphism of $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$ which is the identity on $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))/\Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ is inner, it follows that $\mathfrak{P} = \Omega_1(\boldsymbol{Z}_2(\mathfrak{P})) \cdot \mathfrak{D}$, where $\mathfrak{D} = C_{\mathfrak{P}}(\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})))$. Since $\mathfrak{P}_1$ is cyclic, so is $\mathfrak{D}$, and so $\mathfrak{D} \subseteq \Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$, by virtue of $\mathfrak{P} = [\mathfrak{P}, \mathfrak{A}]$ and $q \nmid p-1$.

CHAPTER III

9. Tamely Imbedded Subsets of a Group

The character ring of a group has a metric structure which is derived from the inner product. Let $\mathfrak{L}$ be a subgroup of the group $\mathfrak{X}$. The purpose of this chapter is to state conditions on $\mathfrak{L}$ and $\mathfrak{X}$ which ensure the existence of an isometry τ that maps suitable subsets of the character ring of $\mathfrak{L}$ into the character ring of $\mathfrak{X}$ and has certain additional properties. If α is in the character ring of $\mathfrak{L}$ and α^τ is defined then these additional properties will yield information concerning $\alpha^\tau(L)$ for some elements L of $\mathfrak{L}$. Once the existence of τ is established it will enable us to derive information about certain generalized characters of $\mathfrak{X}$ provided we know something about the character ring of $\mathfrak{L}$. In this way it is possible to get global information about $\mathfrak{X}$ from local information about $\mathfrak{L}$.

There are two stages in establishing the existence of τ. First we will require that $\mathfrak{L}$ is in some sense "nicely" imbedded in $\mathfrak{X}$. When this requirement is fulfilled it is possible to define α^τ for certain generalized characters α of $\mathfrak{L}$ with $\alpha(1) = 0$. In this situation α^τ is explicitly defined in terms of induced characters of various subgroups of $\mathfrak{X}$. Secondly it is necessary that the character ring of $\mathfrak{L}$ have certain special properties. These properties make it possible to extend the definition of τ to a wider domain. In particular it is then possible to define α^τ for some generalized characters α of $\mathfrak{L}$ with $\alpha(1) \neq 0$. The precise conditions that the character ring of $\mathfrak{L}$ needs to satisfy will be stated later. In this section we are concerned with the imbedding of $\mathfrak{L}$ in $\mathfrak{X}$. The following definition is appropriate.

DEFINITION 9.1. Let $\hat{\mathfrak{L}}$ be a subset of the group $\mathfrak{X}$ such that

$$(9.1) \qquad \langle 1\rangle \subseteq \hat{\mathfrak{L}} \subseteq N(\hat{\mathfrak{L}}) = \mathfrak{L}\,.$$

Let $\mathfrak{L}_0$ be the set of elements L in $\hat{\mathfrak{L}}$ such that $C(L) \subseteq \mathfrak{L}$, and let $\mathfrak{D} = \hat{\mathfrak{L}}^\# - \mathfrak{L}_0$.

We say that $\hat{\mathfrak{L}}$ is *tamely imbedded* in $\mathfrak{X}$ if the following conditions are satisfied:

(i) If two elements of $\hat{\mathfrak{L}}$ are conjugate in $\mathfrak{X}$, they are conjugate in $\mathfrak{L}$.

(ii) If $\mathfrak{D}$ is non empty, then there are non identity subgroups $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ of $\mathfrak{X}$, $n \geqq 1$, with the following properties:

(a) $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ for $i \neq j$;
(b) $\mathfrak{H}_i$ is a S-subgroup of $\mathfrak{N}_i = N(\mathfrak{H}_i)$;
(c) $\mathfrak{N}_i = \mathfrak{H}_i(\mathfrak{L} \cap \mathfrak{N}_i)$ and $\mathfrak{H}_i \cap \mathfrak{L} = 1$;
(d) $(|\mathfrak{H}_i|, |C_{\mathfrak{L}}(L)|) = 1$ for $L \in \hat{\mathfrak{L}}^{\sharp}$;
(e) For $1 \leqq i \leqq n$, define

$$\hat{\mathfrak{N}}_i = \left\{ \bigcup_{H \in \mathfrak{H}_i^{\sharp}} C_{\mathfrak{N}_i}(H) \right\} - \mathfrak{H}_i^{\sharp} .$$

Then $\hat{\mathfrak{N}}_i^{\sharp}$ is a non empty T. I. set in $\mathfrak{X}$ and $\mathfrak{N}_i = N(\hat{\mathfrak{N}}_i)$.

(iii) If $L_0 \in \mathfrak{D}$, then there is a conjugate L of L_0 in $\hat{\mathfrak{L}}$ and an index i such that

$$C(L) = C_{\mathfrak{H}_i}(L) \cdot C_{\mathfrak{L}}(L) \subseteq \mathfrak{N}_i .$$

If $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ then for $1 \leqq i \leqq n$, each of the groups $\mathfrak{H}_i$ is called a *supporting subgroup* of $\hat{\mathfrak{L}}$. The collection $\{\mathfrak{H}_i \mid 1 \leqq i \leqq n\}$ is called a *system of supporting subgroups* of $\hat{\mathfrak{L}}$.

In one important special case, the definition of tamely imbedded subset of $\mathfrak{X}$ is fairly easy to master. Namely, if $\mathfrak{D}$ is empty, the reader can check that $\hat{\mathfrak{L}}$ is a T. I. set.

If $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ with $\mathfrak{L} = N(\hat{\mathfrak{L}})$ then in this section $\mathscr{I}(\hat{\mathfrak{L}})$ denotes the set of generalized characters of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$ and $\mathscr{C}(\hat{\mathfrak{L}})$ denotes the complex valued class functions of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$. Similarly, $\mathscr{I}_0(\hat{\mathfrak{L}})(\mathscr{C}_0(\hat{\mathfrak{L}}))$ is the subset of $\mathscr{I}(\hat{\mathfrak{L}})(\mathscr{C}(\hat{\mathfrak{L}}))$ vanishing at 1. R. Brauer and M. Suzuki noted that if $\hat{\mathfrak{L}}$ is a T. I. set in $\mathfrak{X}$ then the mapping τ from $\mathscr{C}_0(\hat{\mathfrak{L}})$ into the ring of class functions of $\mathfrak{X}$ defined by

$$\alpha^{\tau} = \alpha^{*}$$

is an isometry ([24], p. 662). They were then able to extend this isometry to certain subsets of $\mathscr{C}(\hat{\mathfrak{L}})$. Several authors have since then used this technique and it has played an important role in recent work in group theory.

In this chapter these results will be generalized in two ways. First we will consider tamely imbedded subsets of $\mathfrak{X}$ rather than T. I. sets in $\mathfrak{X}$. Secondly we will show that under a variety of conditions τ can be extended to various large subsets of $\mathscr{C}(\hat{\mathfrak{L}})$. The results proved in this chapter are important for the proof of the main theorem of this paper. However it is unnecessary in general to assume that $\mathfrak{X}$ has odd order or that $\mathfrak{X}$ is a minimal simple group.

The following notation will be used throughout this section.

For a tamely imbedded subset $\hat{\mathfrak{L}}$ of $\mathfrak{X}$ let $\mathfrak{L} = N(\hat{\mathfrak{L}})$ and for

$1 \leqq i \leqq n$ let $\mathfrak{H}_i$ and $\mathfrak{N}_i$ have the same meaning as in Definition 9.1. Define $\mathfrak{H}_0 = 1$ and

$$\mathfrak{L}_i = \{L \mid L \in \mathfrak{D}, C(L) \subseteq \mathfrak{N}_i\} \qquad \text{for } 1 \leqq i \leqq n .$$

For $L \in \mathfrak{L}_i$, $0 \leqq i \leqq n$ let

$$\mathfrak{A}_L = \{LH \mid LH = HL, H \in \mathfrak{H}_i\} = L\{\mathfrak{H}_i \cap C(L)\} . \tag{9.2}$$

Since $\hat{\mathfrak{L}}$ is tamely imbedded in $\mathfrak{X}$ it follows from (9.2) and Definition 9.1 that for $L \in \mathfrak{L}_i$, $0 \leqq i \leqq n$

$$| C(L) | = | C(L) \cap \mathfrak{L} | \; | \mathfrak{A}_L | . \tag{9.3}$$

For $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ and $1 \leqq i \leqq n$ define

$$\alpha_i = \alpha |_{\mathfrak{L} \cap \mathfrak{N}_i} .$$

Let α_{i1} be the class function of $\mathfrak{N}_i/\mathfrak{H}_i$ which satisfies

$$\alpha_{i1} |_{\mathfrak{L} \cap \mathfrak{N}_i} = \alpha_i .$$

Let α_{i2} be the class function of $\mathfrak{N}_i$ induced by α_i. Define

$$\alpha^\tau = \alpha^* + \sum_{i=1}^{n} (\alpha_{i1} - \alpha_{i2})^* . \tag{9.4}$$

If $\alpha \in \mathscr{I}_0(\hat{\mathfrak{L}})$ then (9.4) implies that α^τ is a generalized character of $\mathfrak{X}$. It is an immediate consequence of the definition of induced characters that for $1 \leqq i \leqq n$

$$\begin{aligned} &\alpha_{i1}(A) = \alpha(L) && \text{for } L \in \mathfrak{L}_i, A \in \mathfrak{A}_L \\ &\alpha_{i2}(A) = 0 && \text{for } L \in \mathfrak{L}_i, A \in \mathfrak{A}_L, A \neq L \\ &\alpha_{i2}(L) = | C(L) \cap \mathfrak{H}_i | \alpha(L) && \text{for } L \in \mathfrak{L}_i . \end{aligned} \tag{9.5}$$

LEMMA 9.1. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ let α^τ be defined by* (9.4). *Then $\alpha^\tau(X) = 0$ if X is not conjugate to an element of $\mathfrak{A}_L$ for any $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$, while*

$$\alpha^\tau(A) = \alpha(L) \quad \text{for} \quad A \in \mathfrak{A}_L, L \in \bigcup_{i=0}^{n} \mathfrak{L}_i .$$

Proof. If $N \in \mathfrak{N}_i$ then a complement of $\mathfrak{H}_i$ in $\mathfrak{H}_i\langle N \rangle$ is solvable. Thus ([28] p. 162) for $1 \leqq i \leqq n$ every element of $\mathfrak{N}_i$ is conjugate to an element of the form $HL = LH$ with $L \in \mathfrak{L} \cap \mathfrak{N}_i$, $H \in \mathfrak{H}_i$. Suppose that L is not conjugate to an element of $\hat{\mathfrak{L}}^{\sharp}$; then since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$, (9.4) implies that $\alpha^\tau(HL) = 0$. This implies that $\alpha^\tau(X) = 0$ unless X is conjugate to an element of $\mathfrak{A}_L$ for some $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$.

Let $A \in \mathfrak{A}_L, L \in \mathfrak{L}_i$ for some i with $0 \leqq i \leqq n$. Suppose that $X^{-1}LX \in \hat{\mathfrak{N}}_j$ for some $X \in \mathfrak{X}$ and some j with $1 \leqq j \leqq n, i \neq j$. Then $(|\mathfrak{H}_j|, |C(L)|) \neq 1$. Thus $i \neq 0$ and $L \in \hat{\mathfrak{N}}_i$. Furthermore $C(L) = C_{\mathfrak{Q}}(L)C_{\mathfrak{H}_i}(L)$. By assumption $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ and $(|\mathfrak{H}_j|, |C_{\mathfrak{Q}}(L)|) = 1$. Thus $(|C(L)|, |\mathfrak{H}_j|) = 1$ contrary to the choice of L. Since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$, $\alpha_{j1} - \alpha_{j2}$ vanishes on $\mathfrak{N}_j - \hat{\mathfrak{N}}_j^{\sharp}$. Consequently (9.4) implies that

$$
(9.6) \qquad \begin{aligned} &\alpha^\tau(A) = \alpha^*(A) && \text{for } i = 0 \\ &\alpha^\tau(A) = \alpha^*(A) + (\alpha_{i1} - \alpha_{i2})^*(A) && \text{for } 1 \leqq i \leqq n\,. \end{aligned}
$$

Since $\hat{\mathfrak{N}}_i$ is a T. I. set in $\mathfrak{X}$ with $N(\hat{\mathfrak{N}}_i) = \mathfrak{N}_i$ we get that

$$(\alpha_{i1} - \alpha_{i2})^*(A) = (\alpha_{i1} - \alpha_{i2})(A)\,.$$

Thus (9.6) yields that

$$(9.7) \qquad \alpha^\tau(A) = \alpha^*(A) + (\alpha_{i1} - \alpha_{i2})(A) \quad \text{for } 1 \leqq i \leqq n\,.$$

Assume first that $A = L$. Then $\alpha^*(L) = |C(L) \cap \mathfrak{H}_i|\, \alpha(L)$. Hence (9.5), (9.6) and (9.7) yield that $\alpha^\tau(A) = \alpha(L)$. If $A \neq L$, then $\alpha^*(A) = 0$ and $1 \leqq i \leqq n$. Thus (9.5) and (9.7) yield that also in this case $\alpha^\tau(A) = \alpha(L)$. The proof is complete in all cases.

LEMMA 9.2. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ let α^τ be defined by* (9.4). *Then for $1 \leqq i \leqq n$*

$$\alpha^\tau(N) = \alpha_{i1}(N) \quad \textit{for } N \in \hat{\mathfrak{N}}_i \cup \mathfrak{H}_i\,.$$

Furthermore $\alpha^\tau|_{\mathfrak{N}_i}$ is a linear combination of characters of $\mathfrak{N}_i/\mathfrak{H}_i$.

Proof. If $N \in \mathfrak{H}_i$ then by Lemma 9.1 and the definition of α_{i1}

$$\alpha^\tau(N) = 0 = \alpha_{i1}(1) = \alpha_{i1}(N)\,.$$

If $N \in \hat{\mathfrak{N}}_i$, and $\alpha^\tau(N) \neq 0$, then N is conjugate to an element A of $\mathfrak{A}_L$ for some $L \in \mathfrak{L}_i$. Thus by (9.5) and Lemma 9.1 $\alpha^\tau(N) = \alpha_{i1}(N)$ as required.

Let θ be an irreducible character of $\mathfrak{N}_i$ which does not have $\mathfrak{H}_i$ in its kernel. Then

$$(9.8) \qquad (\alpha^\tau|_{\mathfrak{N}_i}, \theta) = \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha^\tau(N)\overline{\theta(N)} .$$

By Lemma 4.3 θ vanishes on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$ hence (9.8) and the first part of the lemma yield that

$$\begin{aligned}(\alpha^\tau|_{\mathfrak{N}_i}, \theta) &= \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha^\tau(N)\overline{\theta(N)} \\ &= \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha_{i1}(N)\overline{\theta(N)} = (\alpha_{i1}, \theta) .\end{aligned}$$

Since α_{i1} is a linear combination of characters of $\mathfrak{N}_i/\mathfrak{H}_i$ this yields that $(\alpha^\tau|_{\mathfrak{N}_i}, \theta) = 0$. The lemma is proved.

LEMMA 9.3. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ and α^τ is defined by* (9.4) *then*

$$(\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} .$$

Proof. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be all the conjugate classes of $\mathfrak{X}$ which contain elements of $\hat{\mathfrak{L}}^\sharp$. Let $L_1, L_2, \cdots$ be elements in $\bigcup_{i=0}^n \mathfrak{L}_i$ such that $L_j \in \mathfrak{C}_j \cap \hat{\mathfrak{L}}^\sharp$. The number of elements in $\mathfrak{X}$ which are conjugate to an element of $\mathfrak{A}_{L_j}$ is easily seen to be

$$|\mathfrak{C}_j|\,|\mathfrak{A}_{L_j}| = \frac{|\mathfrak{X}|}{|C(L_j)|}|\mathfrak{A}_{L_j}| .$$

Thus by Lemma 9.1 and (9.3)

$$(9.9) \qquad \begin{aligned}(\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} &= \frac{1}{|\mathfrak{X}|}\Sigma_j\frac{|\mathfrak{X}|}{|C(L_j)|}|\mathfrak{A}_{L_j}|\,\alpha(L_j) \\ &= \frac{1}{|\mathfrak{L}|}\Sigma_j\frac{|\mathfrak{L}|}{|C(L_j)\cap\mathfrak{L}|}\alpha(L_j) .\end{aligned}$$

By assumption $\mathfrak{C}_1 \cap \hat{\mathfrak{L}}^\sharp, \mathfrak{C}_2 \cap \hat{\mathfrak{L}}^\sharp, \cdots$ are the conjugate classes of $\mathfrak{L}$ which contain elements of $\hat{\mathfrak{L}}^\sharp$. Since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ this yields that

$$(\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} = \frac{1}{|\mathfrak{L}|}\Sigma_j\frac{|\mathfrak{L}|}{|C(L_j)\cap\mathfrak{L}|}\alpha(L_j) .$$

Therefore (9.9) implies the desired equality.

LEMMA 9.4. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$.*

Let θ be a generalized character of $\mathfrak{X}$ such that for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$, θ is constant on $\mathfrak{A}_L$. If $\alpha, \beta \in \mathscr{I}_0(\hat{\mathfrak{L}})$ and if α^τ, β^τ are defined by (9.4) *then*

$$(\alpha^\tau, \theta)_{\mathfrak{X}} = (\alpha, \theta_{|\mathfrak{L}})_{\mathfrak{L}}$$
$$(\alpha^\tau, \beta^\tau)_{\mathfrak{X}} = (\alpha, \beta)_{\mathfrak{L}} .$$

Proof. Since θ is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$ it follows from Lemma 9.1 that

$$\{\alpha\bar{\theta}_{|\mathfrak{L}}\}^\tau = \alpha^\tau\bar{\theta} .$$

Thus by Lemma 9.3

$$(\alpha^\tau, \theta)_{\mathfrak{X}} = (\alpha^\tau\bar{\theta}, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha\bar{\theta}_{|\mathfrak{L}}, 1_{\mathfrak{L}})_{\mathfrak{L}} = (\alpha, \theta_{|\mathfrak{L}})_{\mathfrak{L}} .$$

By Lemma 9.1 β^τ is a generalized character of $\mathfrak{X}$ which is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. If now θ is replaced by β^τ in the first equation of the lemma the second equation follows.

LEMMA 9.5. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. Let θ be a class function of $\mathfrak{X}$ which is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. Let $\mathfrak{X}_0$ be the set of all elements in $\mathfrak{X}$ which are conjugate to some element of $\mathfrak{A}_L$ with $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. Then*

$$\frac{1}{|\mathfrak{X}|}\Sigma_{\mathfrak{X}_0}\theta(X) = \frac{1}{|\mathfrak{L}|}\Sigma_{\hat{\mathfrak{L}}^\sharp}\theta(L) .$$

Proof. Define $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ by

$$\alpha(L) = \theta(L) \qquad \text{if } L \in \hat{\mathfrak{L}}^\sharp$$
$$\alpha(L) = 0 \qquad \text{if } L \in \hat{\mathfrak{L}} - \hat{\mathfrak{L}}^\sharp .$$

By Lemma 9.1

$$\alpha^\tau(X) = \theta(X) \qquad \text{if } X \in \mathfrak{X}_0$$
$$\alpha^\tau(X) = 0 \qquad \text{if } X \in \mathfrak{X} - \mathfrak{X}_0 .$$

Consequently Lemma 9.3 implies that

$$\frac{1}{|\mathfrak{X}|}\sum_{\mathfrak{X}_0}\Theta(X) = (\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} = \frac{1}{|\mathfrak{L}|}\sum_{\hat{\mathfrak{L}}^\sharp}\alpha(L)$$
$$= \frac{1}{|\mathfrak{L}|}\sum_{\hat{\mathfrak{L}}^\sharp}\Theta(L) .$$

Lemma 9.5 is of great importance. Even the special case in which $\Theta = 1_{\mathfrak{X}}$ is of considerable interest and plays a role in section 26. In this special case, Lemma 9.5 asserts simply that $|\mathfrak{X}_0|/|\mathfrak{X}| = |\hat{\mathfrak{L}}^\sharp|/|\mathfrak{L}|$.

10. Coherent Sets of Characters

Throughout this section let $\hat{\mathfrak{L}}$ be a tamely imbedded subset of the group $\mathfrak{X}$. Let $\mathfrak{L} = N(\hat{\mathfrak{L}})$ and let $\mathscr{I}(\hat{\mathfrak{L}})$ be the set of generalized characters of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$. Let τ be defined by (9.4).

DEFINITION 10.1. A set $\mathscr{S}$ of generalized characters of $\mathfrak{L}$ is *coherent* if and only if

(i) $\mathscr{I}_0(\mathscr{S}) \neq 0$.

(ii) It is possible to extend τ from $\mathscr{I}_0(\mathscr{S})$ to a linear isometry mapping $\mathscr{I}(\mathscr{S})$ into the set of generalized characters of $\mathfrak{X}$.

(iii) $\mathscr{I}_0(\mathscr{S}) \subseteq \mathscr{I}_0(\hat{\mathfrak{L}})$.

It is easily seen that if $\mathscr{S}$ is a coherent set and $\mathscr{T} \subseteq \mathscr{S}$ with $\mathscr{I}_0(\mathscr{T}) \neq 0$ then also $\mathscr{T}$ is a coherent set. It is more difficult to decide whether the union of two coherent subsets of $\mathscr{I}(\hat{\mathfrak{L}})$ is coherent. Examples are known in which $\mathscr{S}$ consists of irreducible characters of $\mathfrak{L}$ and is not coherent though $\mathscr{I}_0(\mathscr{S}) \neq 0$ [25]. In these examples $\hat{\mathfrak{L}}$ is even a T. I. set in $\mathfrak{X}$. The main purpose of this section is to give some sufficient conditions which ensure that a subset $\mathscr{S}$ of $\mathscr{I}(\hat{\mathfrak{L}})$ is coherent.

LEMMA 10.1. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. Let $\mathscr{S} = \{\lambda_i \mid 1 \leq i \leq n\}$ with $n \geq 2$. Assume that for $1 \leq i \leq n$, λ_i is an irreducible character of $\mathfrak{L}$. Furthermore $\lambda_i(L) = \lambda_1(L)$ for $L \in \mathfrak{L} - \hat{\mathfrak{L}}^\sharp$. Then $\mathscr{S}$ is coherent. Furthermore, if τ_1 and τ_2 are extensions of τ to $\mathscr{S}$ then either $\tau_1 = \tau_2$ or $|\mathscr{S}| = 2$ and $\lambda_i^{\tau_1} = -\lambda_{3-i}^{\tau_2}$, $i = 1, 2$.*

Proof. For $1 \leq i, j \leq n$ let $\alpha_{ij} = \lambda_i - \lambda_j$, then $\alpha_{ij} \in \mathscr{I}_0(\mathscr{S})$. Thus $\mathscr{I}_0(\mathscr{S}) \neq 0$ since $n \geq 2$. Furthermore α_{ij}^τ is defined. Since τ is an isometry this yields that

$$(10.1) \qquad (\alpha_{ij}^\tau, \alpha_{i'j'}^\tau) = (\alpha_{ij}, \alpha_{i'j'}) = \delta_{ii'} - \delta_{ji'} - \delta_{ij'} + \delta_{jj'} .$$

In particular (10.1) implies that if $i \neq j$ then $\|\alpha_{ij}^{\tau}\|^2 = 2$. By Lemma 9.1 $\alpha_{ij}^{\tau}(1) = 0$, therefore α_{ij}^{τ} is the difference of two irreducible characters of $\mathfrak{X}$.

If $n > 2$, then it follows from equation (10.1) that $(\alpha_{1i}^{\tau}, \alpha_{1j}^{\tau}) = 1$ if $1 < i, j$ and $i \neq j$. It is now a simple consequence of (10.1) that there exists a unique irreducible character of $\mathfrak{X}$ which is not orthogonal to any α_{1i}^{τ} for $2 \leqq i \leqq n$. Furthermore if Δ_1 is chosen to be plus or minus this character then it may be assumed that

$$(\alpha_{1i}^{\tau}, \Delta_1) = 1 \quad \text{for } 2 \leqq i \leqq n .$$

Now define Δ_i by

$$\alpha_{1i}^{\tau} = \Delta_1 - \Delta_i, \quad 2 \leqq i \leqq n .$$

This implies that

$$\alpha_{ij}^{\tau} = \Delta_i - \Delta_j .$$

Hence (10.1) yields that the generalized characters Δ_i, $1 \leqq i \leqq n$ are pairwise orthogonal and that they each have weight one. It is easily shown that a rational integral linear combination of the characters λ_i of degree zero is a rational integral linear combination of the generalized characters α_{1i}. Hence if $\mathscr{S}^{\tau}$ is the set of generalized characters Δ_i, $1 \leqq i \leqq n$, then the linear mapping sending λ_i into Δ_i is an isometry. Thus, $\mathscr{S}$ is coherent and the extension of τ to $\mathscr{S}$ is unique in this case.

If $n = 2$, define Δ_i for $i = 1, 2$ by $\alpha_{12}^{\tau} = \Delta_1 - \Delta_2$, where Δ_i has weight one. Any rational integral linear combination of λ_1 and λ_2 of degree zero is a multiple of α_{12}. Thus, if τ_1 is any extension of τ to $\mathscr{S}$, $\lambda_i^{\tau_1} = \Delta_i$ or $\lambda_i^{\tau_1} = -\Delta_{3-i}$ for $i = 1, 2$. The proof is complete.

Before proving the main result of this section, another definition is needed. The following notation is introduced temporarily.

Let $\mathscr{S}$ be a subset of $\mathscr{I}(\hat{\mathfrak{L}})$ which consists of pairwise orthogonal characters. If $\mathscr{S}_1 \subseteq \mathscr{S}$, let $x(\mathscr{S}_1)$ denote the smallest weight of any character in $\mathscr{S}_1$ of minimum degree. If $\mathscr{S}_1$ and $\mathscr{T}$ are coherent subsets of $\mathscr{S}$ and τ_1 and τ_2 are extensions of τ to $\mathscr{S}_1$ and $\mathscr{T}$ respectively, define

$$\mathscr{A}(\mathscr{S}_1, \tau_1; \mathscr{T}, \tau_2) = \{\alpha \mid$$

(i) $\alpha \in \mathscr{I}_0(\mathscr{S})$.
(ii) $\alpha^{\tau} = \Delta_1 + \Delta_2$, where
  (a) $\Delta_2 \in \mathscr{C}(\mathscr{T}^{\tau_2})$,
  (b) Δ_1 is not orthogonal to $\mathscr{I}_0(\mathscr{S}_1)^{\tau}$,
  (c) $\|\Delta_1\|^2 \leqq x(\mathscr{S}_1)\}$.

DEFINITION 10.2. Let $\mathscr{S}_1$ be a coherent subset of $\mathscr{S}$ and let τ^* be an extension of τ to $\mathscr{S}_1$. The *pair* $(\mathscr{S}_1, \tau^*)$ *is subcoherent in* $\mathscr{S}$ if the following conditions are satisfied: If $\mathscr{T}$ is any coherent subset of $\mathscr{S}$ which is orthogonal to $\mathscr{S}_1$ and if τ_1 and τ_2 are extensions of τ to $\mathscr{S}_1$ and $\mathscr{T}$ respectively, then

(i) $\mathscr{S}_1^{\tau_1}$ is orthogonal to $\mathscr{T}^{\tau_2}$.

(ii) If $\alpha \in \mathscr{A}(\mathscr{S}_1, \tau_1; \mathscr{T}, \tau_2)$, then α^τ is a sum of two generalized characters, one of which is orthogonal to $\mathscr{S}_1^{\tau^*}$ and the other is in $\pm\mathscr{S}_1^{\tau^*}$.

If $(\mathscr{S}_1, \tau^*)$ is subcoherent in $\mathscr{S}$, we also say that $\mathscr{S}_1$ is subcoherent in $\mathscr{S}$, which causes no confusion in case τ^* has been designated.

Hypothesis 10.1.

(i) $\hat{\mathfrak{L}}$ *is a tamely imbedded subset of the group* $\mathfrak{X}$.

(ii) *For* $1 \leqq i \leqq k$, $\mathscr{S}_i = \{\lambda_{is} \mid 1 \leqq s \leqq n_i\}$ *is a subset of* $\mathscr{I}(\hat{\mathfrak{L}})$.

(iii) $\mathscr{S} = \bigcup_{i=1}^{k} \mathscr{S}_i$ *consists of pairwise orthogonal characters.*

(iv) *For any* i *with* $1 \leqq i \leqq k$, $\mathscr{S}_i$ *is coherent with isometry* τ_i. $\mathscr{S}_i$ *is partitioned into sets* $\mathscr{S}_{ij}$ *such that each* $\mathscr{S}_{ij}$ *either consists of irreducible characters of the same degree and* $|\mathscr{S}_{ij}| \geqq 2$ *or* $(\mathscr{S}_{ij}, \tau_{ij})$ *is subcoherent in* $\mathscr{S}$ *where* $\tau_{ij} = \tau_i$ *on* $\mathscr{S}_{ij}$.

(v) *For* $1 \leqq i \leqq k, 1 \leqq s \leqq n_i$, *there exist integers* ℓ_{is} *such that*

$$1 = \ell_{11} \leqq \ell_{21} \leqq \cdots \leqq \ell_{k1}\,,$$
$$\lambda_{is}(1) = \ell_{is}\lambda_{11}(1),\ \ell_{i1} \mid \ell_{is}\,.$$

(vi) λ_{11} *is an irreducible character of* $\mathfrak{L}$.

(vii) *For any integer* m *with* $1 < m \leqq k$,

$$\sum_{i=1}^{m-1} \sum_{s=1}^{n_i} \frac{\ell_{is}^2}{\|\lambda_{is}\|^2} > 2\ell_{m1}\,. \tag{10.2}$$

THEOREM 10.1. *Suppose that Hypothesis* 10.1 *is satisfied. Then* $\mathscr{S}$ *is coherent. There is an extension* τ^* *of* τ *to* $\mathscr{I}(\mathscr{S})$ *such that either* τ^* *agrees with* τ_i *on* $\mathscr{S}_i$ *or* $\mathscr{S}_i = \{\lambda_1, \lambda_2\}$ *and* $\lambda_j^{\tau_i} = -\lambda_{3-j}^{\tau^*}$ *for* $j = 1, 2$.

Proof. The proof is by induction on k. If $k = 1$ the theorem follows by assumption.

It is easily seen that $\bigcup_{i=1}^{k-1} \mathscr{S}_i$ satisfies the assumption of the theorem. Hence by induction it may be assumed that $\bigcup_{i=1}^{k-1} \mathscr{S}_i$ is coherent. Let τ^* denote an extension of τ to $\bigcup_{i=1}^{k-1} \mathscr{S}_i$, with the property that for

$1 \leq i \leq k-1$, τ^* agrees with τ_i on $\mathscr{S}_i$, or $\mathscr{S}_i = \{\lambda_1, \lambda_2\}$ and $\lambda_j^{\tau^*} = -\lambda_{3-j}^{\tau_i}$, $j = 1, 2$.

Choose the notation so that λ_{k1} has minimum weight among the characters in $\mathscr{S}_k$ of degree $\ell_{k1}\lambda_{11}(1)$. Let $\mathscr{S}_{k1}$ be the subset $\mathscr{S}_{ki}$ which contains λ_{k1}. For $1 \leq s \leq n_k$ define

$$\beta_s = \ell_{ks}\lambda_{11} - \lambda_{ks} .$$

Thus $\beta_s \in \mathscr{I}_0(\mathscr{S})$ and β_s^τ is defined. Define the integer y by

$$(\lambda_{11}^{\tau^*}, \beta_1^\tau) = \ell_{k1} - y . \tag{10.3}$$

If $(i, t) \neq (1, 1)$ and $1 \leq i \leq k-1, 1 \leq t \leq n_i$, then by (10.3)

$$\begin{aligned}(\lambda_{it}^{\tau^*}, \beta_1^\tau) &= (\ell_{it}\lambda_{11}^{\tau^*}, \beta_1^\tau) - (\ell_{it}\lambda_{11}^{\tau^*} - \lambda_{it}^{\tau^*}, \beta_1^\tau) \\ &= \ell_{it}(\ell_{k1} - y) - \ell_{it}\ell_{k1} = -y\ell_{it} .\end{aligned} \tag{10.4}$$

Since λ_{11} is irreducible and τ is an isometry on $\mathscr{I}_0(\mathscr{S})$

$$\| \beta_s^\tau \|^2 = \ell_{ks}^2 + \| \lambda_{ks} \|^2 \quad \text{for } 1 \leq s \leq n_k . \tag{10.5}$$

By (10.4)

$$\beta_1^\tau = \ell_{k1}\lambda_{11}^{\tau^*} - y \sum_{i=1}^{k-1} \sum_{s=1}^{n_i} \frac{\ell_{is}}{\| \lambda_{is} \|^2} \lambda_{is}^{\tau^*} + \Delta \tag{10.6}$$

where $(\Delta, \lambda_{is}^{\tau^*}) = 0$ for $1 \leq i \leq k-1, 1 \leq s \leq n_i$. Equations (10.5) and (10.6) now yield that

$$\ell_{k1}^2 - 2\ell_{k1}y + y^2 \sum_{i=1}^{k-1} \sum_{s=1}^{n_i} \frac{\ell_{is}^2}{\| \lambda_{is} \|^2} + \| \Delta \|^2 = \ell_{k1}^2 + \| \lambda_{k1} \|^2 . \tag{10.7}$$

If $y \neq 0$ then since y is an integer (10.2) and (10.7) imply that

$$0 \leq 2\ell_{k1}(y^2 - y) < \| \lambda_{k1} \|^2 - \| \Delta \|^2 .$$

Therefore

$$\| \Delta \|^2 < \| \lambda_{k1} \|^2 \quad \text{if } y \neq 0 . \tag{10.8}$$

We will show that $y = 0$. By Hypothesis 10.1 (iv), τ can be extended from $\mathscr{I}_0(\mathscr{S}_k)$ to a linear isometry τ_k on $\mathscr{I}(\mathscr{S}_k)$. For $1 \leq s \leq n_k$ let Δ_s be the image of λ_{ks} under this extension. If $(\mathscr{S}_{kj}, \tau_{kj})$ is subcoherent in $\mathscr{S}$, then $\mathscr{S}_{kj}^{\tau_{kj}}$ is orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$. Suppose that $\mathscr{S}_{kj}$ consists of irreducible characters of the same degree. If $\mathscr{S}_{kj}^{\tau_{kj}}$ is not orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$, then there exists $\lambda \in \mathscr{S}_{kj}$ and $\lambda_1 \in \mathscr{S}_{im}$ for some i and m with $1 \leq i \leq k-1$, such that $(\lambda^{\tau_{kj}}, \lambda_1^{\tau^*}) \neq 0$. Assume first that $\mathscr{S}_{im}$ consists of irreducible characters of the same degree.

Then it may be assumed that $\lambda = \lambda_{kt}$, $\lambda_{kt'} \in \mathscr{S}_{kj}$, $\lambda_{kt} \neq \lambda_{kt'}$ and $\lambda_1 = \lambda_{is}$, $\lambda_{is'} \in \mathscr{S}_{im}$, $\lambda_{is} \neq \lambda_{is'}$. Thus $\lambda_{kt}^{\tau kj} = \varepsilon\lambda_{is}^{\tau *}$ for suitable $\varepsilon = \pm 1$. Hence

$$0 = (\lambda_{is}^{\tau *} - \lambda_{is'}^{\tau *}, \lambda_{kt}^{\tau k} - \lambda_{kt'}^{\tau k}) = \varepsilon + (\lambda_{is'}^{\tau *}, \lambda_{kt'}^{\tau k}) .$$

Hence $\lambda_{is'}^{\tau *} = -\varepsilon\lambda_{kt'}^{\tau k}$. Therefore

$$0 = (\lambda_{is}^{\tau *} - \lambda_{is'}^{\tau *})(1) = \varepsilon(\lambda_{kt}^{\tau kj} + \lambda_{kt'}^{\tau kj})(1) = 2\varepsilon\lambda_{kt}^{\tau kj}(1) ,$$

which is not the case as $\| \lambda^{\tau kj} \|^2 = 1$. Suppose now that $\mathscr{S}_{im}$ is subcoherent in $\mathscr{S}$. Then $\mathscr{S}_{im}^{\tau *}$ is orthogonal to $\mathscr{S}_k^{\tau k}$ by definition. Therefore, for $2 \leqq s \leqq n_k$,

$$(10.9) \qquad \left(\Delta, \frac{d_{ks}}{d_{k1}}\Delta_1 - \Delta_s\right) = \left(\beta_1^\tau, \frac{d_{ks}}{d_{k1}}\Delta_1 - \Delta_s\right) = -\frac{d_{ks}}{d_{k1}}\| \lambda_{k1} \|^2 .$$

Thus, Δ is not orthogonal to $\mathscr{I}_0(\mathscr{S}_{k1})^\tau$. If $\mathscr{S}_{k1}$ consists of irreducible characters this yields that $\| \Delta \|^2 \geqq 1$. Hence, $y = 0$ by (10.8). Suppose that $(\mathscr{S}_{k1}, \tau_{k1})$ is subcoherent in $\mathscr{S}$. If $y \neq 0$, (10.8) implies that

$$(10.10) \qquad \beta_1^\tau = \Delta_1 + \Delta$$

where $\Delta \in \pm\mathscr{S}_{k1}^{\tau k1}$ and Δ_1 is orthogonal to $\mathscr{S}_{k1}^{\tau k1}$. By changing notation if necessary it may be assumed that

$$(10.11) \qquad \Delta = \pm\Delta_1$$

by (10.9). Now (10.9), (10.10) and (10.11) yield that

$$(10.12) \qquad \| \lambda_{k1} \|^4 = |(\Delta, \Delta)|^2 \leqq \| \Delta \|^2 \cdot \| \lambda_{k1} \|^2 .$$

Hence, (10.8) and (10.12) imply that $y = 0$ in all cases. Thus, (10.3) becomes

$$(10.13) \qquad (\lambda_{11}^{\tau *}, \beta_1^\tau) = d_{k1} .$$

For $1 \leqq s \leqq n_k$,

$$\beta_s = \frac{d_{ks}}{d_{k1}}\beta_1 + \left(\frac{d_{ks}}{d_{k1}}\lambda_{k1} - \lambda_{ks}\right) .$$

Therefore, (10.13) implies that

$$(10.14) \qquad (\lambda_{11}^{\tau *}, \beta_s^\tau) = d_{ks} , \qquad 1 \leqq s \leqq n_k .$$

For $1 \leqq s \leqq n_k$, define $\lambda_{ks}^{\tau *}$ by

$$(10.15) \qquad \beta_s^\tau = d_{ks}\lambda_{11}^{\tau *} - \lambda_{ks}^{\tau *} ,$$

and extend the definition to $\mathscr{I}(\mathscr{S})$ by linearity. This implies that $\lambda_{ks}^{\tau *} = \lambda_{ks}^{\tau k}$ or $\mathscr{S}_k = \{\lambda_1, \lambda_2\}$ and $\lambda_i^{\tau *} = -\lambda_{3-i}^{\tau k}$ for $i = 1, 2$. Hence, $\mathscr{S}_k^{\tau *}$ is orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$ and thus τ^* is an isometry on $\mathscr{I}(\mathscr{S})$. The proof is complete.

If $\mathscr{S}$ is a coherent subset of $\mathscr{I}(\hat{\mathfrak{L}})$, then τ will be used to denote an extension of τ to $\mathscr{I}(\mathscr{S})$.

Hypothesis 10.2.

(i) *$\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ and $\mathfrak{H}_i$ is a supporting subgroup of $\hat{\mathfrak{L}}$. $\mathfrak{N}_i = N(\mathfrak{H}_i)$.*

(ii) *If θ is any non-principal irreducible character of $\mathfrak{H}_i$ and $\tilde{\theta}$ is the character of $\mathfrak{N}_i$ induced by θ, then $\tilde{\theta}$ is a sum of irreducible characters of $\mathfrak{N}_i$, all of which have the same degree and occur with the same multiplicity in $\tilde{\theta}$.*

LEMMA 10.2. *Suppose that Hypothesis 10.2 is satisfied. For any character α of $\mathfrak{H}_i$ let $\mathscr{S}_\alpha$ be the set of irreducible characters of $\mathfrak{N}_i$ whose restriction to $\mathfrak{H}_i$ coincides with α. If Θ is a generalized character of $\mathfrak{X}$ which is orthogonal to $\mathscr{I}_0(\mathscr{S}_\alpha)^*$ for all α with $(\alpha, 1_{\mathfrak{H}_i}) = 0$ then Θ is constant on the cosets of $\mathfrak{H}_i$ which lie in $\mathfrak{N}_i - \mathfrak{H}_i$.*

Proof. We first remark that by Lemma 4.3 characters in $\mathscr{S}_\alpha$ vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$, and so generalized characters in $\mathscr{I}_0(\mathscr{S}_\alpha)$ vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i$. Suppose that θ_1, θ_2 are distinct characters in $\mathscr{S}_\alpha$ with $(\alpha, 1_{\mathfrak{H}_i}) = 0$. By assumption $(\Theta, (\theta_1 - \theta_2)^*) = 0$. Thus by the Frobenius reciprocity theorem $(\Theta_{|\mathfrak{N}_i}, \theta_1 - \theta_2) = 0$. Hence by Hypothesis 10.2 $\Theta_{|\mathfrak{N}_i} = \tilde{\gamma} + \beta$, where $\tilde{\gamma}$ is a class function of $\mathfrak{N}_i$ induced by a class function γ of $\mathfrak{H}_i$ and β is a generalized character of $\mathfrak{N}_i/\mathfrak{H}_i$. Thus $\Theta(N) = \beta(N)$ for $N \in \mathfrak{N}_i - \mathfrak{H}_i$. The proof is complete.

LEMMA 10.3. *Suppose that Hypothesis 10.2 is satisfied. Let $\mathscr{S}$ be a coherent subset of $\mathscr{I}(\hat{\mathfrak{L}})$ which consists of pairwise orthogonal characters of $\mathfrak{L}$. Assume further that $\mathscr{S}$ contains at least two irreducible characters. Then if $\lambda \in \mathscr{S}$, λ^τ is constant on the cosets of $\mathfrak{H}_i$ which lie in $\mathfrak{N}_i - \mathfrak{H}_i$.*

Proof. Suppose that θ_1, θ_2 are distinct irreducible characters of $\mathfrak{N}_i$ which do not contain $\mathfrak{H}_i$ in their kernel such that $\theta_{1|\mathfrak{H}_i} = \theta_{2|\mathfrak{H}_i}$. We will show that

$$(10.16) \qquad (\lambda^\tau{}_{|\mathfrak{N}_i}, \theta_1 - \theta_2) = 0 .$$

By Lemma 4.3 θ_1 and θ_2 vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$. Since $\hat{\mathfrak{N}}_i$ is a T. I. set in $\mathfrak{X}$ and $\mathfrak{N}_i = N(\hat{\mathfrak{N}}_i)$ the mapping sending $\theta_1 - \theta_2$ into

$(\theta_1 - \theta_2)^*$ defines an isometry on $\mathscr{I}_0(\{\theta_1, \theta_2\})$. By Lemma 10.1 this can be extended to an isometry of $\mathscr{I}(\{\theta_1, \theta_2\})$. Let Θ_1, Θ_2 be the respective images of θ_1, θ_2 under this isometry. By assumption $\mathscr{S}$ contains two irreducible characters λ_1 and λ_2. Since

$$\lambda_j(1)\lambda - \lambda(1)\lambda_j \in \mathscr{I}_0(\mathscr{S})$$

for $j = 1, 2$, Lemma 9.2 implies that if (10.16) is violated then

$$(\lambda^\tau_{j\,|\mathfrak{N}_4}, \theta_1 - \theta_2) \neq 0 \quad \text{for } j = 1, 2 \, .$$

Thus by the Frobenius reciprocity theorem

$$(\lambda^\tau_j, \Theta_1 - \Theta_2) = (\lambda^\tau_j, (\theta_1 - \theta_2)^*) \neq 0 \quad \text{for } j = 1, 2 \, .$$

Thus by changing notation if necessary it may be assumed that $\lambda^\tau_j = \pm\Theta_j$ for $j = 1, 2$, where the sign is independent of j. Hence

$$(10.17) \qquad (\lambda_1(1)\lambda^\tau_2 - \lambda_2(1)\lambda^\tau_1, \Theta_1 - \Theta_2) = \pm(\lambda_1(1) + \lambda_2(1)) \neq 0 \, .$$

Since $\lambda_1(1)\lambda_2 - \lambda_2(1)\lambda_1 \in \mathscr{I}_0(\mathscr{S})$ Lemma 9.2 implies that

$$((\lambda_1(1)\lambda^\tau_2 - \lambda_2(1)\lambda^\tau_1)_{|\mathfrak{N}_4}, \theta_1 - \theta_2) = 0 \, .$$

Thus by the Frobenius reciprocity theorem

$$(\lambda_1(1)\lambda^\tau_2 - \lambda_2(1)\lambda^\tau_1, \Theta_1 - \Theta_2) = (\lambda_1(1)\lambda^\tau_2 - \lambda_2(1)\lambda^\tau_1, (\theta_1 - \theta_2)^*) = 0$$

contrary to (10.17). Therefore (10.16) must hold. The result now follows from Lemma 10.2.

LEMMA 10.4. *Suppose that the assumptions of Lemma* 10.3 *are satisfied. Let a be the least common multiple of all the orders of elements in $\hat{\mathfrak{L}}$. If λ is an irreducible character in $\mathscr{S}$, then $\mathscr{Q}_a$ contains all the values assumed by λ^τ.*

Proof. By assumption $\mathscr{S}$ contains another irreducible character λ_1. Let σ be any automorphism of $\mathscr{Q}_{|\mathfrak{X}|}$ whose fixed field contains $\mathscr{Q}_a$. Then since $\lambda_1(1)\lambda - \lambda(1)\lambda_1 \in \mathscr{I}_0(\mathscr{S})$ it follows directly from (9.4) that

$$\begin{aligned}\sigma[\{\lambda_1(1)\lambda - \lambda(1)\lambda_1\}^\tau] &= \{\lambda_1(1)\sigma(\lambda) - \lambda(1)\sigma(\lambda_1)\}^\tau \\ &= \{\lambda_1(1)\lambda - \lambda(1)\lambda_1\}^\tau \, .\end{aligned}$$

Therefore

$$\lambda_1(1)\sigma(\lambda^\tau) - \lambda(1)\sigma(\lambda^\tau_1) = \lambda_1(1)\lambda^\tau - \lambda(1)\lambda^\tau_1 \, .$$

As $\|\lambda^\tau\|^2 = \|\lambda^\tau_1\|^2 = 1$, this implies that $\sigma(\lambda^\tau) = \lambda^\tau$. As σ may be an arbitrary automorphism of $\mathscr{Q}_{|\mathfrak{X}|}$ whose fixed field contains $\mathscr{Q}_a$ the result is proved.

LEMMA 10.5. *Suppose that* $\hat{\mathfrak{L}}$ *is a tamely imbedded subset of* $\mathfrak{X}$. *Let* $\mathfrak{A}_L$ *have the same meaning as in* (9.2) *and let* Θ *be a generalized character of* $\mathfrak{X}$ *which is constant on* $\mathfrak{A}_L$ *for* $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. *Let* $\mathscr{S}$ *be a coherent subset of* $\mathscr{S}(\hat{\mathfrak{L}})$ *consisting of irreducible characters. Then there exist rational numbers* b, c, *and generalized characters* β, γ *of* $\mathfrak{L}$ *which are orthogonal to* $\mathscr{S}$ *such that if* $L \in \hat{\mathfrak{L}}^{\#}$ *then* $\Theta(L) = b\beta(L)$ *if* Θ *is orthogonal to* $\mathscr{S}^{\tau}$, *and* $\lambda^{\tau}(L) = \lambda(L) + c\gamma(L)$ *if* $\Theta = \lambda^{\tau} \in \mathscr{S}^{\tau}$.

Proof. It is an immediate consequence of Lemma 9.4 that if Θ is orthogonal to $\mathscr{S}^{\tau}$ and if $\xi = \Sigma_i \lambda_i(1)\lambda_i$, where λ_i ranges over $\mathscr{S}$, then

$$\Theta(L) = b_1\xi(L) + b_2\beta_1(L) \quad \text{for } L \in \hat{\mathfrak{L}}^{\#} \tag{10.18}$$

where b_1, b_2 are rational numbers and β_1 is a generalized character of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$. If $\Theta = \lambda^{\tau}$, then Lemma 9.4 yields that

$$\lambda^{\tau}(L) = \lambda(L) + c_1\xi(L) + c_2\gamma_1(L) \quad \text{for } L \in \hat{\mathfrak{L}}^{\#} \tag{19.19}$$

where c_1, c_2 are rational numbers and γ_1 is a generalized character of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$. There exists a generalized character ξ' of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$ such that

$$\xi + \xi' = \rho_{\mathfrak{L}} .$$

Since $\rho_{\mathfrak{L}}(L) = 0$ for $L \in \mathfrak{L}^{\#}$ (10.18) and (10.19) imply respectively that

$$\Theta(L) = -b_1\xi'(L) + b_2\beta_1(L)$$
$$\lambda^{\tau}(L) = \lambda(L) - c_1\xi'(L) + c_2\gamma_1(L) .$$

The lemma follows by a suitable change in notation.

It is worth noting that if the hypotheses of Lemma 10.3 are satisfied for every subgroup in a system of supporting subgroups of $\hat{\mathfrak{L}}$, then that lemma implies that λ^{τ} satisfies the hypotheses of Lemma 10.5. This fact will be used later in this paper.

11. Some Applications of Theorem 10.1

In this section we are concerned with the problem of finding conditions under which it is possible to apply Theorem 10.1. That theorem will then allow us to conclude that certain sets of characters are coherent. To clarify matters the main Hypothesis is stated separately. This also serves to introduce the notation.

Hypothesis 11.1.

(i) $\hat{\mathfrak{L}}_0$ *is a tamely imbedded subset of the group* $\mathfrak{X}$ *and* $\mathfrak{L}_0 = N(\hat{\mathfrak{L}}_0)$ *has odd order.* $\mathfrak{H}_0 \lhd \mathfrak{L}_0$ *and* $\hat{\mathfrak{L}}_0$ *is a union of cosets of* $\mathfrak{H}_0$. *Let* $\mathfrak{L} = \mathfrak{L}_0/\mathfrak{H}_0$ *and let* $\hat{\mathfrak{L}}$ *be the image of* $\hat{\mathfrak{L}}_0$ *in* $\mathfrak{L}$.

(ii) $\mathfrak{H}$ *and* $\mathfrak{K}$ *are normal subgroups of* $\mathfrak{L}$ *such that* $\mathfrak{H}$ *is nilpotent and*

$$(11.1) \qquad \mathfrak{H} \subseteq \bigcup_{H \in \mathfrak{H}^\#} C(H) \cap \mathfrak{K} \subseteq \hat{\mathfrak{L}} \subseteq \mathfrak{K} \subseteq \mathfrak{L} .$$

(iii) $\mathscr{S}$ *is the set of all characters of* $\mathfrak{L}$ *which are induced by non principal irreducible characters of* $\mathfrak{K}$, *each of which vanishes outside* $\hat{\mathfrak{L}}$. *Then* $\mathscr{S}$ *consists of pairwise orthogonal characters.*

(iv) *There exists an integer d such that* $d\,|\mathfrak{L}:\mathfrak{K}| \mid \lambda(1)$ *for* $\lambda \in \mathscr{S}$. *Furthermore* $\mathscr{S}$ *contains an irreducible character of degree* $d\,|\mathfrak{L}:\mathfrak{K}|$.

(v) *Define an equivalence relation on* $\mathscr{S}$ *by the condition that two characters in* $\mathscr{S}$ *are equivalent if and only if they have the same degree and the same weight. Then each equivalence class of* $\mathscr{S}$ *is either subcoherent in* $\mathscr{S}$ *or consists of irreducible characters.*

(vi) *For any subgroup* $\mathfrak{A}$ *of* $\mathfrak{H}$ *which is normal in* $\mathfrak{L}$ *let* $\mathscr{S}(\mathfrak{A})$ *be the subset of* $\mathscr{S}$ *consisting of those characters which are equivalent to some character in* $\mathscr{S}$ *that has* $\mathfrak{A}$ *in its kernel.*

In the application to the main theorem of this paper (11.1) will always be augmented by one of the following conditions.

$$(11.2) \qquad \mathfrak{H} = \hat{\mathfrak{L}} = \mathfrak{K} \subset \mathfrak{L} .$$

$$(11.3) \qquad \mathfrak{H} \subset \hat{\mathfrak{L}} = \mathfrak{K} \subset \mathfrak{L} .$$

$$(11.4) \qquad \mathfrak{H} \subseteq \bigcup_{H \in \mathfrak{H}^\#} C(H) \cap \mathfrak{K} = \hat{\mathfrak{L}} \subseteq \mathfrak{K} \subseteq \mathfrak{L} .$$

THEOREM 11.1. *Suppose that Hypothesis* 11.1 *is satisfied. Let* $\mathfrak{H}_1$ *be a normal subgroup of* $\mathfrak{L}$ *which is contained in* $\mathfrak{H}$ *such that*

$$(11.5) \qquad |\mathfrak{H}:\mathfrak{H}_1| > 4d^2\,|\mathfrak{L}:\mathfrak{K}|^2 + 1 .$$

If $\mathscr{S}(\mathfrak{H}_1)$ *is coherent and contains an irreducible character of degree* $d\,|\mathfrak{L}:\mathfrak{K}|$ *then* $\mathscr{S}$ *is coherent.*

Proof. Let $\mathfrak{H}_2$ be a normal subgroup of $\mathfrak{L}$ which is contained in $\mathfrak{H}_1$ and is minimal with the property that $\mathscr{S}(\mathfrak{H}_2)$ is coherent. Suppose that $\mathfrak{H}_2 \neq \langle 1 \rangle$. Choose $\mathfrak{H}_3 \subset \mathfrak{H}_2$ such that $\mathfrak{H}_2/\mathfrak{H}_3$ is a chief factor of $\mathfrak{L}$. Let $\mathscr{S}(\mathfrak{H}_2) = \mathscr{S}_1 = \{\lambda_{1s} \mid 1 \leqq s \leqq n_1\}$, where λ_{11} is irreducible and $\lambda_{11}(1) = d\,|\mathfrak{L}:\mathfrak{K}|$. Let $\mathscr{S}_2, \cdots \mathscr{S}_k$ be the subsets of $\mathscr{S}(\mathfrak{H}_3) - \mathscr{S}(\mathfrak{H}_2)$ consisting of all characters of a given weight and a given degree. For $2 \leqq i \leqq k$ let $\ell_{i1}\lambda_{11}(1)$ be the common degree of the characters in $\mathscr{S}_i$.

By Hypothesis 11.1 all the assumptions of Theorem 10.1, except possibly inequality (10.2), are satisfied for $\mathscr{S}(\mathfrak{H}_3)$. We will now verify that also inequality (10.2) is satisfied.

Let $\theta_1, \theta_2, \cdots$ be all the irreducible characters of $\mathfrak{K}$ which do not have $\mathfrak{H}$ in their kernel. Let $\tilde{\theta}_j$ denote the character of $\mathfrak{L}$ induced by θ_j. Then each $\tilde{\theta}_j$ is in $\mathscr{S}$ by Lemma 4.3. Furthermore if θ_j ranges only over characters of $\mathfrak{K}/\mathfrak{H}_2$ then

$$\Sigma\theta_j(1)\theta_j = \rho_{\mathfrak{K}/\mathfrak{H}_2} - \rho_{\mathfrak{K}/\mathfrak{H}} .$$

Therefore

$$\Sigma\theta_j(1)\tilde{\theta}_j = \rho_{\mathfrak{L}/\mathfrak{H}_2} - \rho_{\mathfrak{L}/\mathfrak{H}} . \tag{11.6}$$

If $\tilde{\theta}_i \neq \tilde{\theta}_j$ then $(\tilde{\theta}_i, \tilde{\theta}_j) = 0$. Suppose that for a given j there are a_j values of i such that $\tilde{\theta}_j = \tilde{\theta}_i$. Then (11.6) implies that

$$\Sigma\{\theta_j(1)a_j\}^2 \, ||\tilde{\theta}_j||^2 = |\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}| \tag{11.7}$$

where the summation in (11.7) ranges over the distinct ones among the $\tilde{\theta}_j$. Since

$$\{\theta_j(1)a_j\}^2 \, ||\tilde{\theta}_j||^2 = \theta_j(1)^2 \, |\mathfrak{L}:\mathfrak{K}| \, a_j = \tilde{\theta}_j(1)\theta_j(1)a_j = \frac{\tilde{\theta}_j(1)^2}{||\tilde{\theta}_j||^2}$$

(11.7) yields that

$$\sum_s \frac{\lambda_{1s}(1)^2}{||\lambda_{1s}||^2} \geqq |\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}| ,$$

where $\mathscr{S}_1 = \{\lambda_{1s}\}$ or equivalently

$$\sum_s \frac{x_{1s}^2}{||\lambda_{1s}||^2} \geqq \frac{|\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}|}{d^2 \, |\mathfrak{L}:\mathfrak{K}|^2} . \tag{11.8}$$

Since $\mathfrak{H}$ is nilpotent $\mathfrak{H}_2/\mathfrak{H}_3$ is in the center of $\mathfrak{H}/\mathfrak{H}_3$. Every irreducible character of $\mathfrak{K}$ is a constituent of a character induced by an irreducible character of $\mathfrak{H}$. Thus for $2 \leqq m \leqq k$, Lemma 4.1 implies that

$$x_{m1} d \, |\mathfrak{L}:\mathfrak{K}| \leqq \sqrt{|\mathfrak{H}:\mathfrak{H}_2|} \, |\mathfrak{L}:\mathfrak{H}| ,$$

or equivalently

$$x_{m1} \leqq \frac{|\mathfrak{K}:\mathfrak{H}| \sqrt{|\mathfrak{H}:\mathfrak{H}_2|}}{d} . \tag{11.9}$$

Suppose now that inequality (10.2) is violated for some value of m. Then (11.8) and (11.9) yield that

$$\frac{|\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}|}{d^2 \, |\mathfrak{L}:\mathfrak{K}|^2} \leqq \frac{2 \, |\mathfrak{K}:\mathfrak{H}| \sqrt{|\mathfrak{H}:\mathfrak{H}_2|}}{d} .$$

Thus

$$\{|\mathfrak{H}:\mathfrak{H}_2| - 1\} \leqq 2d\,|\mathfrak{L}:\mathfrak{K}|\sqrt{|\mathfrak{H}:\mathfrak{H}_2|}\,,$$

or

$$|\mathfrak{H}:\mathfrak{H}_2|^2 - 2\,|\mathfrak{H}:\mathfrak{H}_2| + 1 \leqq 4d^2\,|\mathfrak{L}:\mathfrak{K}|^2\,|\mathfrak{H}:\mathfrak{H}_2|\,.$$

Since every term is an integer this implies that

$$(11.10) \qquad |\mathfrak{H}:\mathfrak{H}_2| - 1 \leqq 4d^2\,|\mathfrak{L}:\mathfrak{K}|^2\,.$$

However $\mathfrak{H}_2 \subseteqq \mathfrak{H}_1$, thus $|\mathfrak{H}:\mathfrak{H}_2| \geqq |\mathfrak{H}:\mathfrak{H}_1|$. Now (11.5) and (11.10) are incompatible. Therefore inequality (10.2), and thus all the assumptions of Theorem 10.1, are satisfied. Hence by that theorem $\mathscr{S}(\mathfrak{H}_3)$ is coherent contrary to the minimal nature of $\mathfrak{H}_2$. This finally implies that $\mathfrak{H}_2 = \langle 1 \rangle$. Therefore $\mathscr{S} = \mathscr{S}(\mathfrak{H}_2)$ is coherent. The proof is complete.

The remainder of this section consists of applications of Theorem 11.1. Lemmas 11.1 and 11.2 are closely related to Theorem 2 of [8]. By using the argument of that theorem the assumption that $|\mathfrak{L}|$ is odd in the following lemmas can be replaced by suitable weaker assumptions. However the stronger results are not relevant to this paper and will not be proved here.

Hypothesis 11.2.

(i) *Hypothesis* 11.1 *and equation* (11.2) *are satisfied. Thus* $d = 1$.

(ii) $|\mathfrak{L}|$ *is odd and* $\mathfrak{L}/\mathfrak{H}'$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}/\mathfrak{H}'$.

LEMMA 11.1. *Suppose that Hypothesis* 11.2 *is satisfied. If*

$$|\mathfrak{H}:\mathfrak{H}'| > 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

then $\mathscr{S}$ *is coherent.*

Proof. By Lemma 10.1 and 3.16 (iii) $\mathscr{S}(\mathfrak{H}')$ is coherent. The result now follows from Theorem 11.1.

LEMMA 11.2. *Suppose that Hypothesis* 11.2 *is satisfied. Then* $\mathscr{S}$ *is coherent except possibly if* $\mathfrak{H}$ *is a non abelian p-group for some prime p and*

$$|\mathfrak{H}:\mathfrak{H}'| \leqq 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1\,.$$

Proof. If $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are proper normal subgroups of $\mathfrak{L}$, then

$$|\mathfrak{H}_i:\mathfrak{H}_i'| \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} \quad \text{for } i = 1, 2 .$$

Since $|\mathfrak{L}|$ is odd, this implies that

$$|\mathfrak{H}_i:\mathfrak{H}_i'| \geqq 2\,|\mathfrak{L}:\mathfrak{H}| + 1 \quad \text{for } i = 1, 2 .$$

Hence $|\mathfrak{H}:\mathfrak{H}'| > 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1$ and $\mathscr{S}$ is coherent by Lemma 11.1. As $\mathfrak{H}$ is nilpotent this implies that $\mathscr{S}$ is coherent if $\mathfrak{H}$ is not a p-group for any prime p. Since $|\mathfrak{L}|$ is odd

$$|\mathscr{S}| \geqq \frac{|\mathfrak{H}:\mathfrak{H}'|-1}{|\mathfrak{L}:\mathfrak{H}|} \geqq 2 .$$

Thus by Lemma 10.1 $\mathscr{S}$ is coherent if $\mathfrak{H}$ is abelian. The result now follows directly from Lemma 11.1.

LEMMA 11.3. *Suppose that Hypothesis* 11.2 *is satisfied and* $\mathfrak{L}$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}$. *Assume that* $\mathfrak{H}$ *is a p-group for some prime p and* $|\mathfrak{H}:D(\mathfrak{H})| = p^2$. *Then* $\mathscr{S}$ *is coherent.*

Proof. If $\mathfrak{H}$ is abelian Lemma 11.2 implies that $\mathscr{S}$ is coherent. If $\mathfrak{H}$ is not abelian then the second term of the descending central series modulo the third is cyclic. Thus

$$p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

Therefore $(p-1) \geqq 2\,|\mathfrak{L}:\mathfrak{H}|$ as $|\mathfrak{L}|$ is odd. Hence

$$|\mathfrak{H}:\mathfrak{H}'| \geqq p^2 > 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

and the result follows from Lemma 11.1.

LEMMA 11.4. *Suppose that Hypothesis* 11.2 *is satisfied and* $\mathfrak{L}$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}$. *Assume that* $\mathfrak{H}$ *is a p-group for some prime p and* $|\mathfrak{H}:D(\mathfrak{H})| = p^3$. *If*

(11.11) $$p^3 - 1 > 2p\,|\mathfrak{L}:\mathfrak{H}|$$

then $\mathscr{S}$ *is coherent.*

Proof. If $\mathfrak{H}$ is abelian Lemma 11.2 implies that $\mathscr{S}$ is coherent. If $\mathfrak{H}$ is non-abelian let $\mathfrak{H}_1$ be a subgroup of $D(\mathfrak{H})$ such that $D(\mathfrak{H})/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$. As $\mathfrak{H}$ is nilpotent $D(\mathfrak{H})/\mathfrak{H}_1$ is in the center of $\mathfrak{H}/\mathfrak{H}_1$. Thus by Lemma 4.1 the degree of any irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ is either 1 or p. Hence the degree of any character in $\mathscr{S}(\mathfrak{H}_1)$

is either $|\mathfrak{L}:\mathfrak{H}|$ or $|\mathfrak{L}:\mathfrak{H}|p$. Let $\mathscr{S}_1, \mathscr{S}_2$ be the subsets of $\mathscr{S}(\mathfrak{H}_1)$ consisting of all the characters of degree $|\mathfrak{L}:\mathfrak{H}|, |\mathfrak{L}:\mathfrak{H}|p$ respectively. Let $\ell_1 = 1, \ell_2 = p$. By (11.11)

$$|\mathscr{S}_1| \geqq \frac{p^3 - 1}{|\mathfrak{L}:\mathfrak{H}|} > 2p = 2\ell_2 .$$

Thus by Theorem 10.1 $\mathscr{S}(\mathfrak{H}_1)$ is coherent.

If $|D(\mathfrak{H}):\mathfrak{H}_1| = p$ or p^2, then $p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|}$ or

$$p^2 - 1 \equiv 0 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

As $(p^3 - 1, p^2 - 1) = p - 1$ this yields that in either case

$$p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

Therefore $p - 1 \geqq 2|\mathfrak{L}:\mathfrak{H}|$. Hence

$$|\mathfrak{H}:\mathfrak{H}'| \geqq |\mathfrak{H}:D(\mathfrak{H})| = p^3 > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

and $\mathscr{S}$ is coherent by Lemma 11.1. Suppose that $|D(\mathfrak{H}):\mathfrak{H}_1| \geqq p^3$. Then by (11.11)

$$|\mathfrak{H}:\mathfrak{H}_1| \geqq p^6 > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1 .$$

Since $\mathscr{S}(\mathfrak{H}_1)$ is coherent the result now follows from Theorem 11.1.

The next two lemmas involve the following situation:

Hypothesis 11.3.

(i) *Hypothesis* 11.2 *is satisfied.*

(ii) *There exist primes p, q and positive integers a, b such that* $|\mathfrak{L}:\mathfrak{H}| = p^b, |\mathfrak{H}:\mathfrak{H}'| = |\mathfrak{H}:D(\mathfrak{H})| = q^a$. *Thus $|\mathfrak{H}|$ is a power of q.*

LEMMA 11.5. *Suppose that Hypothesis* 11.3 *is satisfied and $a = 2c$ is even. Then $\mathscr{S}$ is coherent except possibly if $q^c + 1 = 2p^b$, q^c is the smallest degree of any non linear irreducible character of $\mathfrak{H}$ whose kernel contains $[\mathfrak{H}, \mathfrak{H}']$ and for no subgroup $\mathfrak{H}_1$ of $\mathfrak{H}'$ with $\mathfrak{H}_1 \neq \mathfrak{H}'$, $\mathfrak{H}_1 \lhd \mathfrak{L}$ is $\mathfrak{L}/\mathfrak{H}_1$ a Frobenius group.*

Proof. Suppose that $\mathscr{S}$ is not coherent. Then by Lemma 11.1 $4p^{2b} + 1 \geqq q^a$. As $(q^c + 1, q^c - 1) = 2$ it follows that $2p^b \mid q^c + 1$ or $2p^b \mid q^c - 1$. If $2p^b \neq q^c + 1$ this implies that $4p^{2b} + 1 < q^a$ contrary to what has been proved above. Therefore $q^c + 1 = 2p^b$.

Let $\mathscr{T}_i = \{\theta_{ij}\}$ be the set of non principal irreducible characters of $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}']$ of degree q^i. Lemma 4.1 implies that $\mathscr{T}_i$ is empty for $i > c$. Let $\mathscr{S}_i = \{\lambda_{ij}\}$ be the set of characters in $\mathscr{S}$ of degree $q^i p^b$.

Since $|\mathscr{S}_0| = 2(q^c - 1) > 2q^{c-1}$, it follows from Hypothesis 11.1 and Theorem 10.1 that $\bigcup_{i=0}^{c-1} \mathscr{S}_i$ is coherent. Suppose that $\bigcup_{i=1}^{c-1} \mathscr{T}_i$ is non empty. Then 3.15 implies that

$$\sum_{i=1}^{c-1} \sum_j \theta_{ij}(1)^2 \geqq q^{2c} .$$

Therefore

$$\sum_{i=0}^{c-1} \sum_j \frac{q^{2i}}{||\lambda_{ij}||^2} = \frac{1}{p^{2b}} \sum_{i=0}^{c-1} \sum_j \frac{\lambda_{ij}(1)^2}{||\lambda_{ij}||^2} = |\mathscr{S}_0| + \frac{1}{p^b} \sum_{i=1}^{c-1} \sum_j \theta_{ij}(1)^2 > 2q^c .$$

Thus by Theorem 10.1

$$\bigcup_{i=0}^{c} \mathscr{S}_i \supseteqq \mathscr{S}([\mathfrak{H}, \mathfrak{H}'])$$

is coherent. Since

$$4|\mathfrak{L}:\mathfrak{H}|^2 + 1 = 4p^{2b} + 1 < q^{a+1} \leqq |\mathfrak{H}:[\mathfrak{H}, \mathfrak{H}']| ,$$

Theorem 11.1 implies that $\mathscr{S}$ is coherent. Thus it may be assumed that q^c is the smallest degree of any non linear irreducible character of $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}']$.

Suppose now that $\mathfrak{H}'$ contains a subgroup $\mathfrak{H}_1 \neq \mathfrak{H}'$ such that $\mathfrak{L}/\mathfrak{H}_1$ is a Frobenius group. Then $\mathfrak{H}_1$ may be chosen so that $\mathfrak{H}'/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$. Thus $[\mathfrak{H}, \mathfrak{H}'] \subseteqq \mathfrak{H}_1$ and by the earlier part of the lemma every irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ has degree either 1 or q^c. As $q^c + 1 = 2p^b$, q^{2c} is the smallest power of q which satisfies $q^{2c} \equiv 1 \pmod{p^b}$. Since $\mathfrak{H}'/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$ this implies that $\mathfrak{H}'/\mathfrak{H}_1$ is in the center of $\mathfrak{H}/\mathfrak{H}_1$ and $|\mathfrak{H}': \mathfrak{H}_1| = q^{2c}$. If θ is an irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ of degree q^c, then the orthogonality relations yield that $\theta(H) = 0$ for $H \in \mathfrak{H}/\mathfrak{H}_1 - \mathfrak{H}'/\mathfrak{H}_1$. As every non linear character of $\mathfrak{H}/\mathfrak{H}_1$ has degree q^c the orthogonality relations may once again be used. They imply that

(11.13) $$|C(H)| = q^{2c} \quad \text{for } H \in \mathfrak{H}/\mathfrak{H}_1 - \mathfrak{H}'/\mathfrak{H}_1 .$$

However

$$\langle H, \mathfrak{H}'/\mathfrak{H}_1 \rangle \subseteqq C(H)$$

which contradicts (11.13). Thus $\mathfrak{H}'$ contains no subgroup $\mathfrak{H}_1 \neq \mathfrak{H}'$ such that $\mathfrak{L}/\mathfrak{H}_1$ is a Frobenius group. All statements in the lemma are proved.

LEMMA 11.6. *Suppose that Hypothesis* 11.3 *is satisfied. Assume further that a is odd and $p = 3$. Then $\mathscr{S}$ is coherent.*

Proof. As a is odd and $q^a \equiv 1 \pmod 3$, it follows that $q \equiv 1 \pmod 3$. Define the integer $c \geqq 1$ by

$$q \equiv 1 \pmod{3^c}, \qquad q \not\equiv 1 \pmod{3^{c+1}}.$$

If $b \leqq c$, then $q \geqq 2.3^b + 1$. Thus if $a \neq 1, 4\cdot 3^{2b} + 1 < q^a$ and $\mathscr{S}$ is coherent by Lemma 11.1. If $a = 1$, then $\mathfrak{H}$ is cyclic. Therefore $\mathscr{S}$ is coherent by Lemma 10.1.

Suppose now that $b > c$. Then since $q^a \equiv 1 \pmod{3^b}$ we must have $a = 3^{b-c}x$ for some integer x. Therefore

$$q^a \geqq (q^{3^{b-c-1}})^3 .$$

Since $q^{3^{b-c-1}} \equiv 1 \pmod{3^{b-1}}$, this yields that

(11.14) $$q^a \geqq (1 + 2\cdot 3^{b-1})^3 .$$

If $4.3^{2b} + 1 < q^a$ then $\mathscr{S}$ is coherent by Lemma 11.1. Thus if $\mathscr{S}$ is not coherent (11.14) implies that

$$4.3^{2b} + 1 \geqq q^a \geqq (1 + 2.3^{b-1})^3 > 8.3^{3(b-1)} + 1 .$$

Therefore $3^3 > 2.3^b$. Hence $b = 1$ or $b = 2$. In either case this implies that $q^a \leqq 4.3^4 + 1 < 7^3$. As $a \equiv 0 \pmod 3$ we get that $q < 7$. However $q \equiv 1 \pmod 3$. This contradiction arose from assuming that $\mathscr{S}$ is not coherent. The proof is complete.

12. Further Results about Tamely Imbedded Subsets

In this section a fairly special situation is studied. Our purpose here is to get some information about certain sets of characters which may not be coherent.

Hypothesis 12.1.

(i) *Let q be a prime and let $\mathfrak{Q}$ be a S_q-subgroup of the group $\mathfrak{X}$. Assume that $\mathfrak{Q} = \hat{\mathfrak{L}}$ is tamely imbedded in $\mathfrak{X}$ and $\mathfrak{L} = N(\mathfrak{Q}) \neq \mathfrak{Q}$ has odd order. Let $\mathfrak{Q}_1 \lhd \mathfrak{L}, \mathfrak{Q}_1 \subset \mathfrak{Q}$ and let $\bar{\mathfrak{Q}} = \mathfrak{Q}/\mathfrak{Q}_1, \bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{Q}_1$.*

(ii) *$\mathscr{L}$ is the set of all characters of $\mathfrak{L}$ which are induced by non-principal irreducible characters of $\bar{\mathfrak{Q}}$. Define an equivalence relation on $\mathscr{L}$ by the condition that two characters are equivalent if and only if they have the same degree and the same weight. Then each equivalence class of $\mathscr{L}$ is either subcoherent in $\mathscr{L}$ or consists of irreducible characters.*

(iii) *Let $1 = q^{f_0} < q^{f_1} \cdots$ be all the integers which are degrees of irreducible characters of $\bar{\mathfrak{Q}}$. Let $n > 0$ be a fixed integer. For $0 \leqq i \leqq n - 1$ let $\mathscr{S}_i$ be the set of all characters in $\mathscr{L}$ of degree*

$q^{f_i}|\mathfrak{L}:\mathfrak{D}|$. *Assume that each* $\mathscr{S}_i$ *consists of irreducible characters. Let* $\mathscr{S}_n$ *be an equivalence class in* $\mathscr{L}$ *consisting of characters of degree* $q^{f_n}|\mathfrak{L}:\mathfrak{D}|$. *Let* $\mathscr{S} = \bigcup_{i=0}^{n} \mathscr{S}_i$.

In case Hypothesis 12.1 is satisfied the following notation will be used.

$$(12.1) \qquad |\bar{\mathfrak{D}}:\bar{\mathfrak{D}}'| = q^a , \qquad |\bar{\mathfrak{L}}:\bar{\mathfrak{D}}| = e > 1 .$$

Since $|\mathfrak{L}|$ is odd, $|\mathscr{S}_i| \geqq 2$ and $\mathscr{I}_0(\mathscr{S}_i) \neq 0$ for $0 \leqq i \leqq n$. Thus by Lemma 10.1 $\mathscr{S}_i$ is coherent for $0 \leqq i \leqq n-1$.

For $0 \leqq i < n$ let a_i be the number of non principal irreducible characters of $\bar{\mathfrak{D}}$ of degree q^{f_i}. By Hypothesis 12.1 $\bar{\mathfrak{L}}/\bar{\mathfrak{D}}$ acts regularly as a permutation group on the non principal irreducible characters of degree q^{f_i} for $0 \leqq i < n$. Since $|\mathfrak{L}|$ is odd, no non principal irreducible character of $\mathfrak{D}$ is real. Thus a_i is even. Therefore

$$(12.2) \qquad a_i \equiv 0 \pmod{2e}, \ |\mathscr{S}_i| = \frac{a_i}{e} \quad \text{for } 0 \leqq i \leqq n-1 .$$

Let $j_0 = 0$. Define j_s inductively to be the largest integer not exceeding $n+1$ such that $\bigcup_{i=j_{s-1}}^{j_s-1} \mathscr{S}_i$ is coherent. Suppose that

$$0 = j_0 < \cdots < j_t < j_{t+1} = n+1 .$$

For $0 \leqq s \leqq t$, define

$$(12.3) \qquad \mathscr{T}_s = \bigcup_{i=j_s}^{j_{s+1}-1} \mathscr{S}_i$$

and let $m_s = f_{j_s}$. Define

$$(12.4) \qquad c_s = \sum_i a_i q^{2(f_i - m_s)} \quad \text{for } 0 \leqq s \leqq t ,$$

where i ranges from j_s to $j_{s+1}-1$. Define

$$(12.5) \qquad d_s = q^{m_{s+1}-m_s} \quad \text{for } 0 \leqq s < t .$$

Then by Theorem 10.1 applied to $\mathscr{T}_s \cup \mathscr{S}_{j_{s+1}}$

$$(12.6) \qquad c_s \leqq 2ed_s \quad \text{for } 0 \leqq s < t .$$

By (12.2)

$$(12.7) \qquad c_s \equiv 0 \pmod{2e} \quad \text{for } 0 \leqq s < t .$$

By 3.15

$$(12.8) \qquad 1 + \sum_{j=0}^{s} c_j q^{2m_j} \equiv 0 \pmod{q^{2m_{s+1}}} \quad \text{for } 0 \leqq s < t .$$

LEMMA 12.1. *Suppose that Hypothesis* 12.1 *is satisfied. Assume that*

$$|\bar{\mathfrak{Q}}:\bar{\mathfrak{Q}}'| = q^a \leqq 4e^2 + 1 .$$

Then

$$\frac{d_s^2 e}{c_s} < e + 1 \quad for\ 0 \leqq s < t .$$

Furthermore if a is odd, $c_s < e^2$ and $c_s \not\equiv 0 \pmod q$, then

$$\frac{d_s^2 e}{c_s} < e - 1 .$$

Proof. We will first prove that

$$1 + \sum_{j=0}^{s-1} c_j q^{2m_j} < eq^{2m_s} \quad \text{for } 0 \leqq s < t . \tag{12.9}$$

This is true if $s = 0$ since $1 < e$. Suppose that $s > 0$. Then by (12.5) and (12.6)

$$\begin{aligned} 1 + \sum_{j=0}^{s-1} c_j q^{2m_j} &\leqq 1 + 2e \sum_{j=0}^{s-1} q^{m_j + m_{j+1}} \\ &\leqq 1 + 2e(1 + q + \cdots + q^{2m_s - 1}) \\ &\leqq 1 + 2e\frac{(q^{2m_s} - 1)}{(q-1)} \leqq 1 + e(q^{2m_s} - 1) < eq^{2m_s} . \end{aligned}$$

Assume now that the lemma is false and choose s minimum to violate the result. Let $c = c_s$, $d = d_s$.

By (12.8) and (12.9)

$$q^{2m_{s+1}} < eq^{2m_s} + cq^{2m_s} .$$

Hence by (12.5)

$$d^2 < e + c . \tag{12.10}$$

Inequalities (12.6) and (12.10) yield that $d^2 < e + 2ed$ or $d^2 - 2ed - e < 0$. This implies that

$$e - \sqrt{e^2 + e} \leqq d \leqq e + \sqrt{e^2 + e} .$$

Consequently

$$d \leqq e + \sqrt{e^2 + e} < 3e . \tag{12.11}$$

Suppose that

$$1 + \sum_{j=0}^{s} c_j q^{2m_j} \geqq 3q^{2m_{s+1}} .$$

Then by (12.9)

$$3q^{2m_{s+1}} < (e + c)q^{2m_s} .$$

Hence by (12.7) $3d^2_s < e + c \leqq 3c/2$. Thus

$$\frac{d^2 e}{c} \leqq \frac{e}{2} < e - 1$$

since $e > 2$. This contradicts the choice of s. Hence

$$1 + \sum_{j=0}^{s} c_j q^{2m_j} < 3q^{2m_{s+1}} .$$

As c_j is even for $0 \leqq j \leqq s$, (12.8) implies that

(12.12) $$1 + \sum_{j=0}^{s} c_j q^{2m_j} = q^{2m_{s+1}} .$$

The group $\bar{\mathfrak{Q}}$ contains a normal subgroup $\bar{\mathfrak{Q}}_0$ of index $q^{2m_{s+1}}$. Every irreducible character of $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$ has degree strictly less than $q^{m_{s+1}}$ and the sum of the squares of the degrees of these characters is equal to $q^{2m_{s+1}}$. Hence (12.12) implies that every character of $\bar{\mathfrak{Q}}$ whose degree is strictly less than $q^{m_{s+1}}$ has $\bar{\mathfrak{Q}}_0$ in its kernel. Thus $\bar{\mathfrak{Q}}_0$ is a normal subgroup of $\bar{\mathfrak{L}}$ and $\bar{\mathfrak{L}}/\bar{\mathfrak{Q}}_0$ is a Frobenius group with Frobenius kernel $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$. Therefore

(12.13) $$q^{2m_{s+1}} \equiv d^2 q^{2m_s} \equiv 1 \pmod{e} ,$$

and the center of $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$ has order at least q^a. Thus by Lemma 4.1 $q^{2m_s} \leqq q^{2m_{s+1}-a}$. This yields that

(12.14) $$q^a \leqq d^2 .$$

Define the integer k by

(12.15) $$c + k = d^2 .$$

By (12.10) $k < e$ and by (12.12) $0 < k$. Thus

(12.16) $$0 < k < e .$$

Define the integer b by

(12.17) $$q^{2m_s} \equiv q^{a-b} \pmod{e} , \qquad 0 \leqq b \leqq a - 1 .$$

Equations (12.7), (12.13), (12.15) and (12.17) imply that

(12.18) $$k \equiv d^2 \equiv q^{b-a} \equiv q^b \pmod{e} .$$

If $b = 0$, then by (12.16) and (12.18) $k = 1$. Thus by (12.15) $c = d^2 - 1$, hence by (12.7)

$$\frac{d^2e}{c} = \frac{(c+1)e}{c} = e + \frac{e}{c} < e + 1 .$$

If $c < e^2$ and a is odd, then

$$d^2 = c + 1 < e^2 + 1 < q^{2a} .$$

Thus by (12.18) $d^2 = q^a$. However this is impossible as a is odd.

Assume now that $b \neq 0$. As d^2 is a power of q, (12.14) and (12.18) imply that either $d^2 = q^{a+b}$ or $d^2 \geqq q^{2a+b}$. Since $b \neq 0$, the latter case leads to

$$d^2 \geqq q^{2a+b} = q^{2a}q^b > 4e^2q > 9e^2 .$$

Hence $d > 3e$ contrary to (12.11). Thus

$$d^2 = q^{a+b}, \quad 2 \leqq a - b . \tag{12.19}$$

The inequality follows from (12.17) and the fact that $a + b$ is even. Now (12.11) and (12.19) yield that

$$q^{2b} = \frac{q^{a+b}}{q^{a-b}} = \frac{d^2}{q^{a-b}} < \frac{9e^2}{q^2} \leqq e^2 .$$

Thus $1 \leqq q^b < e$. (12.16) and (12.18) imply that

$$k = q^b , \quad b > 0 . \tag{12.20}$$

Equation (12.15) now becomes $d^2 = c + q^b$. Hence

$$c \equiv 0 \pmod q .$$

Furthermore by (12.19)

$$c = d^2 - q^b = q^b(q^a - 1) .$$

Consequently

$$\frac{d^2e}{c} = \frac{q^{a+b}e}{q^b(q^a-1)} = \frac{q^ae}{q^a-1} = e + \frac{e}{q^a-1} < e + 1 .$$

THEOREM 12.1. *Suppose that Hypothesis* 12.1 *is satisfied. Assume that for some j with $0 \leqq j \leqq n-1$, $\lambda_1 \in \mathscr{S}_j$ and $\lambda_2 \in \mathscr{S}_{j+1}$. Define*

$$\alpha = q^{f_{j+1}-f_j}\lambda_1 - \lambda_2 .$$

Suppose that $\mathscr{S}_j \subseteqq \mathscr{T}$, and

$$\alpha^\tau = \Delta + \Delta_1$$

where $\Delta_1 \in \mathscr{I}(\mathscr{T}_s^\tau)$ *and* Δ *is orthogonal to* $\mathscr{I}(\mathscr{T}_s^\tau)$. *Then*

$$\|\Delta\|^2 \leqq e + \|\lambda_2\|^2 .$$

Furthermore if a *is odd,* $c = c_s < e^2$ *and* $c \not\equiv 0 \pmod q$ *then*

$$\|\Delta\|^2 \leqq e + \|\lambda_2\|^2 - 2 .$$

Proof. Let $\mathscr{T} = \mathscr{T}_s$. If $\mathscr{S}_{j+1} \subseteq \mathscr{T}$ then $\alpha^\tau \in \mathscr{I}(\mathscr{T})^\tau$ and $\Delta = 0$. Thus the result is trivial in this case. Hence it may be assumed that $\mathscr{S}_{j+1} \not\subseteq \mathscr{T}$. In particular, $\mathscr{S}$ is not coherent, hence $\mathscr{L}$ is not coherent, so by Lemma 11.1 $|\bar{\mathfrak{Q}}:\bar{\mathfrak{Q}}'| \leqq 4e^2 + 1$. Consequently Lemma 12.1 may be applied. Furthermore $f_{j+1} = m_{s+1}$ and $s < t$. Thus $\mathscr{T}$ consists of irreducible characters. Let $\mathscr{T} = \{\lambda_{si} \mid 1 \leqq i \leqq n_s\}$, where the notation is chosen so that $\lambda_1 \neq \lambda_{s1}$ and $\lambda_{si}(1) \mid \lambda_{s\,i+1}(1)$ for $1 \leqq i < n_s$. Suppose that $\lambda_1 = \lambda_{sk}$. Define the integer x by $(\alpha^\tau, \lambda_{s1}^\tau) = -x$. Then since $\alpha \in \mathscr{I}_0(\mathscr{S})$ Lemma 9.4 implies that

$$(\alpha^\tau, \lambda_{si}^\tau) = -x\frac{\lambda_{si}(1)}{\lambda_{s1}(1)} + \delta_{ik}q^{m_{s+1}-f_j} \quad \text{for } 1 \leqq i \leqq n_s .$$

Then

$$\Delta_1 = q^{m_{s+1}-f_j}\lambda_{sk}^\tau - x\sum_{i=1}^{n}\frac{\lambda_{si}(1)}{\lambda_{s1}(1)}\lambda_{si}^\tau .$$

Therefore

$$\begin{aligned} \|\Delta\|^2 = \|\alpha^\tau\|^2 - \|\Delta_1\|^2 = q^{2(m_{s+1}-f_j)} \\ + \|\lambda_2\|^2 - x^2\frac{c}{e} - q^{2(m_{s+1}-f_j)} + 2x\frac{\lambda_{sk}(1)}{\lambda_{s1}(1)}q^{m_{s+1}-f_j} , \end{aligned} \tag{12.21}$$

where $c = c_s$ is defined by (12.4). Let $d = d_s$ be defined by (12.5). Since $\lambda_{s1}(1) = q^{m_s}$ and $\lambda_{sk}(1) = q^{f_j}$ (12.21) yields that

$$\|\Delta\|^2 = \|\lambda_2\|^2 + 2xd - \frac{x^2c}{e} . \tag{12.22}$$

As a function of x, $2xd - (x^2c/e)$ assumes its maximum at $x = ed/c$. Thus (12.22) implies that

$$\|\Delta\|^2 \leqq \|\lambda_2\|^2 + 2\frac{ed^2}{c} - \frac{ed^2}{c} = \|\lambda_2\|^2 + \frac{ed^2}{c} . \tag{12.23}$$

As $\|\Delta\|^2$ is an integer Lemma 12.1 and (12.23) imply that $\|\Delta\|^2 \leqq \|\lambda_2\|^2 + e$. Furthermore if a is odd, $c < e^2$ and $c \not\equiv 0 \pmod q$, then

$$\|\Delta\|^2 \leqq \|\lambda_2\|^2 + e - 2 .$$

The proof is complete.

13. Self Normalizing Cyclic Subgroups

Hypothesis 13.1.

(i) $\mathfrak{W}$ *is a cyclic subgroup of the group* $\mathfrak{X}$ *with* $|\mathfrak{W}| = w$ *odd. Suppose that* $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$, *where* $w_i = |\mathfrak{W}_i|$ *and* $w_i \neq 1$ *for* $i = 1, 2$. *Let*

$$\hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2 .$$

For any non empty subset $\mathfrak{A}$ *of* $\hat{\mathfrak{W}}$

$$C(\mathfrak{A}) = N(\mathfrak{A}) = \mathfrak{W} . \tag{13.1}$$

(ii) *Let* ω_{10}, ω_{01} *be faithful irreducible characters of* $\mathfrak{W}/\mathfrak{W}_2$, $\mathfrak{W}/\mathfrak{W}_1$ *respectively. Define*

$$\omega_{ij} = \omega_{10}^i \omega_{01}^j$$

for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$.

If w_1, w_2 in Hypothesis 13.1 are both primes then (13.1) follows from the assumption that $N(\mathfrak{W}) = \mathfrak{W}$. Thus the situation described above is a generalization of this case.

LEMMA 13.1. *Suppose that Hypothesis* 13.1 *is satisfied. Then* $\hat{\mathfrak{W}}$ *is a* T. I. *set in* $\mathfrak{X}$. *There exists an orthonormal set* $\{\eta_{ij} \mid 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1\}$ *of generalized characters of* $\mathfrak{X}$ *such that for* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$, *the values assumed by* $\eta_{ij}, \eta_{i0}, \eta_{0j}$ *lie in* $\mathcal{Q}_w, \mathcal{Q}_{w_1}, \mathcal{Q}_{w_2}$ *respectively.* $\eta_{00} = 1_{\mathfrak{X}}$ *and*

$$\eta_{ij}(W) = \omega_{ij}(W) \quad \text{for} \quad W \in \hat{\mathfrak{W}} ,$$
$$(1 - \omega_{i0} - \omega_{0j} + \omega_{ij})^* = 1_{\mathfrak{X}} - \eta_{i0} - \eta_{0j} + \eta_{ij} .$$

Furthermore every irreducible character of $\mathfrak{X}$ *distinct from all* $\pm\eta_{ij}$ *vanishes on* $\hat{\mathfrak{W}}$.

Proof. It follows directly from Hypothesis 13.1 that $\hat{\mathfrak{W}}$ is a T. I. set in $\mathfrak{X}$. Define the generalized character α_{ij} of $\mathfrak{W}$ by

$$\alpha_{ij} = (\omega_{00} - \omega_{i0})(\omega_{00} - \omega_{0j}) .$$

Clearly α_{ij} vanishes on $\mathfrak{W} - \hat{\mathfrak{W}}$. Thus

$$\begin{aligned} &\alpha_{ij}^*(W) = \alpha_{ij}(W) \quad \text{for} \quad W \in \hat{\mathfrak{W}} , \\ &(\alpha_{ij}^*, \alpha_{st}^*) = 1 + \delta_{is} + \delta_{jt} + \delta_{is}\delta_{jt} \end{aligned} \tag{13.2}$$

for $1 \leqq i, s \leqq w_1 - 1, 1 \leqq j, t \leqq w_2 - 1$. Therefore $||\alpha^*_{ij}||^2 = 4$ and $(\alpha^*_{ij}, \alpha^*_{it}) = 2$ for $i, j, t \neq 0, j \neq t$. It follows directly from the definition of α_{ij} that the values of α^*_{ij} lie in $\mathcal{Q}_w$.

For any algebraic number field $\mathcal{F}$ and any generalized character α of a group let $\mathcal{F}(\alpha)$ denote the field generated by $\mathcal{F}$ and all the values assumed by α. Since $\mathcal{Q}(\alpha_{ij}) = \mathcal{Q}(\alpha^*_{ij})$ we see that $\mathcal{Q}(\alpha^*_{ij}) = \mathcal{Q}_v$ for some v with $v \mid w$. If $i, j \neq 0$ then $v = v_1 v_2$, where $v_s \mid w_s$ and $v_s > 1$ for $s = 1, 2$. By (13.2)

$$\alpha^*_{ij} = 1_{\mathfrak{X}} \pm \theta_1 \pm \theta_2 \pm \theta_3 ,$$

where $\theta_1, \theta_2, \theta_3$ are distinct irreducible characters of $\mathfrak{X}$.

Suppose that $\mathcal{Q}(\theta_k) \not\subseteq \mathcal{Q}_{v_1}$ for $k = 1, 2, 3$. Let

$$\mathcal{F} = \mathcal{Q}_v(\theta_1, \theta_2, \theta_3) = \mathcal{Q}(\theta_1, \theta_2, \theta_3) .$$

Let $\mathfrak{G}$ be the Galois group of $\mathcal{F}$ over $\mathcal{Q}_{v_1}$. For $k = 1, 2, 3$ let $\mathfrak{G}_k$ be the subgroup of $\mathfrak{G}$ whose fixed field is $\mathcal{Q}_{v_1}(\theta_k)$.

Assume first that $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$. By (13.2) $\mathfrak{G}_s \cap \mathfrak{G}_t = 1$ for $1 \leqq s < t \leqq 3$. If $\mathfrak{G} = \mathfrak{G}_k$ for some k then $\mathcal{Q}(\theta_k) \subseteq \mathcal{Q}_{v_1}$ contrary to assumption. Let $|\mathfrak{G}| = g$ and $|\mathfrak{G}_k| = g_k$ for $k = 1, 2, 3$. Then it may be assumed that $g > g_1 \geqq g_2 \geqq g_3$. Since $g = g_1 + g_2 + g_3 - 1 - 1 - 1 + 1$ we must have $g_1 = g/2$. Therefore

$$1 = |\mathfrak{G}_1 \cap \mathfrak{G}_2| \geqq g_2/2, \quad 1 = |\mathfrak{G}_1 \cap \mathfrak{G}_3| \geqq g_3/2 .$$

Hence

$$g/2 = g - g_1 = g_2 + g_3 - 2, \quad g_2, g_3 \leqq 2 .$$

Therefore $g \leqq 4$. $\mathfrak{G}$ is not cyclic as it is the union of proper subgroups. Hence $\mathfrak{G}$ is the non cyclic group of order 4 and $|\mathfrak{G}_k| = 2$ for $k = 1, 2, 3$. As v_2 is odd this implies that $v_2 = 3$. For $k = 1, 2, 3$ let $\mathfrak{G}_k = \langle \sigma_k \rangle$, where the notation is chosen so that $\mathcal{Q}_v = \mathcal{Q}_{v_1}(\theta_1)$. Therefore $\sigma_1(\alpha^*_{ij}) = \alpha^*_{ij}$. Hence $\sigma_1(\theta_2) = \theta_3$. Consequently $\mathcal{Q}_{v_1}(\theta_2) = \mathcal{Q}_{v_1}(\theta_3)$ as $\mathfrak{G}$ is abelian. This implies that $\sigma_2 = \sigma_3$ which is not the case. Thus $\mathfrak{G} \neq \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$.

If $\sigma \in \mathfrak{G} - \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$, then by (13.2) $(\alpha^*_{ij}, \sigma(\alpha^*_{ij})) \geqq 2$. Hence by choosing the notation suitably it may be assumed that $\sigma(\theta_2) = \theta_3$. If $(\alpha^*_{ij}, \theta_2) \neq (\alpha^*_{ij}, \sigma(\theta_2))$ then replacing σ by σ^{-1} and θ_2 by θ_3 if necessary we get that

$$\alpha^*_{ij} = 1_{\mathfrak{X}} \pm \{\theta_1 + \theta_2 - \sigma(\theta_2)\} .$$

By (13.2) $\sigma(\theta_2) \neq \theta_1, \theta_2$. Hence also $\sigma(\theta_2) \neq \sigma^2(\theta_2)$. Therefore

$$\begin{aligned}2 \leqq (\sigma(\alpha_{ij}^*), \alpha_{ij}^*) &= 1 - 1 + (\theta_1 + \theta_2, \sigma(\theta_1) - \sigma^2(\theta_2)) \\ &= (\theta_1 + \theta_2, \sigma(\theta_1)) - (\theta_1 + \theta_2, \sigma^2(\theta_2)) \\ &\leqq (\theta_1 + \theta_2, \sigma(\theta_1)) \leqq 1\end{aligned}$$

since $\theta_1, \theta_2, \sigma(\theta_1)$ and $\sigma^2(\theta_2)$ are all characters. This contradiction establishes that $(\alpha_{ij}^*, \theta_2) = (\alpha_{ij}^*, \sigma(\theta_2))$. Since $\alpha_{ij}^*(1) = 0$ we see that

$$\alpha_{ij}^* = 1_{\mathfrak{X}} \pm \{\theta_2 + \sigma(\theta_2) - \theta_1\} . \tag{13.3}$$

Furthermore $\mathfrak{G}_2 = \mathfrak{G}_3$ and if $\gamma \in \mathfrak{G} - \mathfrak{G}_1 \cup \mathfrak{G}_2$ then $\theta_1 \neq \gamma(\theta_2)$. By definition $\theta_1 \neq \gamma(\theta_2)$ for $\gamma \in \mathfrak{G}_1 \cup \mathfrak{G}_2$. Therefore

$$\theta_1 \neq \gamma(\theta_2) \quad \text{for } \gamma \in \mathfrak{G} .$$

Suppose that $\gamma(\theta_2) = \theta_2$ for some automorphism γ of $\mathscr{F}$. Then $\gamma\sigma(\theta_2) = \sigma(\theta_2)$ and (13.3) implies that $(\alpha_{ij}^*, \gamma(\alpha_{ij}^*)) \geqq 3$. Thus by (13.2) $\gamma(\alpha_{ij}^*) = \alpha_{ij}^*$. Consequently $\gamma(\theta_1) = \theta_1$ and so

$$\mathscr{Q}_v \subseteq \mathscr{F} = \mathscr{Q}(\theta_2) . \tag{13.4}$$

If now $\gamma \in \mathfrak{G}^\#, \gamma \neq \sigma, \gamma \neq \sigma^{-1}$, then (13.3) yields that

$$2 \leqq (\alpha_{ij}^*, \gamma(\alpha_{ij}^*)) = 1 + (\theta_1, \gamma(\theta_1)) .$$

Therefore $\gamma(\theta_1) = \theta_1$ and $\gamma \in \mathfrak{G}_1$. Thus $|\mathfrak{G}_1| \geqq |\mathfrak{G}| - 2$. Since $\mathfrak{G}_1 \neq \mathfrak{G}$ and $|\mathfrak{G}_1| \mid |\mathfrak{G}|$ we get that $|\mathfrak{G}| \leqq 4$. If $|\mathfrak{G}| = 2$ then $\mathscr{F} \subseteq \mathscr{Q}_v$. Thus (13.2) and (13.3) yield that $2 = (\alpha_{ij}^*, \sigma(\alpha_{ij}^*)) \geqq 3$. Since $|\mathscr{Q}_v : \mathscr{Q}_{v_1}|$ is even we get that $|\mathfrak{G}| = 4$. Thus either $v_2 = 5$ and $\mathscr{F} \subseteq \mathscr{Q}_v$ or $v_2 = 3$. In the latter case (13.2), (13.3) and (13.4) imply that $\sigma(\theta_1) = \theta_1$. Thus $\mathfrak{G} = \mathfrak{G}_1$ or equivalently $\mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_1}$ contrary to assumption.

Suppose now that $v_2 = 5$. Thus $v_1 \neq 5$ and the previous argument with v_1 and v_2 interchanged yields that $\mathscr{Q}(\theta_k) \subseteq \mathscr{Q}_{v_2}$ for $k = 1$ or $k = 2$. Thus by (13.4) $\mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_2}$. By (13.2) and (13.3) $\mathfrak{G} = \langle\sigma\rangle$. Thus $\sigma^2(\theta_1) = \theta_1$ since $(\sigma^2(\alpha_{ij}^*), \alpha_{ij}^*) = 2$. Let γ be in the Galois group of $\mathscr{Q}_v$ over $\mathscr{Q}_{v_2}$. Then $\gamma\sigma^2(\theta_1) = \theta_1$ and γ can be chosen so that

$$(\alpha_{ij}^*, \gamma\sigma^2(\alpha_{ij}^*)) = 1 .$$

Hence (13.3) yields that

$$(\theta_2 + \sigma(\theta_2) - \theta_1, \gamma\sigma^2(\theta_2) + \gamma\sigma^3(\theta_2) - \theta_1) = 0 .$$

Since θ_1 is not conjugate to θ_2 this implies that

$$(\theta_2 + \sigma(\theta_2), \gamma\sigma^2(\theta_2) + \gamma\sigma^3(\theta_2)) = -1$$

contrary to the fact that $\theta_2, \sigma(\theta_2), \gamma\sigma^2(\theta_2)$ and $\gamma\sigma^3(\theta_2)$ are all characters.

Thus in any case there exists a non principal irreducible character

θ_1 of $\mathfrak{X}$ such that $(\theta_1, \alpha_{ij}^*) \neq 0$ and $\mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_1}$. Suppose that $\mathscr{Q}(\theta_1) = \mathscr{Q}$. Since w is odd

$$(\alpha_{ij}^*, \overline{\alpha_{ij}^*}) = (\alpha_{ij}, \overline{\alpha_{ij}}) = 1 .$$

Therefore

$$1 = (1_{\mathfrak{X}} \pm \theta_1 \pm \theta_2 \pm \theta_3, 1_{\mathfrak{X}} \pm \theta_1 \pm \bar{\theta}_2 \pm \bar{\theta}_3) = 2 + (\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) .$$

Hence

$$(\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) = -1 .$$

Since θ_2 and θ_3 are characters this yields that $\theta_k \neq \bar{\theta}_k$ for $k = 2, 3$. Hence $\bar{\theta}_2 = \theta_3$ and so $\bar{\theta}_3 = \theta_2$. Consequently $(\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) = \pm 2$, which is not the case. Therefore

$$\mathscr{Q} \neq \mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_1} . \tag{13.5}$$

Similarly there exists an irreducible character θ_2 of $\mathfrak{X}$ with $(\theta_2, \alpha_{ij}^*) \neq 0$ and $\mathscr{Q} \neq \mathscr{Q}(\theta_2) \subseteq \mathscr{Q}_{v_2}$. Thus by (13.5) $\theta_1 \neq \theta_2$. Now (13.2) yields that

$$\alpha_{ij}^* = 1_{\mathfrak{X}} - \eta_{i0} - \eta_{0j} + \eta_{ij} \tag{13.6}$$

for $1 \leqq i \leqq w_1 - 1, 1 \leqq j \leqq w_2 - 1$. The $\pm\eta_{ij}$ are distinct irreducible characters of $\mathfrak{X}$ whose values lie in the required field. Suppose now that

$$\eta_{s0|\mathfrak{W}} = \sum_{i,j} a_{ij}\omega_{ij} + a\rho_{\mathfrak{W}}$$

with $a_{00} = 0$. Then by the Frobenius reciprocity theorem it follows from (13.6) that

$$-a_{i0} - a_{0j} + a_{ij} = -\delta_{is} ,$$

$$\begin{aligned}\eta_{s0|\mathfrak{W}} &= \sum_{i=1}^{w_1-1} a_{i0}\omega_{i0} + \sum_{j=1}^{w_2-1} a_{0j}\omega_{0j} + \sum_{i=1}^{w_1-1} a_{i0} \sum_{j=1}^{w_2-1} \omega_{ij} \\ &\qquad + \sum_{j=1}^{w_2-1} a_{0j} \sum_{i=1}^{w_1-1} \omega_{ij} - \sum_{j=1}^{w_2-1} \omega_{sj} + a\rho_{\mathfrak{W}} \\ &= \sum_{i=1}^{w_1-1} a_{i0} \sum_{j=0}^{w_2-1} \omega_{ij} + \sum_{j=1}^{w_2-1} a_{0j} \sum_{i=0}^{w_1-1} \omega_{ij} - \sum_{j=1}^{w_2-1} \omega_{sj} + a\rho \quad .\end{aligned}$$

Consequently for $W \in \hat{\mathfrak{W}}$

$$\eta_{s0}(W) = -\sum_{j=1}^{w_2-1} \omega_{sj}(W) = \omega_{s0}(W) .$$

In a similar way it can be shown that $\eta_{0t}(W) = \omega_{0t}(W)$. Then it follows from (13.6) that $\eta_{st}(W) = \omega_{st}(W)$ for $W \in \hat{\mathfrak{W}}$.

This implies that if $W \in \hat{\mathfrak{W}}$ then

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}|\eta_{ij}(W)|^2=\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}|\omega_{ij}(W)|^2=w=|C(W)|\,.$$

The orthogonality relations for the irreducible characters of $\mathfrak{X}$ now yield that every irreducible character of $\mathfrak{X}$ distinct from all $\pm\eta_{ij}$ vanishes on $\hat{\mathfrak{W}}$. This completes the proof of the lemma.

LEMMA 13.2. *Suppose that Hypothesis* 13.1 *is satisfied. If Δ is a generalized character of $\mathfrak{X}$ which vanishes on $\hat{\mathfrak{W}}$ then*

$$\begin{aligned}\Delta = a_{00}1_{\mathfrak{X}} &+ \sum_{i=1}^{w_1-1} a_{i0}\sum_{j=0}^{w_2-1}\eta_{ij}\\ &+ \sum_{j=1}^{w_2-1} a_{0j}\sum_{i=0}^{w_1-1}\eta_{ij} - a_{00}\sum_{i=1}^{w_1-1}\sum_{j=1}^{w_2-1}\eta_{ij} + \Delta_0\end{aligned}$$

where $(\Delta_0,\eta_{ij})=0$ *for* $0\leqq i\leqq w_1-1,\ 0\leqq j\leqq w_2-1$.

Proof. Let

$$\Delta=\Delta_0+\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}a_{ij}\eta_{ij}\,,$$

where $(\Delta_0,\eta_{ij})=0$ for $0\leqq i\leqq w_1-1,\ 0\leqq j\leqq w_2-1$. By Lemma 13.1

$$(\Delta,1_{\mathfrak{X}}-\eta_{i0}-\eta_{0j}+\eta_{ij})=0 \qquad \text{for } 0\leqq i\leqq w_1-1,\ 0\leqq j\leqq w_2-1\,.$$

Hence

$$a_{00}-a_{i0}-a_{0j}+a_{ij}=0 \qquad \text{for } 0\leqq i\leqq w_1-1,\ 0\leqq j\leqq w_2-1\,.$$

This implies the desired result.

Hypothesis 13.2.

(i) *The group* $\mathfrak{L}=\mathfrak{X}$ *satisfies Hypothesis* 13.1.

(ii) $\mathfrak{L}$ *contains a normal subgroup* $\mathfrak{K}$ *such that*

$$\mathfrak{L}=\mathfrak{K}\mathfrak{W}_1,\ \mathfrak{K}\cap\mathfrak{W}_1=\langle 1\rangle$$

and if $\mathfrak{A}$ *is a non empty subset of* $\mathfrak{W}-\mathfrak{W}_2$ *then*

$$C(\mathfrak{A})=N(\mathfrak{A})=\mathfrak{W}\,.$$

Since $\mathfrak{W}_1$ is a S-subgroup of $\mathfrak{W}$, Hypothesis 13.2 (ii) implies that $\mathfrak{W}_1$ is a S-subgroup of $\mathfrak{L}$. Also, if $W\in\mathfrak{W}_1^\#$, then $C(W)\cap\mathfrak{K}=\mathfrak{W}_2$.

LEMMA 13.3. *Suppose that $\mathfrak{L}$ satisfies Hypothesis* 13.2. *Then* $\mathfrak{W}-\mathfrak{W}_2$ *is a* T. I. *set in* $\mathfrak{L}$. *For* $0\leqq i\leqq w_1-1,\ 0\leqq j\leqq w_2-1$ *there exist irreducible characters* μ_{ij} *of* $\mathfrak{L}$ *such that*

$$\mu_{ij|\mathfrak{W}} = \pm\,\omega_{ij} + \sum_{t=0}^{w_2-1} a_t \sum_{s=0}^{w_1-1} \omega_{st}\ ,$$

where $\{a_t\}$ *is a set of integers depending on* j *and the sign depends only on* j.

Proof. Hypothesis 13.2 implies that $\mathfrak{W} - \mathfrak{W}_2$ is a T. I. set in $\mathfrak{L}$. For $0 \leqq i, k \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1, \omega_{ij} - \omega_{kj}$ vanishes on $\mathfrak{W}_2$. Define

$$\mathscr{S}_j = \{\omega_{ij} \mid 0 \leqq i \leqq w_1 - 1\} \quad \text{for } 0 \leqq j \leqq w_2 - 1\ .$$

Then by Lemma 10.1 $\mathscr{S}_j$ is coherent for $0 \leqq j \leqq w_2 - 1$. Let $\mu_{ij} = \pm\omega_{ij}^{\tau}$, where the sign is chosen so that $\mu_{ij}(1) > 0$. Then

$$(\omega_{ij} - \omega_{kj})^{\tau} = (\omega_{ij} - \omega_{kj})^{*} = \pm(\mu_{ij} - \mu_{kj})$$
$$\text{for } 0 \leqq i, k \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1\ .$$

The Frobenius reciprocity theorem now implies the required result since $(\omega_{ij} - \omega_{kj})^{*}$ vanishes on $\mathfrak{W}_2$.

LEMMA 13.4. *Suppose that* $\mathfrak{L}$ *satisfies Hypothesis* 13.2. *Let* λ *be an irreducible character of* $\mathfrak{L}$. *Then there exists an integer* a *such that*

$$\lambda_{|\mathfrak{W}_1} = a\rho_{\mathfrak{W}_1}\ ,$$

or

$$\lambda_{|\mathfrak{W}_1} = \pm\omega_{ij_{|\mathfrak{W}_1}} + a\rho_{\mathfrak{W}_1}$$

for some i, j *with* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$.

Proof. Let μ_{ij} be the characters defined in Lemma 13.3. If $\lambda = \mu_{ij}$ for some i, j with $0 \leqq i \leqq w_1 - 1$, $0 \leqq j \leqq w_2 - 1$ then the result follows from Lemma 13.3. Furthermore Lemma 13.3 implies that

$$\sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} |\mu_{ij}(W)|^2 = w = |C(W)| \quad \text{for } W \in \mathfrak{W}_1^{\sharp}\ .$$

Hence if $\lambda \neq \mu_{ij}$ for all i, j we have that $\lambda(W) = 0$ for $W \in \mathfrak{W}_1^{\sharp}$. This completes the proof of the lemma.

We will use the fact that Lemma 13.4 is valid over fields of characteristic prime to $|\mathfrak{L}|$, provided that λ is absolutely irreducible.

LEMMA 13.5. *Suppose that* $\mathfrak{L}$ *satisfies Hypothesis* 13.2. *For*

$0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ let μ_{ij} be the characters defined by Lemma 13.3. *Define*

$$\xi_j = \sum_{i=0}^{w_1-1} \mu_{ij} \quad \textit{for } 0 \leqq j \leqq w_2 - 1 .$$

Then ξ_j is induced by an irreducible character μ_j of $\mathfrak{K}$. Furthermore

$$\mu_{ij|\mathfrak{K}} = \mu_{0j|\mathfrak{K}} = \mu_j \quad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

Proof. By Lemma 13.4 the characters $\mu_{ij}, 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ are the only irreducible characters of $\mathfrak{L}$ which do not vanish on $\mathfrak{W}_1^{\#}$. Since each μ_{i0} agrees on $\mathfrak{W}_1$ with a suitable linear character of $\mathfrak{L}/\mathfrak{K}$ it follows from Lemma 13.1 that $\{\mu_{i0} \mid 0 \leqq i \leqq w_1 - 1\}$ is the set of irreducible characters of $\mathfrak{L}/\mathfrak{K}$. Therefore $\mu_{i0}\mu_{0j}$ agrees with μ_{ij} on $\hat{\mathfrak{W}}$. Hence Lemma 13.1 implies that $\mu_{i0}\mu_{0j} = \mu_{ij}$. Consequently if $\mu_j = \mu_{0j|\mathfrak{K}}$ then

$$\mu_{ij|\mathfrak{K}} = \mu_{0j|\mathfrak{K}} = \mu_j \quad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

Thus the Frobenius reciprocity theorem implies that μ_{ij} is a constituent of μ_j^* for all values of i, j. Since

$$\mu_j^*(1) = w_1\mu_j(1) = \sum_{i=0}^{w_1-1} \mu_{ij}(1) = \xi_j(1)$$

the lemma is proved.

LEMMA 13.6. *Suppose that $\mathfrak{L}$ satisfies Hypothesis* 13.2, *p is a prime, and $\mathfrak{K}$ is an extra special p-group with $\mathfrak{K}' = \mathfrak{W}_2$. Let $|\mathfrak{K}:\mathfrak{K}'| = p^{2n}$. Then w_1 divides either $p^n + 1$ or $p^n - 1$.*

Proof. It is easily seen that a faithful irreducible character of $\mathfrak{K}$ has degree p^n. Thus by Lemmas 13.4 and 13.5

$$p^n = \mu_{11}(1) = aw_1 \pm 1 .$$

This proves the result.

LEMMA 13.7. *Suppose that $\mathfrak{L}$ satisfies Hypothesis* 13.2. *Let μ_j, ξ_j be defined by Lemma* 13.5. *Then an irreducible character of $\mathfrak{K}$ either induces an irreducible character of $\mathfrak{L}$ or it induces ξ_j for some j with $0 \leqq j \leqq w_2 - 1$.*

Proof. The group $\mathfrak{W}_1$ acts as a permutation group on the conjugate classes of $\mathfrak{K}$. If $W \in \mathfrak{W}_1$ and W leaves some conjugate class of $\mathfrak{K}$ fixed, then since $\mathfrak{W}_1$ is a Hall subgroup of $\mathfrak{L}$, W must centralize some element of this conjugate class. Hence by assumption the only conjugate classes of $\mathfrak{K}$ which are fixed by any $W \in \mathfrak{W}_1^\#$ are those containing an element of $\mathfrak{W}_2$. There are at most w_2 of these. The group $\mathfrak{W}_1$ also acts as a permutation group on the irreducible characters of $\mathfrak{K}$. Therefore by 3.14 there are at most w_2 irreducible characters of $\mathfrak{K}$ which are fixed by any element $W \in \mathfrak{W}_1^\#$. By Lemma 13.5 the w_2 distinct characters μ_j, $0 \leqq j < w_2$ are fixed by every $W \in \mathfrak{W}_1$ and these induce ξ_j, $0 \leqq j < w_2$. Thus every other irreducible character of $\mathfrak{K}$ induces an irreducible character of $\mathfrak{L}$. The proof is complete.

Hypothesis 13.3.

(i) *$\hat{\mathfrak{L}}$ is a tamely imbedded subset of the group $\mathfrak{X}$ and $\mathfrak{L} = N(\hat{\mathfrak{L}})$ has odd order.*

(ii) *$\mathfrak{L}$ satisfies Hypothesis 13.2, and $\mathfrak{X}$ satisfies Hypothesis 13.1 with the same group $\mathfrak{W}$.*

(iii) *$\mathfrak{L}$ contains a normal nilpotent subgroup $\mathfrak{H}$ such that*

$$\mathfrak{W}_2 \subseteqq \mathfrak{H} \subseteqq \bigcup_{H \in \mathfrak{H}^\#} C(H) \cap \mathfrak{K} \subseteqq \hat{\mathfrak{L}} \subseteqq \mathfrak{K} \subset \mathfrak{L} .$$

$$\hat{\mathfrak{L}}_1 = \hat{\mathfrak{L}} \cup \bigcup_{L \in \mathfrak{L}} L^{-1} \hat{\mathfrak{W}} L .$$

(iv) *There exist subgroups $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ such that $\{\mathfrak{H}_s \mid 1 \leqq s \leqq n\}$ is a system of supporting subgroups of $\hat{\mathfrak{L}}$ and $\hat{\mathfrak{L}}_1$. Let $\mathfrak{N}_s = N(\mathfrak{H}_s)$ for $1 \leqq s \leqq n$.*

(v) *For $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ let $\eta_{ij}, \mu_{ij}, \xi_j$ be defined respectively by Lemmas 13.1, 13.3 and 13.5.*

(vi) *Let $\mathscr{S}$ be the set of characters of $\mathfrak{L}$ which are induced by non principal irreducible characters of $\mathfrak{K}$, each of which vanishes outside $\hat{\mathfrak{L}}$.*

LEMMA 13.8. *Suppose that Hypothesis 13.3 is satisfied. Assume that for some i, j, k with $0 \leqq i \leqq w_1 - 1, 1 \leqq j, k \leqq w_2 - 1$, $\mu_{ij}(1) = \mu_{ik}(1)$. Then $\mu_{ij} - \mu_{ik}$ vanishes in $\mathfrak{L} - \hat{\mathfrak{L}}_1^\#$ and*

$$(\mu_{ij} - \mu_{ik})^\tau = \pm(\eta_{ij} - \eta_{ik}) .$$

Proof. By Lemma 13.3 μ_{ij}, μ_{ik} do not contain $\mathfrak{W}_2$ in their kernel, thus they do not contain $\mathfrak{H}$ in their kernel. Hence by Lemma 4.3 μ_{ij}, μ_{ik} vanish on $\mathfrak{K} - \hat{\mathfrak{L}}$. By Lemma 13.3 $\mu_{ij|\mathfrak{W}_1} = \mu_{ik|\mathfrak{W}_1}$. Thus $\mu_{ij} - \mu_{ik}$ vanishes on $\mathfrak{L} - \hat{\mathfrak{L}}_1^\#$. Hence $\| (\mu_{ij} - \mu_{ik})^\tau \|^2 = 2$. By Lemmas 9.1 and 13.3

$$\{(\mu_{ij} - \mu_{ik})^\tau \pm (\eta_{ij} - \eta_{ik})\}(W) = 0 \quad \text{for } W \in \hat{\mathfrak{W}} .$$

Thus the result follows from Lemma 13.1.

LEMMA 13.9. *Suppose that Hypothesis* 13.3 *is satisfied. Choose* k *with* $1 \leqq k \leqq w_2 - 1$. *Let*

$$\mathscr{S}_1 = \{\xi_j \mid 1 \leqq j \leqq w_2 - 1, \xi_j(1) = \xi_k(1)\} .$$

Then $\mathscr{S}_1$ *is coherent and*

$$\xi_j^\tau = \varepsilon \sum_{i=0}^{w_1-1} \eta_{ij}$$

is an extension of τ *to* $\mathscr{S}_1$ *where either* $\varepsilon = 1$ *or* $\varepsilon = -1$.

Proof. Since $|\mathfrak{L}|$ is odd $\xi_j \neq \bar{\xi}_j$. Hence $\mathscr{I}_0(\mathscr{S}_1) \neq 0$. By Lemma 13.5

$$\xi_j - \xi_k = \sum_{i=0}^{w_1-1} (\mu_{ij} - \mu_{ik}) .$$

Hence Lemma 13.8 yields that

$$(\xi_j - \xi_k)^\tau = \sum_{i=0}^{w_1-1} \pm (\eta_{ij} - \eta_{ik}) .$$

By Lemma 9.1 $(\xi_j - \xi_k)^\tau$ vanishes on $\hat{\mathfrak{W}}_1$. Thus Lemma 13.2 implies that

$$(\xi_j - \xi_k)^\tau = \pm \sum_{i=0}^{w_1-1} (\eta_{ij} - \eta_{ik}) . \tag{13.7}$$

Now define

$$\xi_j^\tau = \pm \sum_{i=0}^{w_1-1} \eta_{ij}$$

where the sign is the same as in (13.7). It is easily seen that τ is a linear isometry on $\mathscr{S}_1$. Thus $\mathscr{S}_1$ is coherent.

LEMMA 13.10. *Suppose that Hypothesis* 13.3 *is satisfied. Let* $\mathscr{S}_1$ *have the same meaning as in Lemma* 13.9. *Then* $(\mathscr{S}_1, \tau)$ *is subcoherent in* $\mathscr{S}$ *where* τ *is defined on* $\mathscr{S}_1$ *as in Lemma* 13.9.

Proof. By Lemma 13.9 $\mathscr{S}_1$ is coherent. Let $\mathscr{T}$ be a coherent subset of $\mathscr{S}$ which is orthogonal to $\mathscr{S}_1$. Let τ_2 be an extension of τ to $\mathscr{T}$.

Every generalized character in $\mathscr{S}$ vanishes on $\hat{\mathfrak{W}}$. Thus by Lemma 9.1 every generalized character in $\mathscr{I}_0(\mathscr{S})^\tau$ vanishes on $\hat{\mathfrak{W}}$. If λ is an irreducible character in $\mathscr{T}$, then $\lambda \neq \bar{\lambda}$ as $|\mathfrak{L}|$ is odd. Furthermore $(\lambda - \bar{\lambda})^\tau \in \mathscr{I}_0(\mathscr{S})^\tau$ and thus vanishes on $\hat{\mathfrak{W}}$. Hence $\lambda^{\tau_2} \neq \pm\eta_{ij}$ for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$. Therefore λ^{τ_2} is orthogonal to $\mathscr{S}_1^\tau$. If $\xi_s \in \mathscr{T}$, then since $(\xi_s^{\tau_2}, (\xi_s - \bar{\xi}_s)^\tau) = w_1$, $\xi_s^{\tau_2}$ is a linear combination of η_{is} and $\bar{\eta}_{is}$ with $0 \leqq i \leqq w_1 - 1$. Hence $\xi_s^{\tau_2}$ is orthogonal to $\mathscr{S}_1^\tau$. Consequently $\mathscr{T}^{\tau_2}$ is orthogonal to $\mathscr{S}_1^\tau$.

Suppose now that $\alpha \in \mathscr{I}_0(\mathscr{S})$ with $\alpha^\tau = \Delta_1 + \Delta_2$, where $\Delta_2 \in \mathscr{C}(\mathscr{T}^{\tau_2})$, Δ_1 is not orthogonal to $\mathscr{I}_0(\mathscr{S}_1^\tau)$ and $\|\Delta_1\|^2 \leqq w_1$. Let $\alpha^\tau = \Gamma + \Delta$, where Δ is a linear combination of the generalized characters η_{ij} and $(\Gamma, \eta_{ij}) = 0$ for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$. Let σ be the set of integers s such that $\xi_s \in \mathscr{T}$. Lemma 13.8 implies that every generalized character in $\mathscr{T}^{\tau_2}$ is orthogonal to η_{ij} for $0 \leqq i \leqq w_1 - 1, j \notin \sigma$. Let $\Delta = \Delta_0 + \Delta_1'$, where Δ_0 is a linear combination of η_{is} with $s \in \sigma$ and $(\Delta_1', \eta_{is}) = 0$ for $0 \leqq i \leqq w_1 - 1, s \in \sigma$. Then

$$\|\Delta_1'\|^2 \leqq w_1 . \tag{13.8}$$

By changing notation it may be assumed that $\xi_1, \xi_2 \in \mathscr{S}_1$ and $(\Delta_1', \xi_1^\tau - \xi_2^\tau) > 0$. By Lemma 9.4

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau)_{\mathfrak{X}} = (\alpha^\tau, \xi_1^\tau - \xi_2^\tau)_{\mathfrak{X}} = (\alpha, \xi_1 - \xi_2)_{\mathfrak{L}} .$$

Hence $(\Delta_1', \xi_1^\tau - \xi_2^\tau)$ is a non zero integral multiple of w_1. By (13.8)

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau)^2 \leqq \|\Delta_1'\|^2 \, \|\xi_1^\tau - \xi_2^\tau\|^2 \leqq 2w_1^2 .$$

Therefore

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau) = w_1 . \tag{13.9}$$

By Lemma 13.2

$$\Delta_1' = \varepsilon \sum_{i=0}^{w_1-1} a_{i0}\eta_{i0} + \varepsilon \sum_{i=0}^{w_1-1} \{(a_{i0} + a_{01})\eta_{i1} + (a_{i0} + a_{02})\eta_{i2}\} + \Delta_1'' , \tag{13.10}$$

where ε is as in Lemma 13.9 and where $(\Delta_1'', \eta_{it}) = 0$ for $0 \leqq i \leqq w_1 - 1$, $t = 0, 1, 2$. Now (13.9) yields that $a_{01} - a_{02} = 1$. Thus (13.8) and (13.10) imply

$$\sum_{i=0}^{w_1-1} a_{i0}^2 + \sum_{i=0}^{w_1-1} \{(a_{i0} + a_{01})^2 + (a_{i0} + a_{01} - 1)^2\} \leqq w_1 .$$

Every term in the second summation is non zero. Thus $a_{i0} = 0$ for $0 \leqq i \leqq w_1 - 1$. Hence $a_{01} = 1$ or $a_{01} = 0$. Hence (13.8) and (13.10) yield that $\Delta_1' = \xi_1^\tau$ or $\Delta_1' = -\xi_2^\tau$. This shows that $(\mathscr{S}_1, \tau)$ is subcoherent in $\mathscr{S}$ and completes the proof of the lemma.

In the proof of the main theorem of this paper we will reserve the letter τ to denote the extension of τ to $\mathscr{S}_1$ defined by Lemma 13.9. Thus $(\mathscr{S}_1, \tau)$ will always be subcoherent in $\mathscr{S}$.

DEFINITION. A *Z-group* is a group all of whose Sylow subgroups are cyclic.

Hypothesis 13.4.

(i) *$\mathfrak{L} = \mathfrak{B}\mathfrak{K}$ with $\mathfrak{B} \cap \mathfrak{K} = 1$, $\mathfrak{K} \lhd \mathfrak{L}$ and $\mathfrak{K}$ solvable. Furthermore $\mathfrak{B}$ is a cyclic S-subgroup of $\mathfrak{L}$ and $|\mathfrak{L}|$ is odd.*

(ii) *For $B \in \mathfrak{B}^\#$, $C_{\mathfrak{K}}(B) = C_{\mathfrak{K}}(\mathfrak{B})$. Furthermore $C_{\mathfrak{K}}(\mathfrak{B})$ is a Z-group and $\mathfrak{K} \neq C_{\mathfrak{K}}(\mathfrak{B})$.*

(iii) *$\mathfrak{L}$ is faithfully and irreducibly represented on a vector space $\mathscr{V}$ over a field of characteristic not dividing $|\mathfrak{L}|$. $\mathscr{V}$ contains a vector space $\mathscr{V}_0$ of dimension at most 1 such that if $B \in \mathfrak{B}^\#$, $v \in \mathscr{V}$ then $vB = v$ if and only if $v \in \mathscr{V}_0$.*

LEMMA 13.11. *Suppose that Hypothesis 13.4 is satisfied. Then $\mathfrak{K}$ is nilpotent. Furthermore $|\mathfrak{B}|$ is a prime and the representation of $\mathfrak{L}$ on $\mathscr{V}$ is absolutely irreducible.*

Proof. Let λ be the character of the representation of $\mathfrak{L}$ on $\mathscr{V}$. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{K}$ which is normalized but not centralized by $\mathfrak{B}$. Then either $C_{\mathfrak{P}}(\mathfrak{B}) = 1$ or $\mathfrak{P}\mathfrak{B}$ satisfies Hypothesis 13.2. Thus by Lemma 13.4 only one absolutely irreducible constituent of $\lambda_{|\mathfrak{P}\mathfrak{B}}$ is not linear. Hence λ is absolutely irreducible. Furthermore Lemma 13.4 and 3.16 (iii) imply that $\lambda_{|\mathfrak{B}}$ has $\rho_{\mathfrak{B}}$ as a constituent. Thus $|\mathfrak{B}|$ is a prime.

The nilpotence of $\mathfrak{K}$ is proved by induction on $|\mathfrak{K}|$. We assume without loss of generality that the underlying field is algebraically closed. If $\mathfrak{B} \subseteq \boldsymbol{F}(\mathfrak{L})$ then $\mathfrak{K} \subseteq \boldsymbol{C}(\mathfrak{B})$ contrary to assumption. Thus by 3.3 $\mathfrak{B} \not\subseteq \boldsymbol{C}(\boldsymbol{F}(\mathfrak{L}))$. Let $\mathfrak{F}$ be a minimal nilpotent normal subgroup of $\mathfrak{L}$ which is not centralized by $\mathfrak{B}$. Then $\mathfrak{F}$ is a p-group for some prime p. Furthermore $\mathfrak{F}' = \boldsymbol{D}(\mathfrak{F})$ and $\mathfrak{B} \subseteq \boldsymbol{C}(\boldsymbol{D}(\mathfrak{F}))$. By Lemma 13.4 there is exactly one non linear irreducible constituent of $\lambda_{|\mathfrak{F}\mathfrak{B}}$. Let

$$\lambda_{|\mathfrak{F}\mathfrak{B}} = \sum_{i=1}^{n} \mu_i + \theta ,$$

where each μ_i is a linear character of $\mathfrak{F}\mathfrak{B}$. Assume first that $n \neq 0$. If ν is an irreducible constituent of $\theta_{|\mathfrak{F}}$, then $(\nu, \theta_{|\mathfrak{F}}) = 1$. Since $\nu \neq \mu_{i_{|\mathfrak{F}}}$ for $1 \leqq i \leqq n$, we have $(\lambda_{|\mathfrak{F}}, \mu_{i_{|\mathfrak{F}}}) = 1$. Since $\lambda_{|\mathfrak{F}}$ is a sum of conjugate characters this implies that $\mathfrak{F}$ is abelian and the μ_i are distinct. Thus $\mathfrak{F}\mathfrak{B} = \mathfrak{F}_0 \times \mathfrak{F}_1\mathfrak{B}$, where $|\mathfrak{F}_0| = p$ and $\mathfrak{F}_1\mathfrak{B}$ is a Frobenius group. For $L \in \mathfrak{L}$ let $\mu_i^L(X) = \mu_i(L^{-1}XL)$. If $L \in \mathfrak{L}$ such that $\mu_i^L = \mu_j$ for some i, j then $L \in N(\mathfrak{F}_1)$ since $\mathfrak{F}_1$ is the kernel of each $\mu_{i_{|\mathfrak{F}}}$. Since $\mathfrak{L}$ permutes the constituents of $\lambda_{|\mathfrak{F}}$ transitively this implies that $N(\mathfrak{F}_1)$ acts transitively on $\{\mu_1, \cdots, \mu_n\}$. Hence n is odd. Thus $\lambda(1) = n + |\mathfrak{B}|$

is even contradicting the absolute irreducibility of λ. Therefore $n = 0$ and $\lambda_{|\mathfrak{F}\mathfrak{B}}$ is irreducible.

By Lemma 13.4 this implies that $\lambda(1) = |\mathfrak{B}|$ or $\lambda(1) = 2|\mathfrak{B}| - 1$. If $\lambda(1) = |\mathfrak{B}|$ then $\lambda_{|\mathfrak{K}}$ is reducible since $(|\mathfrak{B}|, |\mathfrak{K}|) = 1$. As $|\mathfrak{B}|$ is a prime this implies that $\lambda_{|\mathfrak{K}}$ is a sum of linear characters and $\mathfrak{K}$ is abelian. Thus we can suppose that $\lambda(1) = 2|\mathfrak{B}| - 1$. By Lemma 13.4 $\lambda_{|\mathfrak{F}}$ is irreducible. Thus if $\mathfrak{H}$ is any proper $\mathfrak{B}$-invariant subgroup of $\mathfrak{K}$ with $\mathfrak{F} \subseteq \mathfrak{H}$ then $\mathfrak{B}\mathfrak{H}$ satisfies the induction assumption and $\mathfrak{H}$ is nilpotent. If $\mathfrak{H} = \mathfrak{P} \times \mathfrak{H}_1$ with $\mathfrak{F} \subseteq \mathfrak{P}$ then since $\lambda_{|\mathfrak{P}}$ is irreducible, $\mathfrak{H}_1 \subseteq \boldsymbol{Z}(\mathfrak{L})$. If $\mathfrak{F}$ is not a S_p-subgroup of $\mathfrak{L}$ then $\mathfrak{F}\mathfrak{K}_1$ is a proper subgroup of $\mathfrak{K}$ where $\mathfrak{K}_1$ is a $\mathfrak{B}$-invariant p-complement in $\mathfrak{K}$. Thus $\mathfrak{K}_1 \subseteq \boldsymbol{Z}(\mathfrak{K})$ and $\mathfrak{K}$ is nilpotent. Suppose now that $\mathfrak{F}$ is a S_p-subgroup of $\mathfrak{L}$.

Since $\boldsymbol{D}(\mathfrak{F}) \subseteq \boldsymbol{C}(\mathfrak{B})$, $\boldsymbol{D}(\mathfrak{F})$ is cyclic. Let $\mathfrak{F}_1$ be the subgroup of index p in $\boldsymbol{D}(\mathfrak{F})$. Then $\mathfrak{F}/\mathfrak{F}_1$ is a p-group of class 2 and hence is a regular p-group. If $\mathfrak{F}/\mathfrak{F}_1$ does not have exponent p then there exists a characteristic subgroup of $\mathfrak{F}$ of index p which is normal in $\mathfrak{L}$ but is not centralized by $\mathfrak{B}$ contrary to the minimality of $\mathfrak{F}$. Thus $\mathfrak{F}/\mathfrak{F}_1$ has exponent p. Therefore $\mathfrak{B}$ acts without fixed points on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$ as $\boldsymbol{C}_{\mathfrak{F}}(\mathfrak{B})$ is cyclic and $\boldsymbol{D}(\mathfrak{F}) \subseteq \boldsymbol{C}(\mathfrak{B})$.

Let $\mathfrak{K}/\mathfrak{H}$ be a chief factor of $\mathfrak{L}$ with $\mathfrak{F} \subseteq \mathfrak{H}$. Suppose first that $\mathfrak{B}$ does not centralize $\mathfrak{K}/\mathfrak{H}$. Then $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is a Frobenius group which is represented on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$. As $\mathfrak{B}$ has no fixed points on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$ Lemma 4.6 implies that $\mathfrak{K}/\mathfrak{H}$ acts trivially on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$. Thus $\mathfrak{K} = \mathfrak{F}\boldsymbol{C}_{\mathfrak{K}}(\mathfrak{F})$ is nilpotent. Assume now that $\mathfrak{L}/\mathfrak{H}$ is abelian. Then $|\mathfrak{K}:\mathfrak{H}| = q$ for some prime $q \neq p$. If $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is represented faithfully on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$, the minimal nature of $\mathfrak{F}$ implies that $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is represented irreducibly on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$. Let $\mathfrak{K}/\mathfrak{H} = \langle Q\mathfrak{H}\rangle$. Then Q acts without fixed points on $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$. Since $\lambda_{|\mathfrak{F}}$ is irreducible, $\boldsymbol{Z}(\mathfrak{F}) \subseteq \boldsymbol{Z}(\mathfrak{L})$. Thus $Q \in \boldsymbol{C}(\Omega_1(\boldsymbol{D}(\mathfrak{F})))$. Hence $Q \in \boldsymbol{C}(\boldsymbol{D}(\mathfrak{F}))$. We will now reach a contradiction from the fact that $Q \notin \boldsymbol{C}(\mathfrak{F})$. Let $\mathfrak{H} = \mathfrak{F} \times \mathfrak{H}_1$. Then $\mathfrak{H}_1 \subseteq \boldsymbol{Z}(\mathfrak{L})$. Thus $\mathfrak{L}/\mathfrak{F}$ is abelian. Let μ be the linear character of $\mathfrak{L}/\mathfrak{F}$ such that $\lambda(H) = \lambda(1)\mu(H)$ for $H \in \mathfrak{H}_1$. Let $\lambda_0 = \lambda\mu^{-1}$. Then $\lambda_0(1) = \lambda(1) = 2|\mathfrak{B}| - 1$ and λ_0 is an irreducible character of $\mathfrak{L}/\mathfrak{H}_1$. The group $\mathfrak{L}/\mathfrak{H}_1$ satisfies Hypothesis 13.2 where $\mathfrak{F}\mathfrak{H}_1/\mathfrak{H}_1$ is the normal subgroup. Thus by Lemma 13.4 no irreducible character of $\mathfrak{L}/\mathfrak{H}_1$ has degree $2|\mathfrak{B}| - 1$. This completes the proof of the lemma in all cases.

DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{B}$ be subgroups of a group $\mathfrak{L}$ with $\mathfrak{B} \subseteq \boldsymbol{N}(\mathfrak{A})$. We say that $\mathfrak{B}$ *is prime on* $\mathfrak{A}$ if

$$\boldsymbol{C}_{\mathfrak{A}}(B) = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{B}) \quad \text{for } B \in \mathfrak{B}^{\#} .$$

If $|\mathfrak{B}|$ is a prime then $\mathfrak{B}$ is necessarily prime on $\mathfrak{A}$.

Lemma 13.12. *Let $\mathfrak{L} = \mathfrak{A}\mathfrak{B}$ with $\mathfrak{A} \lhd \mathfrak{L}$, $\mathfrak{A}$ solvable, $\mathfrak{B}$ cyclic, $(|\mathfrak{A}|, |\mathfrak{B}|) = 1$ and $|\mathfrak{A}\mathfrak{B}|$ odd. Suppose that $\mathfrak{B}$ is prime on $\mathfrak{A}$ and $C_{\mathfrak{A}}(\mathfrak{B})$ is a Z-group. If $C_{\mathfrak{A}}(\mathfrak{B}) \subseteq \mathfrak{A}'$ then $\mathfrak{A}/F(\mathfrak{A})$ is nilpotent. If furthermore $|\mathfrak{B}|$ is not a prime then $\mathfrak{A}$ is nilpotent.*

Proof. Let $\mathfrak{L}$ be a counter example to the result for which $|\mathfrak{A}|$ has minimum order. Since $(|\mathfrak{A}|, |\mathfrak{B}|) = 1$ the hypotheses are satisfied by all $\mathfrak{B}$-invariant factor groups of $\mathfrak{A}$.

Suppose that $|\mathfrak{B}|$ is not a prime. Let $\mathfrak{M}$ be a minimal normal subgroup of $\mathfrak{L}$. Then $\mathfrak{M}$ is a p-group for some prime p and $\mathfrak{M} \subseteq \mathfrak{A}$. By induction $\mathfrak{A}/\mathfrak{M}$ is nilpotent. If $\mathfrak{Q}$ is a $\mathfrak{B}$-invariant S_q-group of $\mathfrak{A}$ for $q \in \pi(\mathfrak{A})$, $q \neq p$, then $\mathfrak{M}\mathfrak{Q} \lhd \mathfrak{A}\mathfrak{B}$ and $\mathfrak{B}$ has no fixed points on $\mathfrak{Q} - \mathfrak{Q}'$. If $\mathfrak{A}$ is not nilpotent then it is possible to choose q so that $\mathfrak{M}\mathfrak{Q}$ is not nilpotent. Let $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{M})$. Then $\mathfrak{B}\mathfrak{Q}/\mathfrak{Q}_1$ is faithfully represented on $\mathfrak{M}$. Hypothesis 13.4 is satisfied with $\mathfrak{M}$ in the role of $\mathscr{V}$. Thus by Lemma 13.11 $|\mathfrak{B}|$ is a prime contrary to assumption.

Assume now that $|\mathfrak{B}|$ is a prime. Suppose that $\mathfrak{L}$ contains two distinct minimal normal subgroups $\mathfrak{M}_1$ and $\mathfrak{M}_2$. For $i = 1, 2$ let $\mathfrak{F}_i$ be the inverse image of $F(\mathfrak{A}/\mathfrak{M}_i)$ in $\mathfrak{A}$. By induction $\mathfrak{A}/\mathfrak{F}_i$ is nilpotent for $i = 1, 2$. The result now follows from the fact that $F(\mathfrak{A}) = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Thus it may be assumed that $\mathfrak{L}$ contains a unique minimal normal subgroup $\mathfrak{M}$. Then $\mathfrak{M} \subseteq O_p(\mathfrak{A}) = F(\mathfrak{A})$ for some prime p. Let $\mathfrak{D} = D(O_p(\mathfrak{A}))$. Then $F(\mathfrak{A}/\mathfrak{D})$ is a p-group. Thus the result follows by induction if $\mathfrak{D} \neq 1$. Assume now that $\mathfrak{D} = 1$. Then $C_{\mathfrak{A}}(\mathfrak{M}) = O_p(\mathfrak{A})$.

Let $\mathfrak{A}_1$ be a $\mathfrak{B}$-invariant $S_{p'}$-subgroup of $\mathfrak{A}$. Then $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented of $\mathfrak{M}$. Hypothesis 13.4 is satisfied with $\mathfrak{M}$ in place of $\mathscr{V}$ unless $\mathfrak{A}_1 \subseteq C_{\mathfrak{A}}(\mathfrak{B})$. Thus by Lemma 13.11 $\mathfrak{A}_1$ is nilpotent or $\mathfrak{A}_1 \subseteq C_{\mathfrak{A}}(\mathfrak{B})$.

Let $\mathfrak{A}_0 = \mathfrak{A}/O_p(\mathfrak{A})$ and let $\mathfrak{P}_0$ be a $\mathfrak{B}$-invariant S_p-group of $\mathfrak{A}_0$. If $\mathfrak{P}_0 \subseteq F(\mathfrak{A}_0)$ then $\mathfrak{A}_0/\mathfrak{P}_0$ is nilpotent since it is a p'-group and the result is proved. Assume that $\mathfrak{P}_0 \not\subseteq F(\mathfrak{A}_0)$. By induction $\mathfrak{A}_0/F(\mathfrak{A}_0)$ is nilpotent. Hence $\mathfrak{B}$ does not centralize $\mathfrak{P}_0$ by assumption.

Let $\mathfrak{P}$ be a p-group in $\mathfrak{A}_0$ which is minimal with the property that $\mathfrak{B}$ normalizes $\mathfrak{P}$ but does not centralize $\mathfrak{P}$. Since $F(\mathfrak{A}_0)$ is a p'-group there is a prime $q \neq p$ such that $\mathfrak{P}\mathfrak{Q}$ contains no normal p-subgroup, where $\mathfrak{Q}$ is a S_q-group of $F(\mathfrak{A}_0)$. Thus $\mathfrak{B}\mathfrak{P}$ acts faithfully on $\mathfrak{Q}$. Let $\mathfrak{M}_1 = C_{\mathfrak{M}}(\mathfrak{B})$. As $\mathfrak{Q}\mathfrak{B}$ is faithfully represented on $\mathfrak{M}$ Lemmas 4.6 and 13.4 imply that $\mathfrak{M}_1 \neq 1$. Let $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{B})$. As $\mathfrak{P}\mathfrak{B}$ is represented faithfully on $\mathfrak{Q}/D(\mathfrak{Q})$, Lemmas 4.6 and 13.4 imply that $\mathfrak{Q}_1 \neq 1$. Thus $C_{\mathfrak{A}}(\mathfrak{B})$ is a Z-group, $\mathfrak{M}_1 \lhd C_{\mathfrak{A}}(\mathfrak{B})$ and $pq \mid |C_{\mathfrak{A}}(\mathfrak{B})|$. Therefore

$$p \equiv 1 \pmod{q}. \tag{13.11}$$

By 3.11 $\mathfrak{P}$ is a special p-group and $D(\mathfrak{P}) \subseteq C_{\mathfrak{P}}(\mathfrak{B})$. Thus $D(\mathfrak{P})$ is cyclic. By Lemma 13.11 the representation of $\mathfrak{P}\mathfrak{B}$ on $\mathfrak{Q}/D(\mathfrak{Q})$ has a unique faithful irreducible constituent and this constituent is absolutely irreducible. Let μ be the character of this constituent. If $D(\mathfrak{P}) \neq 1$ then by Lemma 13.4 $\mu_{|\mathfrak{P}}$ remains absolutely irreducible. Hence $q \equiv 1 \pmod p$ contrary to (13.11). Therefore $\mathfrak{P}$ is an elementary abelian group and $\mathfrak{B}\mathfrak{P}$ is a Frobenius group. Thus $\mu(1) = |\mathfrak{B}|$ is a prime. If $\mathfrak{P}$ is not cyclic then $\mu_{|\mathfrak{P}}$ is reducible in the field of q elements as $\mu_{|\mathfrak{P}}$ is faithful. Thus $q \equiv 1 \pmod p$ contrary to (13.11). Therefore $\mathfrak{P}$ is a cyclic group of order p and $\mathfrak{B}\mathfrak{P}$ is a Frobenius group. Hence

$$p \equiv 1 \pmod{|\mathfrak{B}|} . \tag{13.12}$$

Let $\mathfrak{Q}_0$ be a $\mathfrak{B}\mathfrak{P}$ invariant subgroup of $\mathfrak{Q}$ which is minimal subject to $\mathfrak{P} \not\subseteq C_{\mathfrak{A}_0}(\mathfrak{Q}_0)$. Thus the representation of $\mathfrak{B}\mathfrak{P}$ on $\mathfrak{Q}_0/D(\mathfrak{Q}_0)$ is irreducible. Therefore $\mathfrak{Q}_0 \subseteq (\mathfrak{Q}_0\mathfrak{P})'$. Since $O_p(\mathfrak{A})$ is elementary and $C_{\mathfrak{M}}(\mathfrak{B}) \neq 1$ we get that the hypotheses of the lemma are satisfied. Thus the minimal nature of $\mathfrak{A}$ implies that $\mathfrak{A}_0 = \mathfrak{Q}\mathfrak{P}$ and $\mathfrak{Q} = \mathfrak{Q}_0$. Therefore the representation of $\mathfrak{B}\mathfrak{Q}\mathfrak{P}$ on $\mathfrak{M}$ is irreducible. Let $\mathfrak{Q}_1$ be a minimal normal subgroup of $\mathfrak{B}\mathfrak{Q}\mathfrak{P}$ which is not centralized by $\mathfrak{B}$. Thus $\mathfrak{Q}_1 \subseteq \mathfrak{Q}$. Then $\mathfrak{Q}_1' = D(\mathfrak{Q}_1)$ and $\mathfrak{B} \subseteq C(D(\mathfrak{Q}_1))$. Hence $D(\mathfrak{Q}_1)$ is cyclic. Let λ be the character of the representation of $\mathfrak{B}\mathfrak{Q}_1$ on $\mathfrak{M}$. By Lemma 13.4 λ has exactly one irreducible constituent which does not have $(\mathfrak{B}\mathfrak{Q}_1)'$ in its kernel. Let θ be this constituent and let

$$\lambda = \sum_{i=1}^{n} \lambda_i + \theta .$$

Since each λ_i is a character of a group of exponent $q|\mathfrak{B}|$ it follows from (13.11) and (13.12) that each λ_i is absolutely irreducible. Thus $\lambda_i(1) = 1$ for $1 \leqq i \leqq n$. By Lemma 13.11 θ is absolutely irreducible in the field of p elements. By Lemma 13.4 $\theta(1) \leqq 2|\mathfrak{B}| - 1$. Since $|\mathfrak{B}|p$ is odd (B) and (13.12) yield that

$$|\mathfrak{M}| \geqq p^p \geqq p^{2|\mathfrak{B}|} . \tag{13.13}$$

Thus $n \neq 0$. Let $\theta_{|\mathfrak{Q}_1} = \sum_{j=1}^{m} \nu_j$, where each ν_j is an irreducible character of $\mathfrak{Q}_1$. Thus

$$\lambda = \sum_{i=1}^{n} \lambda_{i|\mathfrak{Q}_1} + \sum_{j=1}^{m} \nu_j . \tag{13.14}$$

Since $\mathfrak{Q}_1 \lhd \mathfrak{Q}_1\mathfrak{B}\mathfrak{P}$, $\{\lambda_{i|\mathfrak{Q}_1}, \nu_j\}$ is a set of conjugate characters. Since $n \neq 0$ they are all linear. Thus $\mathfrak{Q}_1' = 1$. Hence $\mathfrak{Q}_1\mathfrak{B} = \mathfrak{Q}_2 \times \mathfrak{Q}_3\mathfrak{B}$, where $\mathfrak{Q}_3\mathfrak{B}$ is a Frobenius group and $|\mathfrak{Q}_2| = q$. Furthermore

(13.15) $$m = \theta(1) = |\mathfrak{B}| .$$

Since $\mathfrak{Q}_3 \subseteq \ker \lambda_i \neq \mathfrak{Q}_1$ for $1 \leqq i \leqq n$ we see that $\lambda_{i|\mathfrak{Q}_1} \neq \nu_j$ for all i, j. Since $\nu_i \neq \nu_j$ for $i \neq j$ we get that no constituent of $\lambda_{|\mathfrak{Q}_1}$ occurs with multiplicity greater than one. Since $\{\lambda_{i|\mathfrak{Q}_2}\}$ is a set of distinct linear characters of $\mathfrak{Q}_2$ we get that $n \leqq q$. Now (13.13), (13.14) and (13.15) yield that

$$p \leqq \lambda(1) = m + n \leqq |\mathfrak{B}| + q .$$

This contradicts (13.11) and (13.12) since $|\mathfrak{B}|pq$ is odd. The proof is complete.

CHAPTER IV

14. Statement of Results Proved in Chapter IV

In this chapter, we begin the proof of the main theorem of this paper. The proof is by contradiction. If the theorem is false, a minimal counterexample is seen to be a non cyclic simple group all of whose proper subgroups are solvable. Such a group is called a *minimal simple group.* Throughout the remainder of this chapter, $\mathfrak{G}$ is a minimal simple group of odd order. We will eventually derive a contradiction from the assumed existence of $\mathfrak{G}$.

In this section, the results to be proved in this chapter are summarized. Several definitions are required.

Let π^* be the subset of $\pi(\mathfrak{G})$ consisting of all primes p such that if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then either $\mathscr{SCN}_3(\mathfrak{P})$ is empty or $\mathfrak{P}$ contains a subgroup $\mathfrak{A}$ of order p such that $C_{\mathfrak{P}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{B}$ is cyclic. Let π_1^* be the subset of π^* consisting of those p such that if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and a is the order of a cyclic subgroup of $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$, then one of the following possibilities occurs:

(i) a divides $p - 1$.

(ii) $\mathfrak{P}$ is abelian and a divides $p + 1$.

(iii) $|\mathfrak{P}| = p^3$ and a divides $p + 1$.

We now define five types of subgroups of $\mathfrak{G}$. The basic property shared by these five types is that they are all maximal subgroups of $\mathfrak{G}$. Thus, for $x =$ I, II, III, IV, V, any group of type x is by definition a maximal subgroup of $\mathfrak{G}$. The remaining properties are more detailed.

We say that $\mathfrak{M}$ is of type I provided

(i) $\mathfrak{M}$ is of Frobenius type with Frobenius kernel $\mathfrak{H}$.

(ii) One of the following conditions is satisfied:

(a) $\mathfrak{H}$ is a T. I. set in $\mathfrak{G}$.

(b) $\pi(\mathfrak{H}) \subseteq \pi_1^*$.

(c) $\mathfrak{H}$ is abelian and $m(\mathfrak{H}) = 2$.

(iii) If $p \in \pi(\mathfrak{M}/\mathfrak{H})$, then $m_p(\mathfrak{M}) \leq 2$ and a S_p-subgroup of $\mathfrak{M}$ is abelian.

The remaining four types are by definition three step groups. If $\mathfrak{S}$ is a three step group, we use the following notation:

$$\mathfrak{S} = \mathfrak{S}'\mathfrak{W}_1 , \qquad \mathfrak{S}' \cap \mathfrak{W}_1 = 1 , \qquad C_{\mathfrak{S}'}(\mathfrak{W}_1) = \mathfrak{W}_2 .$$

Furthermore, $\mathfrak{H}$ denotes the maximal normal nilpotent S-subgroup of $\mathfrak{S}$. By definition, $\mathfrak{H} \subseteq \mathfrak{S}'$ so we let $\mathfrak{U}$ be a complement for $\mathfrak{H}$ in $\mathfrak{S}'$.

In addition to being a three step group, each of the remaining four types has the property that if $\mathfrak{W}_0$ is any non empty subset of $\mathfrak{W}_1\mathfrak{W}_2 - \mathfrak{W}_1 - \mathfrak{W}_2$, then $N_{\mathfrak{G}}(\mathfrak{W}_0) = \mathfrak{W}_1\mathfrak{W}_2$, by definition. The remaining properties are more detailed.

We say that $\mathfrak{S}$ is of type II provided

(i) $\mathfrak{U} \neq 1$ and $\mathfrak{U}$ is abelian.

(ii) $N_{\mathfrak{G}}(\mathfrak{U}) \not\subseteq \mathfrak{S}$.

(iii) $N_{\mathfrak{G}}(\mathfrak{A}) \subseteq \mathfrak{S}$ for every non empty subset $\mathfrak{A}$ of $\mathfrak{S}'^{\sharp}$ such that $C_{\mathfrak{H}}(\mathfrak{A}) \neq 1$.

(iv) $|\mathfrak{W}_1|$ is a prime.

(v) For every prime p, if $\mathfrak{A}_0$, $\mathfrak{A}_1$ are cyclic p-subgroups of $\mathfrak{U}$ which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{S}$, then either $C_{\mathfrak{H}}(\mathfrak{A}_0) = 1$ or $C_{\mathfrak{H}}(\mathfrak{A}_1) = 1$.

(vi) $\mathfrak{H}C(\mathfrak{H})$ is a T. I. set in $\mathfrak{G}$.

We say that $\mathfrak{S}$ is of type III provided (ii) in the preceding definition is replaced by

(ii)′ $N_{\mathfrak{G}}(\mathfrak{U}) \subseteq \mathfrak{S}$,

and the remaining conditions hold.

We say that $\mathfrak{S}$ is of type IV provided (i) and (ii) in the definition of type II are replaced by

(i)″ $\mathfrak{U}' \neq 1$,

(ii)″ $N_{\mathfrak{G}}(\mathfrak{U}) \subseteq \mathfrak{S}$,

and the remaining conditions hold.

We say that $\mathfrak{S}$ is of type V provided

(i) $\mathfrak{U} = 1$.

(ii) One of the following statements is true:

(a) $\mathfrak{S}'$ is a T. I. set in $\mathfrak{G}$.

(b) $\mathfrak{S}' = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{S}_0$ is cyclic and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ with $p \in \pi_1^*$.

THEOREM 14.1. *Let $\mathfrak{G}$ be a minimal simple group of odd order. Two elements of a nilpotent S-subgroup $\mathfrak{H}$ of $\mathfrak{G}$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $N(\mathfrak{H})$. Either (i) or (ii) is true:*

(i) *Every maximal subgroup of $\mathfrak{G}$ is of type I.*

(ii) (a) *$\mathfrak{G}$ contains a cyclic subgroup $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$ with the property that $N(\mathfrak{W}_0) = \mathfrak{W}$ for every non empty subset $\mathfrak{W}_0$ of $\mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$. Also, $\mathfrak{W}_i \neq 1$, $i = 1, 2$.*

(b) *$\mathfrak{G}$ contains maximal subgroups $\mathfrak{S}$ and $\mathfrak{T}$ not of type I such that*

$$\mathfrak{S} \quad \mathfrak{W}_1\mathfrak{S}' , \qquad \mathfrak{T} = \mathfrak{W}_2\mathfrak{T}' , \qquad \mathfrak{S}' \cap \mathfrak{W}_1 = 1 , \qquad \mathfrak{T}' \cap \mathfrak{W}_2 = 1 ,$$
$$\mathfrak{S} \cap \mathfrak{T} = \mathfrak{W} .$$

(c) *Every maximal subgroup of* $\mathfrak{G}$ *is either conjugate to* $\mathfrak{S}$ *or* $\mathfrak{T}$ *or is of type I.*

(d) *Either* $\mathfrak{S}$ *or* $\mathfrak{T}$ *is of type II.*

(e) *Both* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are of type II, III, IV, or V. (They are not necessarily of the same type.)*

In order to state the next theorem we need further notation. If $\mathfrak{L}$ is of type I, let

$$\hat{\mathfrak{L}} = \hat{\mathfrak{L}}_1 = \bigcup_{H \in \mathfrak{H}^\#} C_{\mathfrak{L}}(H) ,$$

where $\mathfrak{H}$ is the Frobenius kernel of $\mathfrak{L}$.

If $\mathfrak{L}$ is of type II, III, IV, or V, we write $\mathfrak{L} = \mathfrak{L}'\mathfrak{W}_1$, $\mathfrak{L}' \cap \mathfrak{W}_1 = 1$. Let $\mathfrak{H}$ be the maximal normal nilpotent S-subgroup of $\mathfrak{L}$, let $\mathfrak{U}$ be a complement for $\mathfrak{H}$ in $\mathfrak{L}'$ and set $\mathfrak{W} = C_{\mathfrak{L}}(\mathfrak{W}_1)$, $\mathfrak{W}_2 = \mathfrak{W} \cap \mathfrak{L}'$, $\hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$.

If $\mathfrak{L}$ is of type II, let

$$\hat{\mathfrak{L}} = \bigcup_{H \in \mathfrak{H}^\#} C_{\mathfrak{L}'}(H) .$$

If $\mathfrak{L}$ is of type III, IV, or V, let

$$\hat{\mathfrak{L}} = \mathfrak{L}' .$$

If $\mathfrak{L}$ is of type II, III, IV, or V, let

$$\hat{\mathfrak{L}}_1 = \hat{\mathfrak{L}} \cup \bigcup_{L \in \mathfrak{L}} L^{-1}\hat{\mathfrak{W}}L .$$

We next define a set $\mathscr{A} = \mathscr{A}(\mathfrak{L})$ of subgroups associated to $\mathfrak{L}$. Namely, $\mathfrak{M} \in \mathscr{A}$ if and only if $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$ and there is an element L in $\hat{\mathfrak{L}}^\#$ such that $C(L) \nsubseteq \mathfrak{L}$ and $C(L) \subseteq \mathfrak{M}$. Let $\{\mathfrak{N}_1, \cdots, \mathfrak{N}_n\}$ be a subset of $\mathscr{A}$ which is maximal with the property that $\mathfrak{N}_i$ and $\mathfrak{N}_j$ are not conjugate if $i \neq j$. For $1 \leq i \leq n$, let $\mathfrak{H}_i$ be the maximal normal nilpotent S-subgroup of $\mathfrak{N}_i$.

THEOREM 14.2. *If* $\mathfrak{L}$ *is of type I, II, III, IV, or V, then* $\hat{\mathfrak{L}}$ *and* $\hat{\mathfrak{L}}_1$ *are tamely imbedded subsets of* $\mathfrak{G}$ *with*

$$N(\hat{\mathfrak{L}}) = N(\hat{\mathfrak{L}}_1) = \mathfrak{L} .$$

If $\mathscr{A}(\mathfrak{L})$ *is empty,* $\hat{\mathfrak{L}}$ *and* $\hat{\mathfrak{L}}_1$ *are* T. I. *sets in* $\mathfrak{G}$. *If* $\mathscr{A}(\mathfrak{L})$ *is non empty, the subgroups* $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ *are a system of supporting subgroups for* $\hat{\mathfrak{L}}$ *and for* $\hat{\mathfrak{L}}_1$.

The purpose of Chapter IV is to provide proofs for these two theorems.

15. A Partition of $\pi(\mathfrak{G})$

We partition $\pi(\mathfrak{G})$ into four subsets, some of which may be empty:

$\pi_1 = \{p \mid$ A S_p-subgroup of $\mathfrak{G}$ is a non identity cyclic group.$\}$

$\pi_2 = \{p \mid$ 1. A S_p-subgroup of $\mathfrak{G}$ is non cyclic.
2. $\mathfrak{G}$ does not contain an elementary subgroup of order p^3.$\}$

$\pi_3 = \{p \mid$ 1. $\mathfrak{G}$ contains an elementary subgroup of order p^3.
2. If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathsf{M}(\mathfrak{P})$ contains a non identity subgroup.$\}$

$\pi_4 = \{p \mid$ 1. $\mathfrak{G}$ contains an elementary subgroup of order p^3.
2. If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathsf{M}(\mathfrak{P})$ contains only $\langle 1 \rangle$.$\}$

It is immediate that the sets partition $\pi(\mathfrak{G})$. The purpose of Lemma 8.4 (i) is that condition 2 defining π_2 is equivalent to the statement that $\mathscr{SCN}_3(\mathfrak{P})$ is empty if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Lemma 8.5 implies that $3 \notin \pi_1 \cup \pi_2$.

16. Lemmas about Commutators

Following P. Hall [19], we adopt the notation $\gamma\mathfrak{A}\mathfrak{B} = [\mathfrak{A}, \mathfrak{B}]$, $\gamma^{n+1}\mathfrak{A}\mathfrak{B}^{n+1} = [\gamma^n\mathfrak{A}\mathfrak{B}^n, \mathfrak{B}]$, $n = 1, 2, \cdots$, and $\gamma^2\mathfrak{A}\mathfrak{B}\mathfrak{C} = [\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$.

If $\mathfrak{X}$ is a group, $\mathscr{NA}(\mathfrak{X})$ denotes the set of normal abelian subgroups of $\mathfrak{X}$.

The following lemmas parallel Lemma 5.6 of [27] and in the presence of (B) absorb much of the difficulty of the proof of Theorem 14.1.

LEMMA 16.1. *Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A}$ an element of $\mathscr{NA}(\mathfrak{P})$. If $\mathfrak{F}$ is a subgroup of $\mathfrak{G}$ such that*

(i) *$\langle \mathfrak{A}, \mathfrak{F} \rangle$ is a p-group,*

(ii) *$\mathfrak{F}$ centralizes some element of $Z(\mathfrak{P}) \cap \mathfrak{A}^\#$,*

then $\gamma^3\mathfrak{F}\mathfrak{A}^3 = \langle 1 \rangle$.

Proof. Let $Z \in C(\mathfrak{F}) \cap Z(\mathfrak{P}) \cap \mathfrak{A}^\#$, and let $\mathfrak{C} = C(Z)$. By Lemma 7.2 (1) we have $\mathfrak{A} \subseteq O_{p',p}(\mathfrak{C}) = \mathfrak{H}$. As $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{C}$, $\mathfrak{P}_1 = \mathfrak{P} \cap \mathfrak{H}$ is a S_p-subgroup of $\mathfrak{H}$. Since $\mathfrak{A} \lhd \mathfrak{P}$, so also $\mathfrak{A} \lhd \mathfrak{P}_1$, and since $\mathfrak{A}$ is abelian, we see that $\gamma^2\mathfrak{H}\mathfrak{A}^2 \subseteq O_{p'}(\mathfrak{C})$. Since $\mathfrak{H} \lhd \mathfrak{C}$, we have $\gamma\mathfrak{F}\mathfrak{A} \subseteq \mathfrak{H}$ and so $\gamma^3\mathfrak{F}\mathfrak{A}^3 \subseteq O_{p'}(\mathfrak{C})$. Since $\langle \mathfrak{A}, \mathfrak{F} \rangle$ is assumed to be a p-group, the lemma follows.

If $\mathfrak{P}$ is a non cyclic p-group, we define $\mathscr{U}(\mathfrak{P})$ as follows: in case $Z(\mathfrak{P})$ is non cyclic, $\mathscr{U}(\mathfrak{P})$ consists of all subgroups of $Z(\mathfrak{P})$ of type (p, p); in case $Z(\mathfrak{P})$ is cyclic, $\mathscr{U}(\mathfrak{P})$ consists of all normal abelian subgroups of $\mathfrak{P}$ of type (p, p).

LEMMA 16.2. *Let $\mathfrak{P}$ be a non cyclic S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A} \in \mathscr{NA}(\mathfrak{P})$,* *and let* $\mathfrak{F}$ *be a subgroup such that*

(i) $\langle \mathfrak{A}, \mathfrak{F} \rangle$ *is a p-group,*

(ii) $\mathfrak{A}$ *contains a subgroup* $\mathfrak{B}$ *of* $\mathscr{U}(\mathfrak{P})$ *such that* $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{F}) \neq \langle 1 \rangle$.

If $p \geqq 5$, *then* $\gamma^4\mathfrak{F}\mathfrak{A}^4 = \langle 1 \rangle$, *while if* $p = 3$, *then* $\gamma^6\mathfrak{F}\mathfrak{A}^6 = \langle 1 \rangle$. *Also, if* $\mathfrak{A}_1 = \mathfrak{A} \cap \boldsymbol{Z}(\mathfrak{P})$ *and* $p \geqq 5$, *then* $\gamma^3\mathfrak{F}\mathfrak{A}_1^3 = \langle 1 \rangle$.

Proof. If $\mathfrak{B}_0 \subseteqq \boldsymbol{Z}(\mathfrak{P})$, the lemma follows from Lemma 16.1. If $\mathfrak{B}_0 \not\subseteqq \boldsymbol{Z}(\mathfrak{P})$, then $\mathfrak{P}_0 = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{B}_0)$ is of index p in $\mathfrak{P}$ so is of index at most p in a suitable S_p-subgroup $\mathfrak{P}^*$ of $C(\mathfrak{B}_0) = \mathfrak{C}$. In particular, $\mathfrak{P}_0 \lhd \mathfrak{P}^*$.

Let $\mathfrak{H} = \boldsymbol{O}_{p',p}(\mathfrak{C})$, $\mathfrak{P}_1^* = \mathfrak{P}^* \cap \mathfrak{H}$, and $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \mathfrak{H}$. Since $\mathfrak{P}_0 \lhd \mathfrak{P}^*$, so also $\mathfrak{P}_1 \lhd \mathfrak{P}_1^*$. Hence $\gamma\mathfrak{P}_1^*\mathfrak{A} \subseteqq \mathfrak{P}_0 \cap \mathfrak{H} \subseteqq \mathfrak{P}_1$, and so $\gamma^3\mathfrak{P}_1^*\mathfrak{A}^3 = \langle 1 \rangle$, $\mathfrak{A}$ being in $\mathscr{N}\mathscr{A}(\mathfrak{P}_0)$. If $p \geqq 5$, we conclude from (B) that $\mathfrak{A} \subseteqq \mathfrak{H}$, and so $\gamma^3\mathfrak{H}\mathfrak{A}^3 \subseteqq \boldsymbol{O}_{p'}(\mathfrak{C})$. Since $\gamma\mathfrak{F}\mathfrak{A} \subseteqq \mathfrak{H}$, the lemma follows in this case. (Since $\mathfrak{P}_0$ centralizes $\mathfrak{A}_1$, we have $\gamma^3\mathfrak{F}\mathfrak{A}_1^3 = \langle 1 \rangle$.)

Suppose now that $p = 3$. If $\mathfrak{P}_1^* = \mathfrak{P}_1$, then $\gamma^2\mathfrak{P}_1^*\mathfrak{A}^2 = \langle 1 \rangle$, and so by (B), $\mathfrak{A} \subseteqq \mathfrak{H}$ and the lemma follows. If $\mathfrak{P}_1^* \neq \mathfrak{P}_1$, then $\mathfrak{P}^* = \mathfrak{P}_0\mathfrak{P}_1^*$, since $|\mathfrak{P}^* : \mathfrak{P}_0| = p$. In this case, letting $\bar{\mathfrak{A}} = \mathfrak{A}\mathfrak{H}/\mathfrak{H}$, $\bar{\mathfrak{P}}^* = \mathfrak{P}^*\mathfrak{H}/\mathfrak{H}$, we see that $\bar{\mathfrak{A}} \in \mathscr{N}\mathscr{A}(\bar{\mathfrak{P}}^*)$ and so $\bar{\mathfrak{A}} \subseteqq \boldsymbol{O}_{p',p}(\mathfrak{C}/\mathfrak{H})$, that is, $\mathfrak{A} \subseteqq \boldsymbol{O}_{p',p,p',p}(\mathfrak{C}) = \mathfrak{K}$. Hence, $\gamma\mathfrak{F}\mathfrak{A} \subseteqq \mathfrak{K}$ and since $\bar{\mathfrak{A}} \lhd \bar{\mathfrak{P}}^*$, we see that $\gamma^3\mathfrak{F}\mathfrak{A}^3 \subseteqq \boldsymbol{O}_{p',p,p'}(\mathfrak{C})$, and so $\gamma^3\mathfrak{F}\mathfrak{A}^3 \subseteqq \mathfrak{H}$. Continuing, we see that $\gamma^4\mathfrak{F}\mathfrak{A}^4 \subseteqq \boldsymbol{O}_{p'}(\mathfrak{C})\mathfrak{P}_1$ and so $\gamma^6\mathfrak{F}\mathfrak{A}^6 \subseteqq \boldsymbol{O}_{p'}(\mathfrak{C})$, from which the lemma follows.

LEMMA 16.3. *Let* $\mathfrak{P}$ *be a* S_3*-subgroup of* $\mathfrak{G}$ *and let* $\mathfrak{C} \in \mathscr{U}(\mathfrak{P})$. *Let* $\mathfrak{F}$ *be a subgroup of* $\mathfrak{G}$ *such that*

(i) $\langle \mathfrak{F}, \mathfrak{C} \rangle$ *is a 3-group.*

(ii) $\mathfrak{C}_1 = \boldsymbol{C}_{\mathfrak{C}}(\mathfrak{F}) \neq \langle 1 \rangle$.

If $\gamma^2\mathfrak{F}\mathfrak{C}^2 \neq \langle 1 \rangle$, *then* $\gamma^2\mathfrak{F}\mathfrak{C}^2 = \mathfrak{C}_1$, *and* $\mathfrak{C}_1 = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$.

Proof. First suppose $\mathfrak{C}_1 \subseteqq \boldsymbol{Z}(\mathfrak{P})$. Let $\mathfrak{H} = \boldsymbol{C}(\mathfrak{C}_1) \supseteqq \langle \mathfrak{P}, \mathfrak{F} \rangle$. Since $\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{H}$, (B) implies that $\mathfrak{C} \subseteqq \boldsymbol{O}_{3',3}(\mathfrak{H})$. Setting $\mathfrak{P}_1 = \boldsymbol{O}_{3',3}(\mathfrak{H}) \cap \mathfrak{P}$, we have $\boldsymbol{O}_{3',3}(\mathfrak{H}) = \boldsymbol{O}_{3'}(\mathfrak{H})\mathfrak{P}_1$. If $\mathfrak{C} \subseteqq \boldsymbol{Z}(\mathfrak{P})$, then $\mathfrak{C} \subseteqq \boldsymbol{Z}(\mathfrak{P}_1)$ and so $\gamma^2\mathfrak{F}\mathfrak{C}^2 \subseteqq \boldsymbol{O}_{3'}(\mathfrak{H}) \cap \langle \mathfrak{F}, \mathfrak{C} \rangle = \langle 1 \rangle$, since $\langle \mathfrak{F}, \mathfrak{C} \rangle$ is a 3-group. If $\mathfrak{C} \not\subseteqq \boldsymbol{Z}(\mathfrak{P})$, then the definition of $\mathscr{U}(\mathfrak{P})$ implies that $\gamma^2\mathfrak{F}\mathfrak{C}^2 \subseteqq \mathfrak{C}_1\boldsymbol{O}_{3'}(\mathfrak{H})$, so if $\gamma^2\mathfrak{F}\mathfrak{C}^2 \neq \langle 1 \rangle$, we must have $\gamma^2\mathfrak{F}\mathfrak{C}^2 = H^{-1}\mathfrak{C}_1H$ for suitable H in $\boldsymbol{O}_{3'}(\mathfrak{H})$. By definition of $\mathfrak{H}$ it follows that $H^{-1}\mathfrak{C}_1H = \mathfrak{C}_1$.

We can suppose now that $\mathfrak{C}_1 \not\subseteqq \boldsymbol{Z}(\mathfrak{P})$. In this case, the definition of $\mathscr{U}(\mathfrak{P})$ implies that $\mathfrak{C} = \langle \mathfrak{D}, \mathfrak{C}_1 \rangle$, where $\mathfrak{D} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$. Let $\mathfrak{P}_0 = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{C}_1)$ and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{H} = \boldsymbol{C}(\mathfrak{C}_1)$ containing $\mathfrak{P}_0$ and let $\mathfrak{P}_0^* = \mathfrak{P}^* \cap \boldsymbol{O}_{3',3}(\mathfrak{H})$. Since $\mathfrak{P}_0$ is of index at most 3 in $\mathfrak{P}^*$ and since $\mathfrak{P}_0$ centralizes $\mathfrak{C}$, we have $\gamma^2\mathfrak{P}^*\mathfrak{C}^2 = \langle 1 \rangle$, and so $\mathfrak{C} \subseteqq \mathfrak{P}_0^*$. If $\mathfrak{P}_0^* \subseteqq \mathfrak{P}_0$, it follows that $\gamma^2\mathfrak{F}\mathfrak{C}^2 \subseteqq \boldsymbol{O}_{3'}(\mathfrak{H}) \cap \langle \mathfrak{C}, \mathfrak{F} \rangle = \langle 1 \rangle$ and we are done. Hence, we can suppose that $\mathfrak{P}_0^* \not\subseteqq \mathfrak{P}_0$. In this case, it follows that $\mathfrak{P}^* = \mathfrak{P}_0\mathfrak{P}_0^*$, since $|\mathfrak{P}^* : \mathfrak{P}_0| = 3$. We also have $D(\mathfrak{P}_0^*) \subseteq \mathfrak{P}_0$, and so $\mathfrak{C} \subseteq C_{\mathfrak{P}_0^*}(D(\mathfrak{P}_0^*)) = \mathfrak{C}$. If $\mathfrak{C} \subseteq \mathfrak{P}_0$, we have $\mathfrak{C} \subseteq Z(\mathfrak{C})$, and since $Z(\mathfrak{C})$ char $\mathfrak{C}$ char $\mathfrak{P}_0^*$, it follows that $\gamma^2\mathfrak{F}\mathfrak{C}^2 \subseteq O_{3'}(\mathfrak{H}) \cap \langle \mathfrak{C}, \mathfrak{F} \rangle = \langle 1 \rangle$ and we are done. We can therefore suppose that $\mathfrak{C} \not\subseteq Z(\mathfrak{C})$. Choose E in $\mathfrak{C} - C_{\mathfrak{C}}(\mathfrak{C})$. Since $\mathfrak{P}^*$ centralizes $\mathfrak{C}_1$ it follows that E does not centralize $\mathfrak{D} = \langle D \rangle$. Consider $[D, E] = F \neq 1$. Now $\mathfrak{C} \subseteq Z(\mathfrak{P}_0) \lhd \mathfrak{P}^*$, and so $F \in Z(\mathfrak{P}_0)$. On the other hand, F lies in $D(\mathfrak{P}_0^*)$ since both E and D are in $\mathfrak{P}_0^*$. Since $E \in \mathfrak{C}$, it follows that E centralizes F. Since $\langle \mathfrak{P}_0, E \rangle = \mathfrak{P}^*$, it follows that F is in $Z(\mathfrak{P}^*)$. But F is of order 3 and $\mathfrak{C}_1 = \Omega_1(Z(\mathfrak{P}^*))$, since $Z(\mathfrak{P}^*)$ is cyclic. It follows that $\langle F \rangle = \mathfrak{C}_1$, and so E normalizes $\mathfrak{C}$ and with respect to the basis (D, F) of $\mathfrak{C}$ has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. On the other hand, $\mathfrak{P}$ possesses an element which normalizes $\mathfrak{C}$ and with respect to the basis (D, F) has the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since these two matrices generate a group of even order, we have the desired contradiction which completes the proof of this lemma.

17. A Domination Theorem and Some Consequences

In view of other applications, Theorem 17.1 is proved in greater generality than is required for this paper.

Let $\mathfrak{P}$ be a S_p-subgroup of the minimal simple group $\mathfrak{X}$ and let $\mathfrak{A}$ be an element of $\mathscr{SCN}(\mathfrak{P})$. Let q be a prime different from p.

THEOREM 17.1. *Let $\mathfrak{Q}, \mathfrak{Q}_1$ be maximal elements of $\mathsf{M}(\mathfrak{A}; q)$.*

(i) *Suppose that $\mathfrak{Q}$ is not conjugate to $\mathfrak{Q}_1$ by any element of $C_{\mathfrak{X}}(\mathfrak{A})$. Then for each element A in $\mathfrak{A}^\#$, either $C_{\mathfrak{Q}}(A) = 1$ or $C_{\mathfrak{Q}_1}(A) = 1$.*

(ii) *If $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are conjugate by an element of $C(\mathfrak{A})$.*

Proof. The proof of (i) proceeds by a series of reductions. If $\mathfrak{A} = 1$, the theorem is vacuously true, so we may assume $\mathfrak{A} \neq 1$.

Choose Z in $Z(\mathfrak{P})$, and let $\mathfrak{Q}^*$ be any element of $\mathsf{M}(\mathfrak{A}; q)$ which is centralized by Z. By Lemmas 7.4 and 7.8, if $\mathfrak{L}$ is any proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$, then $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$.

Now let $\mathfrak{Q}^*$ denote any element of $\mathsf{M}(\mathfrak{A}; q)$ and let $\mathfrak{L}$ be a proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$. We will show that $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. First, suppose $Z(\mathfrak{P})$ is non cyclic. Then $\mathfrak{Q}^* = \langle C_{\mathfrak{Q}^*}(Z) \mid Z \in Z(\mathfrak{P})^\# \rangle$, so by the preceding paragraph, $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. We can suppose that $Z(\mathfrak{P})$ is cyclic. Let Z be an element of $Z(\mathfrak{P})$ of order p. We only need to show that $[\mathfrak{Q}^*, Z] \subseteq O_{p'}(\mathfrak{L})$, by the preceding paragraph. Replacing $\mathfrak{Q}^*$ by $[\mathfrak{Q}^*, Z]$, we may suppose that $\mathfrak{Q}^* = [\mathfrak{Q}^*, Z]$. Furthermore, we may suppose that $\mathfrak{A}$ acts irreducibly on $\mathfrak{Q}^*/D(\mathfrak{Q}^*)$.

Suppose $Z \in O_{p',p}(\mathfrak{L})$. Then $\mathfrak{Q}^* = [\mathfrak{Q}^*, Z] \subseteq O_{p',p}(\mathfrak{L}) \cap \mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$ and we are done. If $\mathfrak{A}$ is cyclic, then Z is necessarily in $O_{p',p}(\mathfrak{L})$, since $\mathfrak{A} \cap O_{p',p}(\mathfrak{L}) \neq 1$. Thus, we can suppose that $\mathfrak{A}$ is non cyclic.

Let $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{Q}^*) = C_{\mathfrak{A}}(\mathfrak{Q}^*/D(\mathfrak{Q}^*))$, so that $\mathfrak{A}/\mathfrak{A}_1$ is cyclic and $Z \notin \mathfrak{A}_1$. We now choose W of order p in $\mathfrak{A}_1$ such that $\langle Z, W\rangle \lhd \mathfrak{P}$.

Suppose by way of contradiction that $\mathfrak{Q}^* \not\subseteq O_{p'}(\mathfrak{L})$. Then by Lemma 7.8, we can find a subgroup $\mathfrak{K}$ of $\mathfrak{A}C(\mathfrak{A}_1)$ which contains $\mathfrak{A}\mathfrak{Q}^*$ and such that $\mathfrak{Q}^* \not\subseteq O_{p'}(\mathfrak{K})$. In particular, $\mathfrak{Q}^* \not\subseteq O_{p'}(C(W))$. Thus, we suppose without loss of generality that $\mathfrak{L} = C(W)$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{L}$ which contains $\tilde{\mathfrak{P}} = \mathfrak{P} \cap C(W)$. If $\mathfrak{P}^* = \tilde{\mathfrak{P}}$, then $Z \in O_{p',p}(\mathfrak{L})$, by Lemma 1.2.3 of [21], which is not the case. Hence, $\tilde{\mathfrak{P}}$ is of index p in $\mathfrak{P}^*$. Clearly, $\mathfrak{A} \subseteq \tilde{\mathfrak{P}}$ and $Z \in Z(\tilde{\mathfrak{P}})$. Hence, $[\mathfrak{P}^*, Z] \subseteq Z(\tilde{\mathfrak{P}}) \subseteq \mathfrak{A}$. Let $\mathfrak{P}_1^* = \mathfrak{P}^* \cap O_{p',p}(\mathfrak{L})$ so that $\mathfrak{P}_1^*$ is a S_p-subgroup of $O_{p',p}(\mathfrak{L})$. Then $[\mathfrak{P}_1^*, \langle Z\rangle, \mathfrak{Q}^*] \subseteq [\mathfrak{A}, \mathfrak{Q}^*] \cap O_{p',p}(\mathfrak{L}) \subseteq \mathfrak{Q}^* \cap O_{p',p}(\mathfrak{L})$, so that $[\mathfrak{P}_1^*, \langle Z\rangle, \mathfrak{Q}^*] \subseteq O_{p'}(\mathfrak{L})$. Let $\mathfrak{B} = O_{p',p}(\mathfrak{L})/O_{p'}(\mathfrak{L})$ and let $\mathfrak{B}_1 = C_{\mathfrak{B}}(\mathfrak{Q}^*)$. The preceding containment implies that $[\mathfrak{B}, \langle Z\rangle] \subseteq \mathfrak{B}_1$. Let $\mathfrak{B}_2 = N_{\mathfrak{B}}(\mathfrak{B}_1)$. Then Z acts trivially on the $\mathfrak{Q}^*\mathfrak{A}$-admissible group $\mathfrak{B}_2/\mathfrak{B}_1$. Hence, so does $[\langle Z\rangle, \mathfrak{Q}^*] = \mathfrak{Q}^*$, that is, $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. This implies that $\mathfrak{B} = \mathfrak{B}_1$ is centralized by $\mathfrak{Q}^*$ so $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. We have succeeded in showing that if $\mathfrak{Q}^*$ is in $\mathsf{M}(\mathfrak{A}; q)$ and $\mathfrak{L}$ is any proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$, then $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$.

Now let $\mathscr{O}_1, \cdots, \mathscr{O}_t$ be the orbits under conjugation by $C(\mathfrak{A})$ of the maximal elements of $\mathsf{M}(\mathfrak{A}; q)$. We next show that if $\mathfrak{Q} \in \mathscr{O}_i$, $\mathfrak{Q}_1 \in \mathscr{O}_j$ and $i \neq j$, then $\mathfrak{Q} \cap \mathfrak{Q}_1 = 1$. Suppose false and $i, j, \mathfrak{Q}, \mathfrak{Q}_1$ are chosen so that $|\mathfrak{Q} \cap \mathfrak{Q}_1|$ is maximal. Let $\mathfrak{Q}^* = N_{\mathfrak{Q}}(\mathfrak{Q} \cap \mathfrak{Q}_1)$ and $\mathfrak{Q}_1^* = N_{\mathfrak{Q}_1}(\mathfrak{Q} \cap \mathfrak{Q}_1)$. Since $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are distinct maximal elements of $\mathsf{M}(\mathfrak{A}; q)$, $\mathfrak{Q} \cap \mathfrak{Q}_1$ is a proper subgroup of both $\mathfrak{Q}^*$ and $\mathfrak{Q}_1^*$. Let $\mathfrak{L} = N(\mathfrak{Q} \cap \mathfrak{Q}_1)$. By the previous argument, $\langle \mathfrak{Q}^*, \mathfrak{Q}_1^*\rangle \subseteq O_{p'}(\mathfrak{L})$. Let $\mathfrak{R}$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $\mathfrak{Q}^*$ and permutable with $\mathfrak{A}$ and let $\mathfrak{R}_1$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $\mathfrak{Q}_1^*$ and permutable with $\mathfrak{A}$. The groups $\mathfrak{R}$ and $\mathfrak{R}_1$ are available by $D_{p,q}$ in $\mathfrak{A}O_{p'}(\mathfrak{L})$. By the conjugacy of Sylow systems, there is an element C in $O_{p'}(\mathfrak{L})\mathfrak{A}$ such that $\mathfrak{A}^C = \mathfrak{A}$ and $\mathfrak{R}^C = \mathfrak{R}_1$. As $\mathfrak{A}$ has a normal complement in $O_{p'}(\mathfrak{L})\mathfrak{A}$, it follows that C centralizes $\mathfrak{A}$. Let $\hat{\mathfrak{Q}}$ be a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ containing $\mathfrak{R}_1$. Then $\hat{\mathfrak{Q}} \cap \mathfrak{Q}_1 \supseteq \mathfrak{Q}_1^* \supset \mathfrak{Q} \cap \mathfrak{Q}_1$, and so $\hat{\mathfrak{Q}} \in \mathscr{O}_j$. Also, $\hat{\mathfrak{Q}} \cap \mathfrak{Q}^C \supseteq \mathfrak{Q}^{*C} \supset (\mathfrak{Q} \cap \mathfrak{Q}_1)^C$ so that $\hat{\mathfrak{Q}} \in \mathscr{O}_i$ and $i = j$.

To complete the proof of (i), let $\mathfrak{Q}$, $\mathfrak{Q}_1$ be maximal elements of $\mathsf{M}(\mathfrak{A}; q)$ with $\mathfrak{Q} \in \mathscr{O}_i$, $\mathfrak{Q}_1 \in \mathscr{O}_j$. Suppose $A \in \mathfrak{A}^\#$ and $C_{\mathfrak{Q}}(A) \neq 1$, $C_{\mathfrak{Q}_1}(A) \neq 1$. Let $\mathfrak{L} = C(A)$, let $\mathfrak{R}$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $C_{\mathfrak{Q}}(A)$ and permutable with $\mathfrak{A}$, and let $\mathfrak{R}_1$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $C_{\mathfrak{Q}_1}(A)$ and permutable with $\mathfrak{A}$. Then $\mathfrak{R}^C = \mathfrak{R}_1$ for suitable C in $C(\mathfrak{A})$. Let $\mathfrak{Q}^*$ be a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ containing $\mathfrak{R}_1$. Then $\mathfrak{Q}^* \cap \mathfrak{Q}_1 \supseteq C_{\mathfrak{Q}_1}(A) \neq 1$ so $\mathfrak{Q}^* \in \mathscr{O}_j$. Also, $\mathfrak{Q}^* \cap \mathfrak{Q}^C \supseteq (C_{\mathfrak{Q}}(A))^C \neq 1$ so $\mathfrak{Q}^* \in \mathscr{O}_i$ and $i = j$. This completes the proof of (i).

As for (ii), if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then there is an element A in $\mathfrak{A}^\#$

such that $C_{\mathfrak{Q}}(A) \neq 1$ and $C_{\mathfrak{Q}_1}(A) \neq 1$. By (i), $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are conjugate under $C(\mathfrak{A})$.

COROLLARY 17.1. *If $p \in \pi_3 \cup \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then for each prime $q \neq p$ and each maximal element $\mathfrak{Q}$ of $\mathsf{M}(\mathfrak{A}; q)$, there is a S_p-subgroup of $N(\mathfrak{A})$ which normalizes $\mathfrak{Q}$.*

Proof. Let $G \in N(\mathfrak{A})$. Then $\mathfrak{Q}^G$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, since any two maximal elements of $\mathsf{M}(\mathfrak{A}; q)$ have the same order, so $\mathfrak{Q}^G = \mathfrak{Q}^C$ for suitable $C = C(G)$ in $C(\mathfrak{A})$. Hence, GC^{-1} normalizes $\mathfrak{Q}$. Setting $\mathfrak{J} = N(\mathfrak{Q}) \cap N(\mathfrak{A})$, we see that $\mathfrak{J}$ covers $N(\mathfrak{A})/C(\mathfrak{A})$, that is, $N(\mathfrak{Q})$ dominates $\mathfrak{A}$. Now we have $\mathfrak{J}C(\mathfrak{A}) = N(\mathfrak{A})$ and $\mathfrak{J}$ contains $\mathfrak{A}$. Since $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$ where $\mathfrak{D}$ is a p'-group, we have $N(\mathfrak{A}) = \mathfrak{J}C(\mathfrak{A}) = \mathfrak{J}\mathfrak{A}\mathfrak{D} = \mathfrak{J}\mathfrak{D}$, and $\mathfrak{J}$ contains a S_p-subgroup of $N(\mathfrak{A})$ as required.

COROLLARY 17.2. *If $p \in \pi_3 \cup \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and q is a prime different from p, then $\mathfrak{P}$ normalizes some maximal element $\mathfrak{Q}$ of $\mathsf{M}(\mathfrak{A}; q)$. Furthermore if G is an element of $\mathfrak{G}$ such that $\mathfrak{A}^G \subseteq \mathfrak{P}$, then $\mathfrak{A}^G = \mathfrak{A}^N$ for some N in $N(\mathfrak{Q})$.*

Proof. Applying Corollary 17.1, some S_p-subgroup $\mathfrak{P}^*$ of $N(\mathfrak{A})$ normalizes $\mathfrak{Q}_1$, a maximal element of $\mathsf{M}(\mathfrak{A}; q)$. Since $\mathfrak{P}$ is a S_p-subgroup of $N(\mathfrak{A})$, $\mathfrak{P} = \mathfrak{P}^{*X}$ for suitable X in $N(\mathfrak{A})$, and so $\mathfrak{P}$ normalizes $\mathfrak{Q} = \mathfrak{Q}_1^X$, a maximal element of $\mathsf{M}(\mathfrak{A}; q)$.

Suppose $G \in \mathfrak{G}$ and $\mathfrak{A}^G \subseteq \mathfrak{P}$. Then $\mathfrak{A}^G$ normalizes $\mathfrak{Q}$ since $\mathfrak{P}$ does, so $\mathfrak{A}$ normalizes $\mathfrak{Q}^{G^{-1}}$. Now $\mathfrak{Q}^{G^{-1}}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ since any two such have the same order. Hence, $\mathfrak{Q}^{G^{-1}} = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$, by Theorem 17.1 and so $CG = N$ is in $N(\mathfrak{Q})$. Since $\mathfrak{A}^N = \mathfrak{A}^{CG} = \mathfrak{A}^G$, the corollary follows.

COROLLARY 17.3. *If $p \in \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then $\mathsf{M}(\mathfrak{A})$ is trivial.*

Proof. Otherwise, $\mathsf{M}(\mathfrak{A}; q)$ is non trivial for some prime $q \neq p$, by Lemma 7.4, and so $\mathsf{M}(\mathfrak{P}; q)$ is non trivial, contrary to the definition of π_4.

Hypothesis 17.1.

(i) *$p \in \pi_3$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.*

(ii) *q is a prime different from p, $\mathsf{M}(\mathfrak{A}; q)$ is non trivial and $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ normalized by $\mathfrak{P}$.*

REMARK. Most of Hypothesis 17.1 is notation. The hypothesis is that $p \in \pi_3$, for in this case a prime q is available such that (ii) is satisfied. Furthermore, we let

$$\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}), \quad \mathfrak{N} = N(\mathfrak{Q}), \quad \text{and} \quad \mathfrak{N}_1 = N(Z(\mathfrak{B})) .$$

LEMMA 17.1. *Under Hypothesis* 17.1 *if* $G \in \mathfrak{G}$ and $\mathfrak{A}^G \subseteq \mathfrak{P}$, *then* $\mathfrak{A}^G = \mathfrak{A}^N$ *for some element* N *in* $N(\mathfrak{Q}) \cap N(\mathfrak{B})$.

Proof. By Corollary 17.2, $\mathfrak{A}^G = \mathfrak{A}^X$ for some element X in $\mathfrak{N}$. Since $\mathfrak{N}$ is solvable, Lemma 7.2 (1) and Corollary 17.2 imply that $\mathfrak{N} = O_{p'}(\mathfrak{N}) \cdot N_{\mathfrak{N}}(\mathfrak{B})$, so we can write $X = N_1 N$ where $N_1 \in O_{p'}(\mathfrak{N})$ and $N \in N_{\mathfrak{N}}(\mathfrak{B})$. Now $\mathfrak{A}^N$ is in $\mathfrak{B}$, so in particular is in $\mathfrak{P}$. Also $\mathfrak{A}^{N_1 N} = \mathfrak{A}^X$ is in $\mathfrak{P}$. Hence, if A is in $\mathfrak{A}$, then $A^{-N} \cdot A^{N_1 N} = [A, N_1]^N$ is in $\mathfrak{P}$, and in particular is a p-element. Since $[A, N_1]$ is a p'-element, we see that $N_1 \in C(\mathfrak{A})$. Hence $\mathfrak{A}^{N_1 N} = \mathfrak{A}^N$, and the lemma follows.

LEMMA 17.2. *Under Hypothesis* 17.1, $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$.

Proof. Since $Z(\mathfrak{B})$ char $\mathfrak{B}$, and $\mathfrak{B}$ is weakly closed in $\mathfrak{P}$, $\mathfrak{N}_1$ contains $N(\mathfrak{P})$, so Theorem 14.4.1 of [12] applies. We consider the double cosets $\mathfrak{N}_1 X \mathfrak{P}$ distinct from $\mathfrak{N}_1$. Denote by $\mathfrak{K}(X)$ the kernel of the homomorphism of $\mathfrak{P}$ onto the permutation representation of $\mathfrak{P}$ on the cosets of $\mathfrak{N}_1$ in $\mathfrak{N}_1 X \mathfrak{P}$. Let $P = P(X)$ be an element of $\mathfrak{P}$ such that $\mathfrak{K}(X)P$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X))$.

Suppose we are able to show that P can always be taken to lie in $\mathfrak{A}$. In this case, we have $[U, P, P] = 1$ for all U in $\mathfrak{P}$. Since $p \geqq 3$ and $\mathfrak{G}$ is simple we conclude from Theorem 14.4.1 in [12] that $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$.

We now proceed to show that P can always be taken to lie in $\mathfrak{A}$. The only restriction on the element X is that $X \notin \mathfrak{N}_1$, that is, we must have $\mathfrak{K}(X) \neq \mathfrak{P}$.

Now $\mathfrak{A} \subseteq \mathfrak{B}$, so $Z(\mathfrak{B})$ centralizes $\mathfrak{A}$. Since $\mathfrak{A} \in \mathscr{SCN}(\mathfrak{P})$, we have $Z(\mathfrak{B}) \subseteq \mathfrak{A}$. It follows that $\mathfrak{N}_1$ contains $C(\mathfrak{A})$.

It suffices to show that $\mathfrak{A} \not\subseteq \mathfrak{K}(X)$. For if $\mathfrak{A} \not\subseteq \mathfrak{K}(X)$, choose A in $\mathfrak{A}$ so that $(\mathfrak{K}(X) \cap \mathfrak{A})A$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X) \cap \mathfrak{A})$. It follows that $\mathfrak{K}(X)A$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X))$.

Suppose by way of contradiction that $\mathfrak{A} \subseteq \mathfrak{K}(X)$. Then $\mathfrak{A} \subseteq \mathfrak{N}_1^X$ so $\mathfrak{A} \subseteq \mathfrak{P}^{*X}$ for $\mathfrak{P}^*$ a suitable S_p-subgroup of $\mathfrak{N}_1$. But $\mathfrak{P}^* = \mathfrak{P}^Y$ for some Y in $\mathfrak{N}_1$. Setting $X_1 = YX$, we have $\mathfrak{N}_1 X \mathfrak{P} = \mathfrak{N}_1 X_1 \mathfrak{P}$ and $\mathfrak{A} \subseteq \mathfrak{P}^{X_1}$. Hence, $\mathfrak{A}^{X_1^{-1}} \subseteq \mathfrak{P}$, so by Lemma 17.1, $\mathfrak{A}^{X_1^{-1}} = \mathfrak{A}^W$ for some W in $\mathfrak{N} \cap N(\mathfrak{B})$. Since $N(\mathfrak{B}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{A} = \mathfrak{A}^{WX_1}$ and $W \in \mathfrak{N} \cap \mathfrak{N}_1$. Let $WX_1 = X_2$. Since $W \in \mathfrak{N}_1$, we have $\mathfrak{N}_1 X_1 \mathfrak{P} = \mathfrak{N}_1 X_2 \mathfrak{P}$.

Since X_2 normalizes $\mathfrak{A}$, $\mathfrak{A}$ normalizes $\mathfrak{Q}^{X_2^{-1}}$. By Theorem 17.1,

$\mathfrak{Q}^{X_2^{-1}} = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$. Hence $X_2^{-1}C^{-1} = X_3^{-1}$ (this defines X_3) normalizes $\mathfrak{Q}$. Since X_2 and C normalize $\mathfrak{A}$, we see that $X_3 \in \mathfrak{N} \cap N(\mathfrak{A})$. Since C centralizes $\mathfrak{A}$ and $C(\mathfrak{A}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{N}_1 X_2 \mathfrak{P} = \mathfrak{N}_1 X_3 \mathfrak{P}$.

We now write $X_3 = X_3' X_4$, where $X_3' \in \mathfrak{N} \cap N(\mathfrak{B})$ and $X_4 \in O_{p'}(\mathfrak{N})$. Such a representation is possible since $X_3 \in \mathfrak{N}$. Consider the equation $X_4 = X_3'^{-1} X_3$. Since $N(\mathfrak{B}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{N}_1 X_3 \mathfrak{P} = \mathfrak{N}_1 X_4 \mathfrak{P}$. If $A \in \mathfrak{A}$, then $[A, X_4^{-1}]$ is a p'-element since $X_4 \in O_{p'}(\mathfrak{N})$. But $[A, X_3^{-1} X_3'] = [A, X_3'][A, X_3^{-1}]^{X_3'}$, an identity holding in all groups. Since $X_3' \in N(\mathfrak{B})$, $[A, X_3'] \in \mathfrak{B}$. Since $X_3 \in N(\mathfrak{A})$, $[A, X_3^{-1}] \in \mathfrak{A} \subseteq \mathfrak{B}$, so $[A, X_3^{-1}]^{X_3'} \in \mathfrak{B}$, a p-group. Hence

$$[A, X_4^{-1}] = [A, X_3^{-1} X_3'] = 1 .$$

Since A is an arbitrary element of $\mathfrak{A}$, we have $X_4 \in C(\mathfrak{A}) \subseteq \mathfrak{N}_1$. Now, however, we have

$$\mathfrak{N}_1 X \mathfrak{P} = \mathfrak{N}_1 X_1 \mathfrak{P} = \mathfrak{N}_1 X_2 \mathfrak{P} = \mathfrak{N}_1 X_3 \mathfrak{P} = \mathfrak{N}_1 X_4 \mathfrak{P} = \mathfrak{N}_1 ,$$

so $X \in \mathfrak{N}_1$, contrary to assumption.

LEMMA 17.3. *Under Hypothesis* 17.1, $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot (\mathfrak{N}_1 \cap \mathfrak{N})$, *and* $\mathfrak{N} = O^p(\mathfrak{N})$.

Proof. We must show that $\mathfrak{N}$ contains at least one element from each coset $\mathfrak{C} = O_{p'}(\mathfrak{N}_1) W$, $W \in \mathfrak{N}_1$, from which the lemma follows directly.

Let $\mathfrak{H} = \mathfrak{P} \cap O_{p',p}(\mathfrak{N}_1)$, $\mathfrak{K} = N_{\mathfrak{N}_1}(\mathfrak{H})$, and $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, $\mathfrak{D}$ being a p'-group. Notice that $\mathfrak{D} \subseteq O_{p'}(\mathfrak{N}_1)$ by Lemma 7.4 together with $C(\mathfrak{A}) \subseteq \mathfrak{N}_1$. (This was the point in taking $Z(\mathfrak{B})$ in place of $\mathfrak{B}$.)

By Sylow's theorem, $\mathfrak{K}$ contains some element of $\mathfrak{C}$, so suppose $W \in \mathfrak{K}$. Since $\mathfrak{A}$ is contained in $\mathfrak{H}$ by Lemma 7.2 (1), we have $\mathfrak{A}^W \subseteq \mathfrak{H} \subseteq \mathfrak{P}$, and $\mathfrak{A}^W$ normalizes $\mathfrak{Q}$. Hence, $\mathfrak{A}$ normalizes $\mathfrak{Q}^{W^{-1}}$ and by Theorem 17.1, $\mathfrak{Q}^{W^{-1}} = \mathfrak{Q}^S$ for some S in $C(\mathfrak{A})$. Write $S = AD$ where $A \in \mathfrak{A}$, $D \in \mathfrak{D}$, so that $\mathfrak{Q}^S = \mathfrak{Q}^D$, since $\mathfrak{A}$ normalizes $\mathfrak{Q}$. Hence, DW normalizes $\mathfrak{Q}$. But $DW \in \mathfrak{C}$, since $D \in O_{p'}(\mathfrak{N}_1)$, so $DW \in \mathfrak{N} \cap \mathfrak{N}_1$ and $\mathfrak{N}$ contains an element of $\mathfrak{C}$.

LEMMA 17.4. *Under Hypothesis* 17.1, *if* $\mathfrak{H}$ *is a subgroup of* $\mathfrak{P}$ *which contains* $\mathfrak{A}$, *then* $N(\mathfrak{H}) \subseteq \mathfrak{N}_1$.

Proof. Let $G \in N(\mathfrak{H})$. Since $\mathfrak{P}$ normalizes $\mathfrak{Q}$, so does $\mathfrak{H}$. Hence, $\mathfrak{H}^G$ normalizes $\mathfrak{Q}^G$. But $\mathfrak{H}^G = \mathfrak{H}$ and $\mathfrak{H}$ contains $\mathfrak{A}$, so $\mathfrak{A}$ normalizes $\mathfrak{Q}^G$. By Theorem 17.1, $\mathfrak{Q}^G = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$. Let $GC^{-1} = N \in \mathfrak{N}$. Now $N = N_1 N_2$ where $N_1 \in O_{p'}(\mathfrak{N})$ and $N_2 \in \mathfrak{N} \cap \mathfrak{N}_1$. Consider the equation $GC^{-1}N_2^{-1} = N_1$. Let $Z \in Z(\mathfrak{B})$.

We have $GC^{-1}N_2^{-1}ZN_2CG^{-1} = GZ_1G^{-1}$, where $Z_1 = Z^{N_2C}$ is in $Z(\mathfrak{B})$;

hence, $Z^{-1}GC^{-1}N_2^{-1}ZN_2CG^{-1} = [Z, N_2CG^{-1}] = Z^{-1}GZ_1G^{-1}$ is a p-element of $\mathfrak{H}$, since $Z_1 \in \boldsymbol{Z}(\mathfrak{B}) \subseteq \mathfrak{A} \subseteq \mathfrak{H}$, so that $GZ_1G^{-1} \in G\mathfrak{H}G^{-1} = \mathfrak{H}$. But $Z^{-1}N_1ZN_1^{-1} \in \boldsymbol{O}_{p'}(\mathfrak{N})$. Hence, $[Z, N_2CG^{-1}] = [Z, N_1^{-1}] = 1$. Since Z is an arbitrary element of $\boldsymbol{Z}(\mathfrak{B})$, it follows that N_1 centralizes $\boldsymbol{Z}(\mathfrak{B})$, so N_1 is contained in $\mathfrak{N}_1$. But now the elements N_1, N_2 and C normalize $\boldsymbol{Z}(\mathfrak{B})$. Since $G = N_1N_2C$, the lemma follows.

LEMMA 17.5. *Under Hypothesis* 17.1, *if* $\mathfrak{K}$ *is a proper subgroup of* $\mathfrak{G}$ *which contains* $\mathfrak{P}$, *then* $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{K})$.

Proof. If $\mathfrak{P}_1 = \mathfrak{P} \cap \boldsymbol{O}_{p',p}(\mathfrak{K})$, and $\mathfrak{K}_1 = \boldsymbol{N}_{\mathfrak{K}}(\mathfrak{P}_1)$, it suffices to show that $\mathfrak{B} \subseteq \mathfrak{P}_1$. By Lemma 7.2 (1), we have $\mathfrak{A} \subseteq \mathfrak{P}_1$, and so by Lemma 17.4, $\mathfrak{K}_1 \subseteq \mathfrak{N}_1$. Thus it suffices to show that $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N}_1)$. By Lemma 17.3, it suffices to show that $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N})$. However, this last containment follows from Lemma 7.2 (1) and Corollary 17.1.

LEMMA 17.6. *Under Hypothesis* 17.1, *if* $\mathfrak{K}$ *is a proper subgroup of* $\mathfrak{G}$, *and* $\mathfrak{P}_0$ *is a* S_p*-subgroup of* $\mathfrak{K}$, *then* $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_0) \subseteq \boldsymbol{O}_{p',p}(\mathfrak{K})$.

Proof. Suppose false, and that $\mathfrak{K}$ is chosen to maximize $|\mathfrak{K}|_p$ and with this restriction to minimize $|\mathfrak{K}|_{p'}$. Let $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \boldsymbol{O}_{p',p}(\mathfrak{K})$. By minimality of $|\mathfrak{K}|_{p'}$ we have $\mathfrak{P}_1 \lhd \mathfrak{K}$. By maximality of $|\mathfrak{K}|_p$, $\mathfrak{P}_0$ is a S_p-subgroup of $\boldsymbol{N}(\mathfrak{P}_1)$. We assume without loss of generality that $\mathfrak{P}_0 \subseteq \mathfrak{P}$. In this case, Lemma 7.9 implies that $\mathfrak{A} \subseteq \mathfrak{P}_1$. Since $\mathfrak{A} \subseteq \mathfrak{P}_1$, by Lemma 17.4 we have $\mathfrak{K} \subseteq \mathfrak{N}_1$; by Lemma 17.5, $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N}_1)$, so in particular, $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_0) \subseteq \mathfrak{P}_1$, as required.

18. Configurations

The necessary E-theorems emerge from a study of the following objects:

(C)

1. A proper subgroup $\mathfrak{K}$ of $\mathfrak{G}$.
2. A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{K}$.
3. A p-subgroup $\mathfrak{A}$ of $\mathfrak{G}$.
4. $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, $\mathfrak{M} = [\boldsymbol{O}_{p',p\,p'}(\mathfrak{K}), \mathfrak{B}]$, $\mathfrak{W} = \boldsymbol{O}_{p',p}(\mathfrak{K})/\boldsymbol{O}_{p'}(\mathfrak{K})$.

DEFINITION 18.1. A *configuration* is any 6-tuple $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ satisfying (C). The semi-colon indicates that $\mathfrak{B}, \mathfrak{M}, \mathfrak{W}$ are determined when $\mathfrak{K}, \mathfrak{P}, \mathfrak{A}$ are given.

DEFINITION 18.2.

$$\mathscr{C}(p) = \{\mathfrak{A} \mid$$

(i) $\mathfrak{A}$ is a p-subgroup of $\mathfrak{G}$.
(ii) for every configuration $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$,
(a) $\mathfrak{M}$ centralizes $Z(\mathfrak{W})$.
(b) If $Z(\mathfrak{W})$ is cyclic, then $\mathfrak{M}$ centralizes $Z_2(\mathfrak{W})/Z(\mathfrak{W})$.}

DEFINITION 18.3.

$$\mathscr{SCN}_m(p) = \cup \mathscr{SCN}_m(\mathfrak{P}), \qquad \mathscr{U}(p) = \cup \mathscr{U}(\mathfrak{P}),$$

$\mathfrak{P}$ ranging over all S_p-subgroups of $\mathfrak{G}$ in both unions.

LEMMA 18.1. *If $p \geq 5$, then $\mathscr{U}(p) \cup \mathscr{SCN}_3(p) \subseteq \mathscr{C}(p)$.*

Proof. Let $\mathfrak{A} \in \mathscr{U}(p) \cup \mathscr{SCN}_3(p)$, and let $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ be a configuration. Suppose by way of contradiction that either $\mathfrak{M}$ fails to centralize $Z(\mathfrak{W})$ or $Z(\mathfrak{W})$ is cyclic and $\mathfrak{M}$ fails to centralize $Z_2(\mathfrak{W})/Z(\mathfrak{W})$. Since $O_{p',p}(\mathfrak{K})$ centralizes both $Z(\mathfrak{W})$ and $Z_2(\mathfrak{W})/Z(\mathfrak{W})$, it follows that some element of $\mathfrak{M}$ induces a non identity p'-automorphism of either $Z(\mathfrak{W})$ or $Z_2(\mathfrak{W})/Z(\mathfrak{W})$, so in both cases, some non identity p'-automorphism is induced on $Z_2(\mathfrak{W})$ by some element of $\mathfrak{M}$. By 3.6, some non identity p'-automorphism is induced on $\Omega_1(Z_2(\mathfrak{W})) = \mathfrak{W}_1$ by some element of $\mathfrak{M}$. Let $\mathfrak{W}_0 = \Omega_1(Z(\mathfrak{W})) \subseteq \mathfrak{W}_1$ and let $\mathfrak{W}_{-1} = \langle 1 \rangle$.

Let $\mathfrak{M}_0 = \ker(O_{p',p,p'}(\mathfrak{K}) \rightarrow \mathrm{Aut}\, \mathfrak{W}_0)$, $\mathfrak{M}_1 = \ker(O_{p',p,p'}(\mathfrak{K}) \rightarrow \mathrm{Aut}(\mathfrak{W}_1/\mathfrak{W}_0))$. By definition of $\mathfrak{M}$, $\mathfrak{M}$ is contained in $\mathfrak{M}_i$ if and only if $\mathfrak{B}$ acts trivially on $O_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$, $i = 0$ or 1. Suppose that $\mathfrak{B}$ does not act trivially on $O_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$. Let $\mathfrak{B} = \mathfrak{A}^G$ be a conjugate of $\mathfrak{A}$ which lies in $\mathfrak{B}$ and does not centralize $O_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$ ($\mathfrak{B}$ depends on i). In accordance with 3.11, we find a subgroup $\mathfrak{N}_i$ of $O_{p',p,p'}(\mathfrak{K})$ such that $\mathfrak{N}_i/\mathfrak{M}_i$ is a special q-group, is $\mathfrak{B}$-admissible, and such that $\mathfrak{B}$ acts trivially on $\mathfrak{D}_i/\mathfrak{M}_i$, irreducibly and non trivially on $\mathfrak{N}_i/\mathfrak{D}_i$, where $\mathfrak{D}_i = D(\mathfrak{N}_i \bmod \mathfrak{M}_i)$. Let $\mathfrak{B}_i = \ker(\mathfrak{B} \rightarrow \mathrm{Aut}(\mathfrak{N}_i/\mathfrak{M}_i))$, so that $\mathfrak{B}_i$ acts trivially on $\mathfrak{N}_i/\mathfrak{M}_i$ and $\mathfrak{B}/\mathfrak{B}_i$ is cyclic.

Let $\mathfrak{X}_i$ be a subgroup of $\mathfrak{W}_i/\mathfrak{W}_{i-1}$ of minimal order subject to being $\mathfrak{B}\mathfrak{N}_i$-admissible and not centralized by $\mathfrak{N}_i$. The minimal nature of $\mathfrak{X}_i$ guarantees that $\mathfrak{B}_i$ acts trivially on $\mathfrak{X}_i$. If $\mathfrak{B}_i B_i$ is a generator for $\mathfrak{B}/\mathfrak{B}_i$, then (B) guarantees that the minimal polynomial of B_i on $\mathfrak{X}_i$ is $(x-1)^r$ where $r = r_i = |\mathfrak{B} : \mathfrak{B}_i|$.

Suppose $i = 0$. Since $\mathfrak{X}_0$ is a p-group, while $O_{p'}(\mathfrak{K})$ is a p'-group, we can find a p-subgroup $\mathfrak{H}_0$ of $\mathfrak{K}$ such that $\mathfrak{H}_0$ and $\mathfrak{X}_0$ are incident, and such that $\mathfrak{H}_0$ is $\mathfrak{B}$-admissible. In particular, $\mathfrak{B}_0$ centralizes $\mathfrak{H}_0$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $N(\mathfrak{B})$, so that $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{G}$. If $\mathfrak{B}_0 \cap Z(\mathfrak{P}^*)^\#$ is non empty, we apply Lemma 16.1 and have a contradiction. Otherwise, Lemma 16.2 gives the contradiction.

We can now suppose that $Z(\mathfrak{W})$ is cyclic. In particular, $\mathfrak{W}_0$ is of order p. Since $\mathfrak{X}_1$ is of the form $\mathfrak{Y}_1/\mathfrak{W}_0$ where $\mathfrak{Y}_1$ is a suitable subgroup of $\mathfrak{W}_1$, we can find a p-subgroup $\mathfrak{H}_1$ of $\mathfrak{K}$ incident with $\mathfrak{Y}_1$ and $\mathfrak{B}$-admissible.

Choose B in $\mathfrak{B}_1$. Since $\mathfrak{B}_1$ centralizes $\mathfrak{Y}_1/\mathfrak{W}_0$ and since $\mathfrak{W}_0$ is of order p, it follows that $\mathfrak{H}_2 = C_{\mathfrak{H}_1}(B)$ is of index 1 or p in $\mathfrak{H}_1$. If $\mathfrak{B}_1 \cap Z(\mathfrak{P}^*)^\#$ is non empty, application of Lemma 16.1 gives $\gamma^3\mathfrak{H}_2\mathfrak{B}^3 = \langle 1\rangle$, and so $\gamma^4\mathfrak{H}_1\mathfrak{B}^4 = \langle 1\rangle$, the desired contradiction. Otherwise, we apply Lemma 16.2 and conclude that $\gamma^4\mathfrak{H}_2\mathfrak{B}^4 = \langle 1\rangle$, and so $\gamma^5\mathfrak{H}_1\mathfrak{B}^5 = \langle 1\rangle$, from which we conclude that $|\mathfrak{B}:\mathfrak{B}_1| = 5$. In this case, however, setting $\mathfrak{Z} = Z(\mathfrak{P}^*) \cap \mathfrak{B}$, we have $\mathfrak{B} = \langle\mathfrak{B}_1, \mathfrak{Z}\rangle$, and so the extra push comes from Lemma 16.2 which asserts that $\gamma^3\mathfrak{H}_2\mathfrak{Z}^3 = \langle 1\rangle$, and so $\gamma^4\mathfrak{H}_1\mathfrak{Z}^4 = \langle 1\rangle$, completing the proof of the lemma.

19. An E-theorem

It is convenient to assume Burnside's theorem that groups of order p^aq^b are solvable. The interested reader can reword certain of the lemmas to yield a proof of the main theorem of this paper without using the theorem of Burnside.

If $p, q \in \pi_3 \cup \pi_4$, we write $p \sim q$ provided $\mathfrak{G}$ contains elementary subgroups $\mathfrak{E}$ and $\mathfrak{F}$ of orders p^3 and q^3 respectively such that $\langle\mathfrak{E}, \mathfrak{F}\rangle \subset \mathfrak{G}$. Clearly, $\sim$ is reflexive and symmetric.

Hypothesis 19.1.
(i) $p \in \pi_3 \cup \pi_4$, $q \in \pi(\mathfrak{G})$ *and* $p \neq q$.
(ii) *A* S_p*-subgroup* $\mathfrak{P}$ *of* $\mathfrak{G}$ *centralizes every element of* $\mathsf{M}(\mathfrak{P}; q)$.

LEMMA 19.1. *Under Hypothesis* 19.1, *if* $\mathfrak{B} \in \mathscr{U}(p)$, *then* $\mathfrak{B}$ *centralizes every element of* $\mathsf{M}(\mathfrak{B}; q)$.

Proof. Suppose false, and that $\mathfrak{Q}$ is an element of $\mathsf{M}(\mathfrak{B}; q)$ minimal with respect to $\gamma\mathfrak{B}\mathfrak{Q} \neq \langle 1\rangle$. From 3.11 we conclude that $\mathfrak{B}$ centralizes $D(\mathfrak{Q})$ and acts irreducibly and non trivially on $\mathfrak{Q}/D(\mathfrak{Q})$, so in particular, $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$ and $\mathfrak{B}_0 = \ker(\mathfrak{B} \to \mathrm{Aut}\,\mathfrak{Q}) \neq \langle 1\rangle$. Let $\mathfrak{C} = C(\mathfrak{B}_0)$, let $\mathfrak{P}$ be a S_p-subgroup of $N(\mathfrak{B})$, and let $\mathfrak{P}_0 = C(\mathfrak{B}) \cap \mathfrak{P}$. Since $\mathfrak{B} \in \mathscr{U}(p)$, $\mathfrak{P}_0$ is of index at most p in a S_p-subgroup $\mathfrak{P}_1$ of $\mathfrak{C}$, and so $\mathfrak{P}_0 \lhd \mathfrak{P}_1$. Hence $\gamma\mathfrak{P}_1\mathfrak{B} \subseteq \mathfrak{P}_0$. Since $\mathfrak{P}_0$ centralizes $\mathfrak{B}$, we have $\gamma^2\mathfrak{P}_1\mathfrak{B}^2 = \langle 1\rangle$, so $\mathfrak{B} \subseteq O_{p',p}(\mathfrak{C}) = \mathfrak{K}$. Let $\mathfrak{L} = O_{p'}(\mathfrak{C})$. Since $\mathfrak{B} \subseteq \mathfrak{K} \lhd \mathfrak{C}$, $\gamma\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{K}$, so $\gamma\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{K} \cap \mathfrak{Q} \subseteq \mathfrak{L}$. Since $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$, we have $\mathfrak{Q} \subseteq \mathfrak{L}$.

By Lemma 8.9, $\mathfrak{B}$ is contained in an element $\mathfrak{A}$ of $\mathscr{SCN}_3(\mathfrak{P})$. Since $\mathfrak{A}$ centralizes $\mathfrak{B}$, we have $\mathfrak{A} \subseteq \mathfrak{P}_0$. Let $\mathfrak{D} = \mathfrak{A}\mathfrak{L}$, and observe that $\mathfrak{L}$ is a normal p-complement for $\mathfrak{A}$ in $\mathfrak{D}$. By Hypothesis 19.1 (ii), Theorem 17.1, Corollary 17.2, and $D_{p,q}$ in $\mathfrak{D}$, $\mathfrak{A}$ centralizes a S_q-subgroup of $\mathfrak{D}$, so $\mathfrak{D}$ satisfies $E^n_{p,q}$ and every p, q-subgroup of $\mathfrak{D}$

is nilpotent. But $\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{D}$, and $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B} \neq \langle 1\rangle$, so $\mathfrak{Q}\mathfrak{B}$ is not nilpotent. This contradiction completes the proof of this lemma.

Hypothesis 19.2.
(i) *$p, q \in \pi_3 \cup \pi_4$ and $p \neq q$.*
(ii) *$p \sim q$.*
(iii) *A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ centralizes every element of И$(\mathfrak{P}; q)$ and a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ centralizes every element of И$(\mathfrak{Q}; p)$.*

THEOREM 19.1. *Under Hypothesis 19.2, $\mathfrak{G}$ satisfies $E_{p,q}^{n}$.*

We proceed by way of contradiction, proving the theorem by a sequence of lemmas. Lemmas 19.2 through 19.14 all assume Hypothesis 19.2. We remark that Hypothesis 19.2 is symmetric in p and q.

LEMMA 19.2. $\langle \mathfrak{A}, \mathfrak{B}\rangle \quad \mathfrak{G}$, *whenever $\mathfrak{A} \in \mathscr{U}(p)$ and $\mathfrak{B} \in \mathscr{U}(q)$.*

Proof. Suppose $\langle \mathfrak{A}, \mathfrak{B}\rangle = \mathfrak{K} \subset \mathfrak{G}$, where $\mathfrak{A} \in \mathscr{U}(p)$, $\mathfrak{B} \in \mathscr{U}(q)$, and $\mathfrak{K}$ is minimal. By $D_{p,q}$ in $\mathfrak{K}$, it follows that $\mathfrak{K}$ is a p, q-group.

By the previous lemma $\mathfrak{A}^{\mathfrak{K}}$ centralizes $\boldsymbol{O}_q(\mathfrak{K})$ and $\mathfrak{B}^{\mathfrak{K}}$ centralizes $\boldsymbol{O}_p(\mathfrak{K})$. Since $\mathfrak{B}$ and $\mathfrak{A}$ are abelian, $\mathfrak{K}/\mathfrak{A}^{\mathfrak{K}}$ and $\mathfrak{K}/\mathfrak{B}^{\mathfrak{K}}$ are abelian, so $\mathfrak{K}'$ centralizes $\boldsymbol{O}_p(\mathfrak{K}) \times \boldsymbol{O}_q(\mathfrak{K}) = \boldsymbol{F}(\mathfrak{K})$. Hence $\mathfrak{K}' \subseteq \boldsymbol{Z}(\boldsymbol{F}(\mathfrak{K}))$ by 3.3.

Let $\mathscr{C}$ be a chief series for $\mathfrak{K}$, one of whose terms is $\mathfrak{K}'$, and let $\mathfrak{C}/\mathfrak{D}$ be a chief factor of $\mathscr{C}$. If $\mathfrak{K}' \subseteq \mathfrak{D}$, then $\mathfrak{C}/\mathfrak{D}$ is obviously a central factor. If $\mathfrak{C} \subseteq \mathfrak{K}'$, and $\mathfrak{C}/\mathfrak{D}$ is a p-group, then $\mathfrak{B}^{\mathfrak{K}}$ centralizes $\mathfrak{C}/\mathfrak{D}$, and since $\mathfrak{C}/\mathfrak{D}$ is a chief factor, $\mathfrak{A}$ must also centralize $\mathfrak{C}/\mathfrak{D}$, so $\mathfrak{C}/\mathfrak{D}$ is a central factor. The situation being symmetric in p and q, every chief factor of $\mathscr{C}$ is central, and so $\mathfrak{K}$ is nilpotent, and $\mathfrak{K} = \mathfrak{A} \times \mathfrak{B}$.

Let $\mathfrak{N} = \boldsymbol{N}(\mathfrak{A})$, let $\mathfrak{M}$ be a $S_{p,q}$-subgroup of $\mathfrak{N}$ with Sylow system $\mathfrak{P}, \mathfrak{Q}$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$, since $\mathfrak{A} \in \mathscr{U}(p)$. By $D_{p,q}$ in $\mathfrak{N}$, $\mathfrak{B}_1 = \mathfrak{B}^N \subseteq \mathfrak{Q}$ for suitable N in $\mathfrak{N}$. Let $\mathfrak{M}_1$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{M}$, with Sylow system $\mathfrak{P}$, $\mathfrak{Q}_1$ where $\mathfrak{Q} \subseteq \mathfrak{Q}_1$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_1$. Finally, let $\mathfrak{V} = \boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{Q}_1)$ and observe that $\mathfrak{B}_1 \subseteq \mathfrak{V}$. By Hypothesis 19.2, $\mathfrak{P}$ centralizes $\boldsymbol{O}_q(\mathfrak{M}_1)$ By the previous lemma, $\mathfrak{V}$ centralizes $\boldsymbol{O}_p(\mathfrak{M}_1)$.

We next show that $\mathfrak{V} \subseteq \boldsymbol{F}(\mathfrak{M}_1)$. Consider $\boldsymbol{O}_{q,p}(\mathfrak{M}_1)$, and let $\mathfrak{P}_1 = \mathfrak{P} \cap \boldsymbol{O}_{q,p}(\mathfrak{M}_1)$. Since $\mathfrak{P}$ centralizes $\boldsymbol{O}_q(\mathfrak{M}_1)$, so does $\mathfrak{P}_1$, so $\boldsymbol{O}_{q,p}(\mathfrak{M}_1) = \mathfrak{P}_1 \times \boldsymbol{O}_q(\mathfrak{M}_1)$ is nilpotent. But now $\mathfrak{V}$ centralizes $\mathfrak{P}_1$, and so Lemma 1.2.3 of [21] implies that $\mathfrak{V} \subseteq \boldsymbol{O}_q(\mathfrak{M}_1)$. It follows that $\mathfrak{V} \triangleleft \mathfrak{M}_1$. Since $\mathfrak{V}$ is weakly closed in a S_q-subgroup of $\mathfrak{M}_1$, it follows that $\mathfrak{M}_1$ is a $S_{p,q}$-subgroup of $\mathfrak{G}$.

Again, $\mathfrak{P}$ centralizes $\boldsymbol{O}_q(\mathfrak{M}_1)$, and now $\mathfrak{Q}_1$ centralizes $\boldsymbol{O}_p(\mathfrak{M}_1)$ both assertions being a consequence of Hypothesis 19.2 (iii). It follows readily that every chief factor of $\mathfrak{M}_1$ is central, and so $\mathfrak{M}_1$ is nilpotent. Since we are advancing by way of contradiction, we accept this lemma.

LEMMA 19.3. *If* $\mathfrak{A} \in \mathscr{U}(p)$, *then either* $C(\mathfrak{A})$ *is a* q'*-group or a* S_q*-subgroup* $\mathfrak{E}$ *of* $C(\mathfrak{A})$ *is of order* q, *and* $\mathfrak{E}$ *has the property that it does not centralize any* $\mathfrak{B} \in \mathscr{U}(q)$.

Proof. Let $\mathfrak{E}$ be a S_q-subgroup of $C(\mathfrak{A})$, and suppose $\mathfrak{E} \neq \langle 1 \rangle$. By Lemma 19.2, no element of $\mathfrak{E}^\#$ centralizes any $\mathfrak{B} \in \mathscr{U}(q)$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{E}$ and let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q})$. Then $C_{\mathfrak{Q}}(\mathfrak{B})$ is of index 1 or q in $\mathfrak{Q}$ and is disjoint from $\mathfrak{E}$. $|\mathfrak{E}| = q$ follows.

Lemmas 19.2 and 19.3 remain valid if p and q are interchanged throughout. In Lemmas 19.4 through 19.14 this symmetry is destroyed by the assumption that $p > q$ (which is not an assumption but a choice of notation).

We now define a family of subgroups of $\mathfrak{G}$, $\mathscr{F} = \mathscr{F}(p)$. First, $\mathscr{F}$ is the set theoretic union of the subfamilies $\mathscr{F}(\mathfrak{P})$, where $\mathfrak{P}$ ranges over the S_p-subgroups of $\mathfrak{G}$. Next, $\mathscr{F}(\mathfrak{P})$ is the set theoretic union of the subfamilies $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$, where $\mathfrak{A}$ ranges through the elements of $\mathscr{SCN}_3(\mathfrak{P})$. We proceed to build up $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$. Form $V(\mathfrak{A}) = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$. Consider the collection $\mathscr{K} = \mathscr{K}(\mathfrak{A}) = \mathscr{K}(\mathfrak{A}, q)$ of all p, q-subgroups $\mathfrak{K}$ of $\mathfrak{G}$ which have the following properties:

1. $\mathfrak{P} \subseteq \mathfrak{K}$.

(K) 2. $V(\mathfrak{A}) \subseteq O_{q,p}(\mathfrak{K})$.

3. Every characteristic abelian subgroup of $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$ is cyclic.

If $\mathscr{K}(\mathfrak{A}, q)$ is empty, we define $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$ to consist of all subgroups of $\mathfrak{A}$ of type (p, p). If $\mathscr{K}(\mathfrak{A}, q)$ is non empty, we define $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$ to consist of all subgroups of $\mathfrak{A}$ of type (p, p) together with all subgroups of $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$ of type (p, p) which contain $\Omega_1(Z(\mathfrak{P} \cap O_{q,p}(\mathfrak{K})))$, and $\mathfrak{K}$ ranges over $\mathscr{K}(\mathfrak{A}, q)$.

Notice that $\mathscr{F}(p)$ depends on q, too, but we write $\mathscr{F}(p)$ to emphasize that its elements are p-subgroups of $\mathfrak{G}$. The nature of $\mathscr{F}$ is somewhat limited by

LEMMA 19.4. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{G}$, $\mathscr{K}(\mathfrak{A}_1)$ *and* $\mathscr{K}(\mathfrak{A}_2)$ *are non empty, and if* $\mathfrak{K}_i \in \mathscr{K}(\mathfrak{A}_i)$, $i = 1, 2$, *then* $\mathfrak{P} \cap O_{q,p}(\mathfrak{K}_1) = \mathfrak{P} \cap O_{q,p}(\mathfrak{K}_2)$.

Proof. Let $\mathfrak{P}_i = \mathfrak{P} \cap O_{q,p}(\mathfrak{K}_i)$, $i = 1, 2$. Then $\mathfrak{P}_i \lhd \mathfrak{P}$, $i = 1, 2$. From 3.5 and the definition of $\mathscr{F}(p)$, we have $cl(\mathfrak{P}_i) = 2$, $i = 1, 2$. Hence $\gamma^3 \mathfrak{P}_1 \mathfrak{P}_2^3 = \langle 1 \rangle$ and $\gamma^3 \mathfrak{P}_2 \mathfrak{P}_1^3 = \langle 1 \rangle$. From (B), we conclude that $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$ and $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, as required.

Using Lemma 8.9 and Lemma 19.4, we arrive at an alternative definition of $\mathscr{F}(\mathfrak{P})$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$. If $\mathscr{K}(\mathfrak{A})$ is empty for all $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathscr{F}(\mathfrak{P})$ is the set of all subgroups $\mathfrak{B}$ of $\mathfrak{P}$ of type (p, p) such that $\mathfrak{B}^{\mathfrak{P}}$ is abelian. If $\mathscr{K}(\mathfrak{A})$ is non empty for some $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, then $\mathscr{F}(\mathfrak{P})$ consists of all subgroups of type (p, p) in $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ which contain $\Omega_1(Z(O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}))$, together with all subgroups $\mathfrak{B}$ of $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ of type (p, p) such that $\mathfrak{B}^{\mathfrak{P}}$ is abelian. Here we are also using (B) to conclude that $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ contains every element of $\mathscr{SCN}^{*}(\mathfrak{P})$.

LEMMA 19.5. *Let $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, where $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Let $\mathfrak{P}_0 = \mathfrak{P} \cap O_{q,p}(\mathfrak{K})$. If $\mathfrak{M}$ is any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, then $\mathfrak{P}_0 O_{p'}(\mathfrak{M}) \lhd \mathfrak{M}$.*

Proof. Since $\gamma^3 \mathfrak{P} \mathfrak{P}_0^3 = 1$, it follows from (B) that $\mathfrak{P}_0 \subseteq \mathfrak{P} \cap O_{p',p}(\mathfrak{M}) = \mathfrak{P}_1$, say. By Sylow's theorem, $\mathfrak{M} = O_{p'}(\mathfrak{M}) N_{\mathfrak{M}}(\mathfrak{P}_1)$, so it suffices to show that $\mathfrak{P}_0 \lhd N_{\mathfrak{M}}(\mathfrak{P}_1) = \mathfrak{N}$. Choose N in $\mathfrak{N}$. Then $[\mathfrak{P}_0^N, \mathfrak{P}_0, \mathfrak{P}_0, \mathfrak{P}_0] = 1$. Since $\mathfrak{P}_0 \subseteq \mathfrak{P}_1 \subseteq \mathfrak{P}^N \subseteq \mathfrak{K}^N$, it follows from (B) applied to $\mathfrak{K}^N$ that $\mathfrak{P}_0 \subseteq \mathfrak{P}_0^N$, so that $\mathfrak{P}_0 = \mathfrak{P}_0^N$, as required.

LEMMA 19.6. *Let $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$, and let $\mathfrak{L}$ be a subgroup of index p in $\mathfrak{P}_0 = O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$. Then $\mathfrak{V} = V(ccl_{\mathfrak{G}}(\mathfrak{L}); \mathfrak{P}) \subseteq \mathfrak{P} \cap O_{q,p}(\mathfrak{K})$.*

Proof. Since $\mathscr{SCN}_3(\mathfrak{P})$ is non empty, (B) implies that $\mathfrak{L}$ is non abelian. Now $\Omega_1(Z(\mathfrak{P}))$ is of order p and is contained in $\mathfrak{L}$. By 3.5 $\mathfrak{L}/\Omega_1(Z(\mathfrak{P}))$ is abelian.

Let $\mathfrak{L}^G = \mathfrak{L}_1$ be a conjugate of $\mathfrak{L}$ contained in $\mathfrak{P}$, $G \in \mathfrak{G}$. First, suppose that $(\Omega_1(Z(\mathfrak{P})))^G = \mathfrak{Z}$ is contained in $\mathfrak{P}_0$. Then $C_{\mathfrak{P}_0}(\mathfrak{Z}) = \mathfrak{C}_1$ is of index 1 or p in $\mathfrak{P}_0$. Set $\mathfrak{C}_2 = C(\mathfrak{Z})$. By Lemma 19.5, with $\mathfrak{C}_2$ in the role of $\mathfrak{M}$, $\mathfrak{P}^G$ in the role of $\mathfrak{P}$, $\mathfrak{P}_0^G$ in the role of $\mathfrak{P}_0$, we see that $\gamma^3 \mathfrak{C}_1 \mathfrak{L}_1^3 = \langle 1 \rangle$, and it follows that $\gamma^4 \mathfrak{P}_0 \mathfrak{L}_1^4 = \langle 1 \rangle$, so by (B), $\mathfrak{L}_1 \subseteq \mathfrak{P}_0$. (Recall that $p \geq 5$.)

Thus, if $\mathfrak{L}_1 \not\subseteq \mathfrak{P}_0$, but $\mathfrak{L}_1 \subseteq \mathfrak{P}$, then $\mathfrak{Z} \not\subseteq \mathfrak{P}_0$. But $\mathfrak{L}_1$ normalizes $\mathfrak{P}_0$, so $\mathfrak{P}_0 \cap \mathfrak{L}_1 \lhd \mathfrak{L}_1$. Since $\mathfrak{L}_1$ is of index p in $\mathfrak{P}_0^G$, any non cyclic normal subgroup of $\mathfrak{L}_1$ contains $\mathfrak{Z}$. Hence, $\mathfrak{P}_0 \cap \mathfrak{L}_1$ is cyclic and disjoint from $\mathfrak{Z}$. If now $\Omega_1(\mathfrak{P}_0)$ is extra special of order p^{2r+1}, we see that $\Omega_1(\mathfrak{L}_1)$ contains an extra special subgroup $\mathfrak{T}$ of order p^{2r-1} which is disjoint from $\mathfrak{P}_0$.

Consider now the configuration $(\mathfrak{K}, \mathfrak{P}, \mathfrak{L}; \mathfrak{V}, \mathfrak{M}, \mathfrak{W})$, and observe that $\mathfrak{W} \cong \mathfrak{P}_0$. $\mathfrak{T}$ is disjoint from $\mathfrak{P}_0$, so is faithfully represented on $\mathfrak{F} = O_{q,p,q}(\mathfrak{K})/O_{q,p}(\mathfrak{K})$, a q-group. Furthermore, $\mathfrak{F}$ is faithfully represented on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$, which makes sense, since $O_{q,p}(\mathfrak{K})$ acts trivially on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$. Let $\mathfrak{F}_1$ be the subgroup of $\mathfrak{F}$ which acts trivially on $\Omega_1(Z(\mathfrak{W}))$, which also makes sense, since $O_{q,p}(\mathfrak{K})$ acts trivially on $\Omega_1(Z(\mathfrak{W}))$. Then $\mathfrak{F}/\mathfrak{F}_1$ is cyclic and $\mathfrak{T}$ acts trivially on $\mathfrak{F}/\mathfrak{F}_1$ since $p > q$.

Since $\mathfrak{T}$ is a p-group, $\mathfrak{T}$ acts faithfully on $\mathfrak{F}_1$, so acts faithfully on $\mathfrak{F}_1/D(\mathfrak{F}_1)$. If $|\mathfrak{F}_1 : D(\mathfrak{F}_1)| = q^n$, then $|\mathfrak{T}|$ divides $(q^n-1)(q^{n-1}-1)\cdots(q-1)$, and so $|\mathfrak{T}| < q^n$, by Lemma 5.2.

On the other hand, $\mathfrak{F}_1$ acts faithfully on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$, and trivially on $\Omega_1(Z(\mathfrak{W}))$, so $\mathfrak{F}_1$ is isomorphic to a subgroup of the symplectic group $Sp(2r, p)$. Hence, $|\mathfrak{F}_1|$ divides $|Sp(2r,p)|_{p'} = (p^{2r}-1)\cdots(p^2-1)$ [6], so by Lemma 5.2 (ii), $|\mathfrak{F}_1| < p^{2r-1}$. Combining this with the previous paragraph, we have $|\mathfrak{T}| = p^{2r-1} < q^n \leqq |\mathfrak{F}_1| < p^{2r-1}$, a contradiction, completing the proof of the lemma.

We can now translate this information about $\mathfrak{L}$ to the general p, q-subgroup of $\mathfrak{G}$. To do this, we let $\mathscr{L}(p)$ be the set theoretic union of sets $\mathscr{L}(\mathfrak{P})$, $\mathfrak{P}$ ranging over the S_p-subgroups of $\mathfrak{G}$. $\mathscr{L}(\mathfrak{P})$ is the set of all subgroups $\mathfrak{L}$ which can occur in the previous lemma. Formally, $\mathscr{L}(\mathfrak{P})$ is the set of all subgroups of index p in $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$, where $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.

LEMMA 19.7. *If* $\mathfrak{L} \in \mathscr{L}(p)$ *and* $\mathfrak{H}$ *is a* p, q*-subgroup of* $\mathfrak{G}$, *then* $\mathfrak{B}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{L}); \mathfrak{H}) \subseteq O_{q,p}(\mathfrak{H})$.

Proof. Let $(\mathfrak{H}, \mathfrak{P}_1, \mathfrak{L}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ be a configuration. The lemma is clearly equivalent to the statement that $\mathfrak{B} \subseteq O_{q,p}(\mathfrak{H})$. Let $\mathfrak{P}_2$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_1$ and let $\mathfrak{L}_1 = \mathfrak{L}^G$ be a conjugate of $\mathfrak{L}$ contained in $\mathfrak{P}_1$. Since $\mathfrak{L}_1 \in \mathscr{L}(p)$, we have $\mathfrak{L}_1 \in \mathscr{L}(\mathfrak{P}_3)$ for some S_p-subgroup $\mathfrak{P}_3$ of $\mathfrak{G}$. Now $\mathfrak{P}_3 = \mathfrak{P}_2^X$ for some X in $\mathfrak{G}$, and so $\mathfrak{L}_1^X \subseteq \mathfrak{P}_3$. By Lemma 19.6, we have $\gamma^3\mathfrak{P}_3(\mathfrak{L}_1^X)^3 = \langle 1 \rangle$, and so $\gamma^3\mathfrak{P}_2\mathfrak{L}_1^3 = \langle 1 \rangle$; in particular, $\gamma^3\mathfrak{P}_1\mathfrak{L}_1^3 = \langle 1 \rangle$, so (B) and $p \geqq 5$ imply this lemma.

LEMMA 19.8. *If* $\mathfrak{A} \in \mathscr{SCN}_3(p)$, *then* $\mathfrak{B} \subseteq O_{q,p}(\mathfrak{K})$ *for every configuration* $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ *for which* $\mathfrak{K}$ *is a* p, q*-group.*

Proof. Suppose false, and that $\mathfrak{K}$ is chosen to maximize $\mathfrak{P}$, and, with this restriction to minimize $|\mathfrak{K}|_q$. It follows readily that $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$ and that $\mathfrak{P}$ is a S_p-subgroup of every p, q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{K}$.

By Lemma 18.1 and the isomorphism $O_p(\mathfrak{K}) \cong O_{q,p}(\mathfrak{K})/O_q(\mathfrak{K}) = \mathfrak{W}$, we conclude that $\mathfrak{M}$ centralizes $Z(O_p(\mathfrak{K}))$. By minimality of $|\mathfrak{K}|_q$, we also have $\mathfrak{K} = \mathfrak{P}\mathfrak{M}$.

If $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, we see that $Z(\mathfrak{P}^*)$ centralizes $O_p(\mathfrak{K})$, and so $Z(\mathfrak{P}^*) \subseteq Z(O_p(\mathfrak{K}))$, by maximality of $\mathfrak{P}$. It now follows that $\mathfrak{K}$ centralizes $Z(\mathfrak{P}^*)$, and maximality of $\mathfrak{P}$ yields $\mathfrak{P} = \mathfrak{P}^*$.

Since $\mathfrak{B}$ does not act trivially on $O_{q,p,q}(\mathfrak{K})/O_{q,p}(\mathfrak{K})$, and since $p > q$, it follows that $\mathfrak{M}$ contains an elementary subgroup of order q^3. But

$\mathfrak{M}$ centralizes $Z(O_p(\mathfrak{K})) = \mathfrak{Z}$ and if $\mathfrak{Z}$ is non cyclic, then $\mathfrak{Z}$ contains an element of $\mathscr{U}(\mathfrak{P})$, in violation of Lemma 19.3. Hence, $\mathfrak{Z}$ is cyclic. In this case, we conclude from Lemma 18.1 that a S_q-subgroup of $\mathfrak{M}$ centralizes $Z_2(O_p(\mathfrak{K})) = \mathfrak{Z}_2$. But $\mathfrak{Z}_2$ contains an element of $\mathscr{U}(\mathfrak{P})$, so once again Lemma 19.3 is violated. This contradiction completes the proof of this lemma.

LEMMA 19.9. *If* $\mathfrak{L} \in \mathscr{L}(p) \cup \mathscr{S}\mathscr{C}\mathscr{N}_3(p)$, *then* $\mathfrak{L} \subseteq O_p(\mathfrak{K})$ *for every* p, q*-subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$ *which contains* $\mathfrak{L}$.

Proof. By Lemmas 19.7 and 19.8, it suffices to show that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$. If $\mathfrak{L} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(p)$, Theorem 17.1, Corollary 17.2 and Hypothesis 19.2 imply that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$. If $\mathfrak{L} \in \mathscr{L}(p)$, then $\mathfrak{L} \in \mathscr{L}(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. In this case, if $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$, the definition of $\mathscr{L}(\mathfrak{P})$ implies that $\mathfrak{A} \cap \mathfrak{L} = \mathfrak{A}_0$ is non cyclic. Hence, $O_q(\mathfrak{K})$ is generated by its subgroups $C(A) \cap O_q(\mathfrak{K})$ as A ranges over $\mathfrak{A}_0^\#$. By the preceding argument, $\mathfrak{A}$ is contained in $O_p(\mathfrak{K}_0)$ for every p, q-subgroup $\mathfrak{K}_0$ of $C(A)$ which contains $\mathfrak{A}$. Lemma 7.5 implies that $\mathfrak{A}_0$ centralizes $O_q(\mathfrak{K})$. In particular, $\Omega_1(Z(\mathfrak{P}))$ centralizes $O_q(\mathfrak{K})$.

Consider $C(\Omega_1(Z(\mathfrak{P}))) \supseteq \langle \mathfrak{P}, O_q(\mathfrak{K}) \rangle$. Since $\mathfrak{L} \subseteq O_p(\mathfrak{K}_1)$ for every p, q-subgroup $\mathfrak{K}_1$ of $\langle \mathfrak{P}, O_q(\mathfrak{K}) \rangle$ which contains $\mathfrak{L}$ by (B) and Hypothesis 19.2, a second application of Lemma 7.5 shows that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$, as required.

LEMMA 19.10. *If* $\mathfrak{B} \in \mathscr{F}(p)$, *then* $\mathfrak{B}$ *centralizes every element of* $\mathsf{N}(\mathfrak{B}; q)$.

Proof. Suppose false, and $\mathfrak{Q}$ is chosen minimal subject to $\mathfrak{Q} \in \mathsf{N}(\mathfrak{B}; q)$ and $\gamma\mathfrak{Q}\mathfrak{B} \neq \langle 1 \rangle$, so that we have $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$ and $\mathfrak{B}_0 = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\mathfrak{Q}) \neq \langle 1 \rangle$. Let $\mathfrak{C} = C(\mathfrak{B}_0)$. Since $\mathfrak{B} \in \mathscr{F}(p)$, we have $\mathfrak{B} \in \mathscr{F}(\mathfrak{P})$ for a suitable S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. By definition of $\mathscr{F}(\mathfrak{P})$, either $C(\mathfrak{B})$ contains an element $\mathfrak{A}_1$ of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$ or else $C(\mathfrak{B})$ contains a subgroup $\mathfrak{P}_1$ of index p in $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$, $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$ and $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{A}_1$ in the first case, and $\mathfrak{P}_1$ in the second case. Lemma 19.9 implies that $\mathfrak{A}_1 \subseteq O_p(\mathfrak{H})$ in the first case and $\mathfrak{P}_1 \subseteq O_p(\mathfrak{H})$ in the second case. In both cases, we have $\mathfrak{B} \subseteq O_p(\mathfrak{H})$. Now let $\mathfrak{H}_1$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{B}\mathfrak{Q}$. By Lemma 7.5, we have $\mathfrak{B} \subseteq O_p(\mathfrak{H}_1)$ and so $\gamma\mathfrak{Q}\mathfrak{B} \subseteq O_p(\mathfrak{H}_1) \cap \mathfrak{Q} = \langle 1 \rangle$, contrary to assumption.

LEMMA 19.11. *If* $\mathfrak{B} \in \mathscr{F}(p)$, $\mathfrak{A} \in \mathscr{U}(q)$, *then* $\mathfrak{G} = \langle \mathfrak{A}, \mathfrak{B} \rangle$.

Proof. Suppose $\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{K} \subset \mathfrak{G}$, and $\mathfrak{A}$ and $\mathfrak{B}$ are chosen to minimize $\mathfrak{K}$. By the minimal nature of $\mathfrak{K}$, $\mathfrak{K}$ is a p, q-group. By the previous lemmas, $\mathfrak{A}^{\mathfrak{K}}$ centralizes $O_p(\mathfrak{K})$, and $\mathfrak{B}^{\mathfrak{K}}$ centralizes $O_q(\mathfrak{K})$. It follows readily that $\mathfrak{K}$ is nilpotent, so $\mathfrak{K} = \mathfrak{A} \times \mathfrak{B}$. But now $C(\mathfrak{A})$ contains $\mathfrak{B}$ in violation of Lemma 19.3, with p and q interchanged. This interchange is permissible since Lemma 19.3 was proved before we discarded the symmetry in p and q.

LEMMA 19.12. *If $\mathfrak{D}$ is a p, q-subgroup of $\mathfrak{G}$ and if $\mathfrak{D}$ possesses an elementary subgroup of order p^3, then a S_p-subgroup of $\mathfrak{D}$ is normal in $\mathfrak{D}$.*

Proof. Case 1. $\mathfrak{D}$ contains a S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{D}$, let $\mathfrak{Q}_1 = \mathfrak{Q} \cap O_{p,q}(\mathfrak{D})$, let $\tilde{\mathfrak{Q}}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, let $\mathfrak{B} \in \mathscr{U}(\tilde{\mathfrak{Q}})$, and $\mathfrak{Q}_2 = C_{\mathfrak{Q}_1}(\mathfrak{B})$. Then $\mathfrak{Q}_2$ is of index 1 or q in $\mathfrak{Q}_1$.

Next, let $\mathfrak{K} = O_p(\mathfrak{D})$, and assume by way of contradiction that $\mathfrak{K} \subset \mathfrak{P}$. By the preceding lemmas, $\mathfrak{K}$ contains $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$ for every $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. By the preceding lemma, no element of $\mathfrak{Q}_2^{\#}$ centralizes any element of $\mathscr{F}(p)$.

If $\mathfrak{K}$ contains a non cyclic characteristic subgroup $\mathfrak{C}$, then every subgroup of $\mathfrak{C}$ of type (p, p) belongs to $\mathscr{F}(\mathfrak{P})$, and so $C_{\mathfrak{C}}(Q)$ is cyclic for $Q \in \mathfrak{Q}_2$. This implies that $\mathscr{SCN}_3(\mathfrak{Q}_2)$ is empty, and if $\mathfrak{Q}_2$ possesses a subgroup of type (q, q), then $p \equiv 1 \pmod q$. However, if $\mathfrak{K}$ does not contain any non cyclic characteristic abelian subgroup, then every subgroup of $\mathfrak{K}$ of type (p, p) which contains $\Omega_1(Z(\mathfrak{K}))$ lies in $\mathscr{F}(\mathfrak{P})$, and we again conclude that $\mathscr{SCN}_3(\mathfrak{Q}_2)$ is empty, and if $\mathfrak{Q}_2$ is non cyclic, then $p \equiv 1 \pmod q$.

Now $\mathfrak{Q}_1 \cong O_{p,q}(\mathfrak{D})/\mathfrak{K}$ admits a non trivial p-automorphism since $\mathfrak{K} \subset \mathfrak{P}$, so $\mathscr{SCN}_3(\mathfrak{Q}_1)$ is non empty, by Lemma 8.4 (ii) and $p > q$. Hence, $\mathfrak{Q}_2$ is non cyclic, being of index at most q in $\mathfrak{Q}_1$, and this yields $p \equiv 1 \pmod q$. We apply Lemma 8.8 and conclude that $p = 1 + q + q^2$, and $\mathfrak{Q}_1$ is elementary of order q^3. This implies that any two subgroups of $\mathfrak{Q}_1$ of the same order are conjugate in $\mathfrak{D}$. Since at least one subgroup of $\mathfrak{Q}_1$ of order q centralizes $\mathfrak{B}$, every subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{U}(q)$. Since at least one subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{F}(\mathfrak{P})$, every subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{F}(p)$. This conflicts with Lemma 19.11.

Case 2. $\mathfrak{D}$ does not contain a S_p-subgroup of $\mathfrak{G}$. Among all $\mathfrak{D}$ which satisfy the hypotheses but not the conclusion of this lemma, choose $\mathfrak{D}$ so that $|\mathfrak{D} \cap \Omega_1(\mathfrak{A})|$ is a maximum, where $\mathfrak{A}$ ranges over all elements of $\mathscr{SCN}_3(p)$, and with this restriction, maximize $|\mathfrak{D}|_p$.

Let $\mathfrak{D}_1$ be a S_p-subgroup of $\mathfrak{D}$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{D}_1$.

First, assume $\mathfrak{D}_1$ centralizes $O_q(\mathfrak{D})$. In this case, $O_p(\mathfrak{D})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{D})$. By maximality of $|\mathfrak{D}|_p$, $\mathfrak{D}_1$ is a S_p-subgroup of $N(O_p(\mathfrak{D}))$. This implies that $\mathfrak{D}_1$ contains every element of $\mathscr{SCN}_3(\mathfrak{P})$. To see this, let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, and let $\mathfrak{A}_1 = \mathfrak{A} \cap \mathfrak{D}_1$. Since $O_p(\mathfrak{D})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{D})$, it follows that $\mathfrak{A} \cap \mathfrak{D}_1 \subseteq O_p(\mathfrak{D})$. If $\mathfrak{A}_1$ were a proper subgroup of $\mathfrak{A}$, then $\mathfrak{D}_1$ would be a proper subgroup of $N_{\mathfrak{A}\mathfrak{D}_1}(O_p(\mathfrak{D}))$. Since this is not possible, we have $\mathfrak{A} = \mathfrak{A}_1$. But now, $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{D}_1) \lhd \mathfrak{D}$, and by maximality of $|\mathfrak{D}|_p$, $\mathfrak{D}_1 = \mathfrak{P}$ follows, and we are in the preceding case.

We can now assume that $\mathfrak{D}_1$ does not centralize $O_q(\mathfrak{D})$. Suppose $\mathfrak{D}_1$ contains some element $\mathfrak{B}$ of $\mathscr{F}(p)$. By Lemma 19.10, $\mathfrak{B}$ centralizes $O_q(\mathfrak{D})$. Since $\mathfrak{D}_1$ does not centralize $O_q(\mathfrak{D})$, $|O_q(\mathfrak{D})| > q$, and so Lemma 19.11 is violated in $C(Q)$, Q being a suitable element of $O_q(\mathfrak{D})$. Thus, we can suppose that $\mathfrak{D}_1$ does not contain any element of $\mathscr{F}(p)$. In particular, $\mathfrak{D} \cap \Omega_1(\mathfrak{A})$ is of order 1 or p for all $\mathfrak{A} \in \mathscr{SCN}_3(p)$. Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, and $\mathfrak{D}_2 = C_{\mathfrak{D}_1}(\mathfrak{B})$. Since $\mathscr{SCN}_3(\mathfrak{D}_1)$ is non empty by hypothesis, $\mathfrak{D}_2$ is non cyclic. Let $\mathfrak{E}$ be a subgroup of $\mathfrak{D}_2$ of type (p, p). Since $\mathfrak{B} \not\subseteq \mathfrak{D}_1$, $\langle \mathfrak{E}, \mathfrak{B} \rangle$ is elementary of order at least p^3. If $\mathfrak{E}$ does not centralize $O_q(\mathfrak{D})$, then there is an element E in $\mathfrak{E}^\#$ such that $\mathfrak{E}$ does not centralize $C(E) \cap O_q(\mathfrak{D})$. But in this case, a $S_{p,q}$-subgroup of $C(E)$ is larger than $\mathfrak{D}$ in our ordering since $\mathfrak{B} \subseteq C(E)$, $C(E)$ possesses an elementary subgroup of order p^3, and a S_p-subgroup of a $S_{p,q}$-subgroup $\mathfrak{F}$ of $C(E)$ is not normal in $\mathfrak{F}$. This conflict forces every subgroup of $\mathfrak{D}_2$ of type (p, p) to centralize $O_q(\mathfrak{D})$. Thus, $\Omega_1(\mathfrak{D}_2) = \mathfrak{D}^*$ centralizes $O_q(\mathfrak{D})$, since $\mathfrak{D}^*$ is generated by its subgroups of type (p, p). However, we now have $N(\mathfrak{D}^*) \supseteq \langle \mathfrak{D}_1, \mathfrak{B}, O_q(\mathfrak{D}) \rangle$ and a $S_{p,q}$-subgroup $\mathfrak{F}_1$ of $N(\mathfrak{D}^*)$ is larger than $\mathfrak{D}$ in our ordering, possesses an elementary subgroup of order p^3, and has the additional property that its S_p-subgroups are not normal in $\mathfrak{F}_1$. This conflict completes the proof of this lemma.

Lemma 19.12 gives us a fairly good idea of the structure of the p, q-subgroups of $\mathfrak{G}$. The remaining analysis is still somewhat detailed, but the moves are more obvious.

For the remainder of this section, $\mathfrak{P}$ denotes a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{Q}$ a S_q-subgroup of $N(\mathfrak{P})$, and $\tilde{\mathfrak{Q}}$ a S_q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{Q}$.

LEMMA 19.13. *$\mathscr{SCN}_3(\mathfrak{Q})$ is non empty.*

Proof. We apply Hypothesis 19.2 (ii) and let $\mathfrak{D}$ be a maximal p, q-subgroup of $\mathfrak{G}$ which contains elementary subgroups of order p^3 and q^3. By Lemma 19.12, $\mathfrak{D}_p \lhd \mathfrak{D}$, $\mathfrak{D}_p$ being a S_p-subgroup of $\mathfrak{D}$. Since $\mathfrak{D}$ is a maximal p, q-subgroup of $\mathfrak{G}$, $\mathfrak{D}_p$ is a S_p-subgroup of $\mathfrak{G}$, so $\mathfrak{D}_p = \mathfrak{P}^G$ and the lemma follows.

We now choose $\mathfrak{B}$ in $\mathscr{U}(\tilde{\mathfrak{Q}})$ and set $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{B})$.

LEMMA 19.14.
(i) $\mathscr{SCN}_3(\mathfrak{Q}_1)$ *is empty.*
(ii) $\mathfrak{Q}$ *contains* $\Omega_1(Z(\tilde{\mathfrak{Q}}))$.
(iii) $p \equiv 1 \pmod q$.
(iv) $\mathfrak{Q}^\#$ *contains an element* Y *which centralizes an element of* $\mathscr{F}(\mathfrak{P})$, *and has the additional property that* $C_{\mathfrak{Q}}(Y)$ *contains an elementary subgroup of order* q^3.
(v) *If* $X \in \mathfrak{Q}^\#$ *and* X *centralizes an element of* $\mathscr{F}(\mathfrak{P})$, *then* X *does not centralize any element of* $\mathscr{U}(\tilde{\mathfrak{Q}})$, *and* $C(X)$ *does not contain an elementary subgroup of order* q^4.

Proof. Let $\mathfrak{E}$ be an elementary subgroup of $\mathfrak{Q}$ of order q^3, and choose $\mathfrak{E}$ in $\mathfrak{Q}_1$ if possible. If $\mathfrak{P}$ possesses a non cyclic characteristic abelian subgroup $\mathfrak{C}$, then some element of $\mathfrak{E}$ has a non cyclic fixed point set on $\mathfrak{C}$. Since every subgroup of $\mathfrak{C}$ of type (p, p) lies in $\mathscr{F}(\mathfrak{P})$, (iv) is established in this case.

If every characteristic abelian subgroup of $\mathfrak{P}$ is cyclic, then some non cyclic subgroup $\mathfrak{E}_1$ of $\mathfrak{E}$ centralizes $Z(\mathfrak{P})$. Since any non cyclic subgroup of $\mathfrak{P}$ which contains $\Omega_1(Z(\mathfrak{P}))$ is normal in $\mathfrak{P}$, by 3.5, some element of $\mathfrak{E}_1$ centralizes an element of $\mathscr{F}(\mathfrak{P})$, so (iv) is proved.

If $\mathfrak{E} \subseteq \mathfrak{Q}_1$, then Lemma 19.11 is violated in $C(E)$, $E \in \mathfrak{E}^\#$, E centralizing an element of $\mathscr{F}(\mathfrak{P})$. Hence, (i) is proved.

On the other hand, $\mathscr{SCN}_3(\mathfrak{Q})$ is non empty, so $\mathfrak{Q}_1$ possesses a subgroup $\mathfrak{F}_1$ of type (q, q). If $p \not\equiv 1 \pmod q$, then some element of $\mathfrak{F}_1$ is seen to centralize an element of $\mathscr{F}(\mathfrak{P})$. Since this is forbidden by Lemma 19.11, (iii) follows.

We now turn attention to (v). In view of Lemma 19.11, we only need to show that if X in $\mathfrak{Q}^\#$ centralizes an element of $\mathscr{F}(\mathfrak{P})$, then $C(X)$ does not contain an elementary subgroup of order q^4.

Let $\mathfrak{A}$ be an element of $\mathscr{F}(\mathfrak{P})$ centralized by X, let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $C(X)$ and let $\mathfrak{R}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. By $D_{p,q}$ in $C(X)$, $\mathfrak{A}_1 = \mathfrak{A}^G \subseteq \mathfrak{H} \subseteq \mathfrak{R}$, for some G in $C(X)$. Suppose by way of contradiction that $C(X)$ contains an elementary subgroup of order q^4. By $D_{p,q}$ in $C(X)$, $\mathfrak{H}$ contains an elementary subgroup of order q^4; thus, $\mathfrak{R}$ contains such a subgroup.

We first show that a S_p-subgroup of $\mathfrak{R}$ is not normal in $\mathfrak{R}$. Suppose false. In this case, since $\mathfrak{R}$ is a maximal p, q-subgroup of $\mathfrak{G}$, a S_p-subgroup of $\mathfrak{R}$ is conjugate to $\mathfrak{P}$, and so $\mathfrak{R}$ is conjugate to $\mathfrak{P}\mathfrak{Q}$. However, (i) implies that $\mathfrak{Q}$ does not contain an elementary subgroup of order q^4, since $|\mathfrak{Q} : \mathfrak{Q}_1| = q$, so $\mathfrak{R}$ does not contain one either.

We now apply Lemma 19.12 and conclude that $\mathfrak{R}$ does not possess an elementary subgroup of order p^3. It follows directly from Lemma 8.13 that $\mathfrak{K}$ has p-length one. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$ containing $\mathfrak{A}_1$, and let $\mathfrak{B}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{A}_1); \mathfrak{K}_p)$. By Lemma 19.10, $\mathfrak{B}_1$ centralizes $O_q(\mathfrak{K})$. Since $\mathfrak{K}$ has p-length one, $\mathfrak{B}_1 \lhd \mathfrak{K}$. But then $N(\mathfrak{B}_1) = \mathfrak{N}$ contains S_p-subgroups of larger order than $|\mathfrak{K}_p|$, and $\mathfrak{N}$ also contains $\mathfrak{K}$, contrary to the assumption that $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$. This contradiction proves (v).

We now turn to (ii). Choose Y to satisfy (iv) and let $\mathfrak{E}$ be an elementary subgroup of $C_{\mathfrak{Q}}(Y)$ of order q^3. If $\Omega_1(Z(\tilde{\mathfrak{Q}})) = \Omega_1$ were not contained in $\mathfrak{E}$, then $\langle \mathfrak{E}, \Omega_1 \rangle$ would contain an elementary subgroup of order q^4, and (v) would be violated. This completes the proof of this lemma.

We remark that Lemma 19.2 and Lemma 19.14 (ii) imply that $Z(\tilde{\mathfrak{Q}})$ is cyclic.

Theorem 19.1 can now be proved fairly easily. We again denote by $\mathfrak{E}$ an elementary subgroup of $\mathfrak{Q}$ of order q^3, and we let Y be an element of $\mathfrak{E}^{\#}$ which centralizes an element of $\mathscr{F}(\mathfrak{P})$. Let $\mathfrak{E}_1 = C_{\mathfrak{E}}(\mathfrak{B})$. Since $\Omega_1 = \Omega_1(Z(\tilde{\mathfrak{Q}}))$ centralizes $\mathfrak{B}$, Ω_1 does not centralize any element of $\mathscr{U}(\mathfrak{P})$, by Lemma 19.2, and so does not centralize $\mathfrak{P}$. Thus, we can find an element E in $\mathfrak{E}_1^{\#}$ with the property that Ω_1 does not centralize $C_{\mathfrak{P}}(E)$. Consider $\mathfrak{C} = C(E)$. We see that $\mathfrak{C}$ contains both Y and $\mathfrak{B}$. Since Y does not centralize $\mathfrak{B}$, $\langle Y, \mathfrak{B} \rangle$ is a non abelian group of order q^3, with center Ω_1. Let $\mathfrak{L}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ which contains $\langle Y, \mathfrak{B} \rangle$; since $\mathfrak{L}$ contains $\mathfrak{B}$, $\mathfrak{L}$ does not contain an elementary subgroup of order p^3. Since Ω_1 is contained in the derived group of $\langle Y, \mathfrak{B} \rangle$, Ω_1 is contained in $\mathfrak{L}'$. We apply Lemma 8.13 and conclude that Ω_1 centralizes every chief p-factor of $\mathfrak{L}$. It follows that $\gamma^n \mathfrak{L} \Omega_1^n = \langle 1 \rangle$ for suitably large n, and so $\Omega_1 \subseteq O_q(\mathfrak{L})$. But now if $\mathfrak{H}$ is any $S_{p,q}$-subgroup of $\mathfrak{C}$ which contains Ω_1, we have $\Omega_1 \subseteq O_q(\mathfrak{H})$, by Lemma 7.5, and so $[\Omega_1, C_{\mathfrak{P}}(E)]$ is both a p-group and q-group, so is $\langle 1 \rangle$, contrary to construction. This completes the proof of Theorem 19.1.

COROLLARY 19.1. *If $p, q \in \pi_3 \cup \pi_4$, $p \neq q$, and $p \sim q$, then either $p \in \pi_3$ or $q \in \pi_3$.*

Proof. If $\mathfrak{G}$ satisfies $E_{p,q}^n$, then both p and q are in π_3. Otherwise, Hypothesis 19.2 is violated and the corollary follows.

20. An E-theorem for π_3

Hypothesis 20.1 $p, q \in \pi_3$, $p \neq q$, *and* $p \sim q$.

THEOREM 20.1. *Under Hypothesis* 20.1, $\mathfrak{G}$ *satisfies* $E_{p,q}$.

The proof of this theorem is by contradiction. The following lemmas assume that Hypothesis 20.1 is satisfied but $\mathfrak{G}$ does not satisfy $E_{p,q}$.

LEMMA 20.1. *If* $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{G}$ *and* $\mathfrak{Q}$ *is a* S_q*-subgroup of* $\mathfrak{G}$, *then either* $\mathfrak{P}$ *normalizes but does not centralize some* q*-subgroup of* $\mathfrak{G}$, *or* $\mathfrak{Q}$ *normalizes but does not centralize some* p*-subgroup of* $\mathfrak{G}$.

Proof. This lemma is an immediate consequence of Hypothesis 20.1, Theorem 19.1, and the assumption that $\mathfrak{G}$ does not satisfy $E_{p,q}$.

We assume now that notation is chosen so that $\mathfrak{P}$, a S_p-subgroup of $\mathfrak{G}$, does not centralize $\mathfrak{Q}_1$, a maximal element of $\mathsf{M}(\mathfrak{P}; q)$. Let $\mathfrak{Q}^*$ be a S_q-subgroup of $N(\mathfrak{Q}_1)$ permutable with $\mathfrak{P}$, and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$.

LEMMA 20.2. $O_p(\mathfrak{P}\mathfrak{Q}^*) \neq \langle 1 \rangle$.

Proof. Suppose false. Let $\mathfrak{A}$ be an element of $\mathscr{SCN}_3(\mathfrak{Q})$. By Lemma 7.9, we have $\mathfrak{A} \subseteq O_q(\mathfrak{P}\mathfrak{Q}^*)$. We apply Lemma 17.4 and conclude that $N(\mathfrak{Q}_1) \subseteq N(\mathfrak{Z})$, where $\mathfrak{Z} = Z(\mathfrak{B})$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{Q})$, and so $\mathfrak{G}$ satisfies $E_{p,q}$, contrary to assumption.

Let $\mathfrak{P}_1 = O_p(\mathfrak{P}\mathfrak{Q}^*)$.

LEMMA 20.3. $\mathfrak{Q}^*$ *is a* S_q*-subgroup of every proper subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$ *which contains* $\mathfrak{P}_1\mathfrak{Q}^*$.

Proof. Let $\mathfrak{T}$ be a $S_{p,q}$-subgroup of $\mathfrak{K}$ with Sylow system $\mathfrak{Q}_2$, $\mathfrak{P}_2$ where $\mathfrak{Q}^* \subseteq \mathfrak{Q}_2$ and $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, and let $F(\mathfrak{T}) = \mathfrak{T}_1 \times \mathfrak{T}_2$, where $\mathfrak{T}_1 = O_p(\mathfrak{T})$, $\mathfrak{T}_2 = O_q(\mathfrak{T})$.

We first show that $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$. Suppose by way of contradiction that $\mathfrak{T}_1 \cap \mathfrak{P}_1 \subset \mathfrak{T}_1$. Since $\mathfrak{Q}^*$ and $\mathfrak{P}_1$ both normalize $\mathfrak{T}_1 \cap \mathfrak{P}_1$ and both normalize $\mathfrak{T}_1$, setting $\mathfrak{T}_1^* = N_{\mathfrak{T}_1}(\mathfrak{T}_1 \cap \mathfrak{P}_1)$, we see that $\mathfrak{T}_1^*\mathfrak{Q}^*\mathfrak{P}_1$ is a group, and that $\mathfrak{Q}^*\mathfrak{P}_1$ normalizes $\mathfrak{T}_1^*$. Let $\mathfrak{T}^*/\mathfrak{T}_1 \cap \mathfrak{P}_1$ be a chief factor of $\mathfrak{T}_1^*\mathfrak{Q}^*\mathfrak{P}_1$ with $\mathfrak{T}^* \subseteq \mathfrak{T}_1^*$. Since $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}^*$, it follows that $\mathfrak{P}_1$ centralizes $\mathfrak{T}^*/\mathfrak{T}_1 \cap \mathfrak{P}_1$, that is $\gamma\mathfrak{T}^*\mathfrak{P}_1 \subseteq \mathfrak{T}_1 \cap \mathfrak{P}_1$. In particular $\mathfrak{T}^*$ normalizes $\mathfrak{P}_1$. Now $\mathfrak{P}\mathfrak{Q}^*$ is a maximal p, q-subgroup of $\mathfrak{G}$ by Lemma 7.3, so $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{P}_1)$. A second application of Lemma 7.3 yields that $\mathfrak{P}_1$ is a S_p-subgroup of $O_{q'}(N(\mathfrak{P}_1))$. But $\mathfrak{P}_1\mathfrak{T}^*$ is normalized by $\mathfrak{Q}^*$, so a third application of Lemma 7.3 yields $\mathfrak{P}_1\mathfrak{T}^* \subseteq O_{q'}(N(\mathfrak{P}_1))$, so $\mathfrak{T}^* \subseteq \mathfrak{P}_1$, contrary to our choice of $\mathfrak{T}^*$. Thus, $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$.

We next show that $\mathfrak{T}_2 \subseteq \mathfrak{Q}^*$. To do this, it suffices to show that

$\mathfrak{P}_1$ centralizes $\mathfrak{T}_2$, for if this is the case, then $\mathfrak{T}_2 \subseteq C(\mathfrak{P}_1) \subseteq N(\mathfrak{P}_1)$, and so $\mathfrak{T}_2\mathfrak{Q}^*$ is a q-subgroup of $N(\mathfrak{P}_1)$. Since $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{P}_1)$, $\mathfrak{T}_2 \subseteq \mathfrak{Q}^*$ follows.

To show that $\mathfrak{P}_1$ centralizes $\mathfrak{T}_2$, we first show that $\mathfrak{P}_1$ centralizes $C_{\mathfrak{T}_2}(\mathfrak{Q}_1)$. By definition, $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{Q}_1)$, and since $\mathfrak{Q}^*C_{\mathfrak{T}_2}(\mathfrak{Q}_1)$ is a q-subgroup of $N(\mathfrak{Q}_1)$, we have $C_{\mathfrak{T}_2}(\mathfrak{Q}_1) \subseteq \mathfrak{Q}^*$. Hence, $[C_{\mathfrak{T}_2}(\mathfrak{Q}_1), \mathfrak{P}_1] \subseteq \mathfrak{T}_2 \cap [\mathfrak{Q}^*, \mathfrak{P}_1] \subseteq \mathfrak{T}_2 \cap \mathfrak{P}_1 = \langle 1 \rangle$. Suppose that $\mathfrak{P}_1$ does not centralize $\mathfrak{T}_2$ and that $\mathfrak{T}_3$ is a $\mathfrak{P}_1\mathfrak{Q}_1$-invariant subgroup of $\mathfrak{T}_2$, minimal subject to the condition $\gamma\mathfrak{T}_3\mathfrak{P}_1 \neq \langle 1 \rangle$. By minimality of $\mathfrak{T}_3$, we have $\mathfrak{T}_3 = \gamma\mathfrak{T}_3\mathfrak{P}_1$. Since $\mathfrak{T}_3$ is a q-group, $\gamma\mathfrak{T}_3\mathfrak{Q}_1 \subset \mathfrak{T}_3$, and so $\gamma^2\mathfrak{T}_3\mathfrak{Q}_1\mathfrak{P}_1 = \langle 1 \rangle$. Since $\gamma\mathfrak{Q}_1\mathfrak{P}_1 = \langle 1 \rangle$, we also have $\gamma^2\mathfrak{Q}_1\mathfrak{P}_1\mathfrak{T}_3 = \langle 1 \rangle$. The three subgroups lemma now yields $\gamma^2\mathfrak{P}_1\mathfrak{T}_3\mathfrak{Q}_1 = \langle 1 \rangle$, so $\mathfrak{Q}_1$ centralizes $\gamma\mathfrak{P}_1\mathfrak{T}_3 = \mathfrak{T}_3$. By what we have already shown this implies that $\mathfrak{P}_1$ centralizes $\mathfrak{T}_3$. This conflict forces $\gamma\mathfrak{P}_1\mathfrak{T}_2 = \langle 1 \rangle$.

We next show that $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$. To do this, consider $C_{\mathfrak{T}}(\mathfrak{T}_1) = \mathfrak{C} \lhd \mathfrak{T}$. Since $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$, we see that $\mathfrak{Q}_1 \subseteq \mathfrak{C}$. On the other hand, $Z(\mathfrak{P}_1)$ centralizes both $\mathfrak{T}_1$ and $\mathfrak{T}_2$, so $Z(\mathfrak{P}_1) \subseteq \mathfrak{T}_1$, by 3.3. Hence, $\mathfrak{C} \subseteq C_{\mathfrak{T}}(Z(\mathfrak{P}_1)) \subseteq C(Z(\mathfrak{P}_1)) \subseteq N(Z(\mathfrak{P}_1))$. Since $\mathfrak{Q}_1 = O_q(\mathfrak{P}\mathfrak{Q}^*)$, Lemma 7.5 implies that $\mathfrak{Q}_1 \subseteq O_q(\mathfrak{C})$ char $\mathfrak{C} \lhd \mathfrak{T}$, and so $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$.

Consider finally $C_{\mathfrak{T}}(\mathfrak{T}_2)$. Since $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$, we have $C_{\mathfrak{T}}(\mathfrak{T}_2) \subseteq C_{\mathfrak{T}}(\mathfrak{Q}_1) \subseteq C(\mathfrak{Q}_1) \subseteq N(\mathfrak{Q}_1)$. Since $\mathfrak{P}_1 = O_p(\mathfrak{P}\mathfrak{Q}^*)$, Lemma 7.5 implies that $\mathfrak{P}_1 \subseteq O_p(C_{\mathfrak{T}}(\mathfrak{T}_2))$ char $C_{\mathfrak{T}}(\mathfrak{T}_2) \lhd \mathfrak{T}$, and so $\mathfrak{P}_1 \subseteq \mathfrak{T}_1$. Since we have already shown that $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$, we have $\mathfrak{T}_1 = \mathfrak{P}_1 \lhd \mathfrak{T}$, and so $\mathfrak{Q}^*$ is a S_q-subgroup of $\mathfrak{T}$, as required.

To prove Theorem 20.1 recall that $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$. Choose $\mathfrak{A}$ in $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q})$, and let $\mathfrak{A}^* = \mathfrak{A} \cap \mathfrak{Q}^*$. We first show that $\mathfrak{A}^* \subset \mathfrak{A}$. Suppose by way of contradiction that $\mathfrak{A}^* = \mathfrak{A}$. Then $\mathfrak{A}$ normalizes $\mathfrak{P}_1$. Lemma 7.3 and the previous lemma imply that $\mathfrak{P}_1$ is a maximal element of $\mathsf{N}(\mathfrak{A}; p)$. By Corollary 17.1, $N(\mathfrak{P}_1)$ contains a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{G}$ satisfies $E_{p,q}$. Since we are advancing by contradiction, we have $\mathfrak{A}^* \subset \mathfrak{A}$.

We next show that $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1 \rangle$. To do this, we observe that $\mathfrak{A}^* \cap \mathfrak{Q}_1 \lhd \mathfrak{Q}^*$, so if $\mathfrak{A}^* \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$, then $\mathfrak{A}^* \cap \mathfrak{Q}_1 \cap Z(\mathfrak{Q}^*) \neq \langle 1 \rangle$. In this case, however, $C(\mathfrak{A}^* \cap \mathfrak{Q}_1 \cap Z(\mathfrak{Q}^*))$ contains $\mathfrak{P}_1$ and also contains $\mathfrak{Q}^*\mathfrak{A}$, contrary to the previous lemma. Thus, $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1 \rangle$. Since $\mathfrak{A}^*$ and $\mathfrak{Q}_1$ are both normal in $\mathfrak{Q}^*$, we have $\gamma\mathfrak{A}^*\mathfrak{Q}_1 = \langle 1 \rangle$.

Let $\mathfrak{A}_1 = N_{\mathfrak{A}}(\mathfrak{Q}^*)$, so that $\mathfrak{A}^* \subset \mathfrak{A}_1 \subseteq \mathfrak{A}$. Observe that $\gamma\mathfrak{A}_1\mathfrak{Q}^* \subseteq \mathfrak{Q}^* \cap \mathfrak{A} \subset \mathfrak{A}_1$ and so $\mathfrak{Q}^*$ normalizes $\mathfrak{A}_1$. Let $\mathfrak{B}$ be any subgroup of $\mathfrak{A}_1$ which contains $\mathfrak{A}^*$ properly. Since $[\mathfrak{B}, \mathfrak{Q}_1\mathfrak{A}^*] \subseteq \mathfrak{A}^*$, we see that $\mathfrak{B}$ normalizes $\mathfrak{Q}_1\mathfrak{A}^* = \mathfrak{Q}_1 \times \mathfrak{A}^*$. Since $\mathfrak{Q}^*$ normalizes $\mathfrak{B}$, $\mathfrak{Q}^*$ also normalizes $C_{\mathfrak{Q}_1}(\mathfrak{B}) = \mathfrak{D}$, say. If $\mathfrak{D} \neq \langle 1 \rangle$, then $\mathfrak{D} \cap Z(\mathfrak{Q}^*) \neq \langle 1 \rangle$. But then the previous lemma is violated in $C(\mathfrak{D} \cap Z(\mathfrak{Q}^*))$. Hence, $\mathfrak{D} = \langle 1 \rangle$. Since $C(\mathfrak{A}_1) \cap \mathfrak{Q}_1\mathfrak{A}^* \supseteq \mathfrak{A}^*$, we have $C(\mathfrak{B}) \cap \mathfrak{Q}_1\mathfrak{A}^* = \mathfrak{A}^*$.

Since $\mathfrak{B}$ normalizes $\mathfrak{Q}_1 \times \mathfrak{A}^*$, $\mathfrak{B}$ also normalizes $(\mathfrak{Q}_1 \times \mathfrak{A}^*)' = \mathfrak{Q}_1'$. Since $\mathfrak{B}$ has no fixed points on $\mathfrak{Q}_1^\#$ by the above argument, $\mathfrak{Q}_1$ is abelian. But now $\mathfrak{Q}_1\mathfrak{A}^*$ and $\mathfrak{B}$ are normal abelian subgroups of $\langle \mathfrak{Q}_1, \mathfrak{B} \rangle$, so $\langle \mathfrak{Q}_1, \mathfrak{B} \rangle$ is of class two, so is regular. It follows that if $B \in \mathfrak{B}$, $Q \in \mathfrak{Q}_1$, then $[B^q, Q] = [B, Q^q] = [B, Q]^q$. But $\mathfrak{B}$ is an arbitrary subgroup of $\mathfrak{A}_1$ which contains $\mathfrak{A}^*$ properly, so we can choose $\mathfrak{B}$ such that $\mho^1(\mathfrak{B}) \subseteq \mathfrak{A}^*$. For such a $\mathfrak{B}$, the element B centralizes $\mho^1(\mathfrak{Q}_1)$. It now follows that $\mathfrak{Q}_1$ is elementary.

We take a different approach for an instant. $\mathfrak{P}$ does not centralize the elementary abelian group $\mathfrak{Q}_1$, and $N(\mathfrak{Q}_1)$ has no normal subgroup of index p, by Lemma 17.3. It follows that $\mathfrak{Q}_1$ is not of order q.

Returning to the groups $\mathfrak{A}^*$ and $\mathfrak{B}$, since $\mathfrak{B}$ has no fixed points on $\mathfrak{Q}_1$, if $B \in \mathfrak{B}$, $B \notin \mathfrak{A}^*$, then the mapping $\phi_B : \mathfrak{Q}_1 \to \mathfrak{A}^*$ defined by $\phi_B(Q) = [B, Q]$, Q in $\mathfrak{Q}_1$, is an isomorphism of $\mathfrak{Q}_1$ onto a subgroup of $\mathfrak{A}^*$. Hence, $\mathfrak{A}^*$ is not cyclic.

From the definition of $\mathfrak{A}^*$, we see that $\mathfrak{A}^*$ contains $Z(\mathfrak{Q})$. We wish to show that $\mathfrak{A}^*$ contains an element of $\mathscr{U}(\mathfrak{Q})$. This is immediate if $Z(\mathfrak{Q})$ is non cyclic, so suppose $Z(\mathfrak{Q})$ is cyclic. If $\mathfrak{A}^*$ does not contain any element of $\mathscr{U}(\mathfrak{Q})$, then the element B above can be taken to lie in some element of $\mathscr{U}(\mathfrak{Q})$. However, $[Q, B] \in \Omega_1(Z(\mathfrak{Q}))$, so ϕ_B could not map $\mathfrak{Q}_1$ onto a subgroup of order exceeding q. We conclude that $\mathfrak{A}^*$ contains $Z(\mathfrak{Q})$ and also some element of $\mathscr{U}(\mathfrak{Q})$.

We will now show that for each element Z of $Z(\mathfrak{Q})^\#$, we can find a p-subgroup $\mathfrak{H}(Z)$ in $\mathsf{N}(\mathfrak{A}; p)$ which is not centralized by Z. Namely, $\mathfrak{A}^*$ is faithfully represented on $\mathfrak{P}_1$, since $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1 \rangle$ and $\mathfrak{A}^*$ is a normal abelian subgroup of $\mathfrak{Q}^*$. We first consider the case in which $Z(\mathfrak{Q})$ is non cyclic. Let $\mathfrak{E}$ be a subgroup of $Z(\mathfrak{Q})$ of type (q, q) which has non trivial intersection with $\langle Z \rangle$, that is let $\mathfrak{E}$ contain $\mathfrak{Z}_1 = \Omega_1(\langle Z \rangle)$. Since $\mathfrak{Z}_1$ acts non trivially on $\mathfrak{P}_1$, $\mathfrak{Z}_1$ acts non trivially on $C_{\mathfrak{P}_1}(E)$ for suitable E in $\mathfrak{E}^\#$. Let $\mathfrak{C} = C(E)$, and let $\mathfrak{R}$ be a S_p-subgroup of $\mathfrak{C}$ permutable with $\mathfrak{Q}$. It is easy to see that $\mathfrak{Z}_1$ does not centralize $O_p(\mathfrak{Q}\mathfrak{R}) \in \mathsf{N}(\mathfrak{A}; p)$.

If $Z(\mathfrak{Q})$ is cyclic, we use the fact that $\mathfrak{A}^*$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{Q})$. We can find an element U in $\mathfrak{U}^\#$ such that $\mathfrak{Z}_1 = \Omega_1(Z(\mathfrak{Q}))$ does not centralize $C_{\mathfrak{P}_1}(U)$. Let $\mathfrak{C} = C(U)$. By (B), it follows that $\mathfrak{U} \subseteq O_{q',q}(\mathfrak{C})$, and so $[\mathfrak{Z}_1, C_{\mathfrak{P}_1}(U)] \subseteq O_{q'}(\mathfrak{C})$. Thus, $\mathfrak{C}$ contains an element of $\mathsf{N}(\mathfrak{A}; p)$ which $\mathfrak{Z}_1$ does not centralize.

It now follows from Theorem 17.1 and the preceding argument that if $\tilde{\mathfrak{P}}$ is a maximal element of $\mathsf{N}(\mathfrak{Q}; p)$, then $Z(\mathfrak{Q})$ is faithfully represented on $\tilde{\mathfrak{P}}$. If $\hat{\mathfrak{P}}$ is a S_p-subgroup of $N(\tilde{\mathfrak{P}})$ permutable with $\mathfrak{Q}$, then Lemma 20.2 is violated with p and q interchanged. This completes the proof of Theorem 20.1.

21. A C^*-theorem for π_3 and a C-theorem for π_3

It is convenient to introduce another proposition which is "between" C_π and D_π.

C_π^*: $\mathfrak{X}$ satisfies C_π, and if $\mathfrak{T}$ is a π-subgroup of $\mathfrak{X}$ with the property that $|\mathfrak{T}|_p = |\mathfrak{X}|_p$ for at least one prime p in π, then $\mathfrak{T}$ is contained in a S_π-subgroup of $\mathfrak{X}$.

THEOREM 21.1 *If $p, q \in \pi_3$ and $p \sim q$, then $\mathfrak{G}$ satisfies $C_{p,q}^*$.*

Proof. We can suppose $p \neq q$. We first show that $\mathfrak{G}$ satisfies $C_{p,q}$. By Theorem 20.1, $\mathfrak{G}$ satisfies $E_{p,q}$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{Q}$, where $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. We assume notation is chosen so that $|\mathfrak{P}| > |\mathfrak{Q}|$. Then $O_p(\mathfrak{H}) \neq \langle 1 \rangle$ by Lemma 5.2. Lemma 7.3 implies that $O_p(\mathfrak{H})$ is a maximal element of $\mathsf{V}(\mathfrak{Q}; p)$. If $\mathfrak{H}_1$ is another $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, then $O_p(\mathfrak{H}_1)$ is also a maximal element of $\mathsf{V}(\mathfrak{Q}; p)$. From Section 17 we conclude that $O_p(\mathfrak{H}_1) = G^{-1}O_p(\mathfrak{H})G$ for suitable G in $\mathfrak{G}$. Hence, $G\mathfrak{H}_1G^{-1}$ and $\mathfrak{H}$ both normalize $O_p(\mathfrak{H})$ so are conjugate in $N(O_p(\mathfrak{H}))$.

Turning to $C_{p,q}^*$, we drop the hypothesis $|\mathfrak{P}| > |\mathfrak{Q}|$, and let $\mathfrak{T}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$.

First, assume that $O_q(\mathfrak{T}) \neq 1$. In this case, $O_q(\mathfrak{T})$ is a maximal element of $\mathsf{V}(\mathfrak{P}; q)$. If $O_q(\mathfrak{H}) \neq 1$, then $O_q(\mathfrak{H})$ is also a maximal element of $\mathsf{V}(\mathfrak{P}; q)$. Thus, Theorem 17.1 yields that $\mathfrak{H}$ is conjugate to $\mathfrak{T}$. (Here, as elsewhere, we are using the fact that every maximal element of $\mathsf{V}(\mathfrak{P}; q)$ is also a maximal element of $\mathsf{V}(\mathfrak{A}; q)$ for all $\mathfrak{A}$ in $\mathscr{SCN}_3(\mathfrak{P})$.) Thus, suppose $O_q(\mathfrak{H}) = 1$. In this case, if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then $\mathfrak{B} \lhd \mathfrak{H}$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, by Lemma 17.5, so $|\mathfrak{G}|_q = |N(\mathfrak{B}) : C(\mathfrak{B})|_q$. But $N(O_q(\mathfrak{T}))$ dominates $\mathfrak{B}$, so $|N(O_q(\mathfrak{T}))|_q > |\mathfrak{G}|_q$, which is absurd.

We can now suppose that $O_q(\mathfrak{T}) = 1$. We apply Lemma 17.5 and conclude that $\mathfrak{B} \lhd \mathfrak{T}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. Let $\mathfrak{Q}_0$ be a S_q-subgroup of $\mathfrak{T}$. Since $\mathfrak{T}$ is a maximal p, q-subgroup of $\mathfrak{G}$, $\mathfrak{Q}_0$ is a S_q-subgroup of $N(\mathfrak{B})$.

Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{H}$. Let $\mathfrak{Q}_1 = O_q(\mathfrak{H})$. If $\mathfrak{Q}_1 = \langle 1 \rangle$, then $\mathfrak{H} \subseteq N(\mathfrak{B})$, by Lemma 17.5, and we are done. Otherwise, $\mathfrak{H} = \mathfrak{Q}_1 N_{\mathfrak{H}}(\mathfrak{B})$, again by Lemma 17.5, and we assume without loss of generality that $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{T}$.

Assume that $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$. Then in particular, $\mathfrak{T} \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$, contrary to $O_q(\mathfrak{T}) = \langle 1 \rangle$. Hence, $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 = \langle 1 \rangle$.

We will now show directly that $N_{\mathfrak{H}}(\mathfrak{B}) = \mathfrak{T}$. Since $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{T}$, it suffices to show that $|N_{\mathfrak{H}}(\mathfrak{B})|_q \geqq |\mathfrak{T}|_q$. Now $N(\mathfrak{Q}_1) = O_{p'}(N(\mathfrak{Q}_1)) \cdot (N(\mathfrak{Q}_1) \cap N(\mathfrak{B}))$, by Lemma 17.1, and since $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 = \langle 1 \rangle$, it follows easily that $|N_{\mathfrak{H}}(\mathfrak{B})|_q = |N(\mathfrak{Q}_1) \cap N(\mathfrak{B})|_q$.

Let $\mathfrak{N}_1 = N(Z(\mathfrak{B}))$. By Lemma 17.3 we have $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\cdot(N(\mathfrak{Q}_1)\cap\mathfrak{N}_1)$. Let $\mathfrak{M} = N(\mathfrak{Q}_1)\cap\mathfrak{N}_1$. Since $\mathfrak{M}$ contains $\mathfrak{P}$, $O_{p'}(\mathfrak{M}) = O_{p'}(\mathfrak{N}_1)\cap\mathfrak{M}$. By Lemma 17.5, we now have $\mathfrak{M} = (O_{p'}(\mathfrak{N}_1)\cap\mathfrak{M})\cdot(N(\mathfrak{B})\cap\mathfrak{M})$, which yields $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\cdot(N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))$. Now $\mathfrak{N}_1$ contains $\mathfrak{T}$ and $\mathfrak{T}\cap O_{p'}(\mathfrak{N}_1) = \langle 1\rangle$, since $O_q(\mathfrak{T}) = \langle 1\rangle$. Thus, $\mathfrak{Q}_0$ is mapped isomorphically into $\mathfrak{N}_1/O_{p'}(\mathfrak{N}_1) \cong (N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))/(O_{p'}(\mathfrak{N}_1)\cap N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))$, and it follows that $|N(\mathfrak{Q}_1)\cap N(\mathfrak{B})|_q \geqq |\mathfrak{Q}_0| = |\mathfrak{T}|_q$, as required.

Since $N_{\mathfrak{H}}(\mathfrak{B}) = \mathfrak{T}$, it follows that $\mathfrak{T}\subsetneqq\mathfrak{H}$, proving the theorem.

THEOREM 21.2. *Let σ be a subset of π_3. Assume that $\mathfrak{G}$ satisfies $E_{p,q}$ for all p, q in σ. Then $\mathfrak{G}$ satisfies C_σ.*

Proof. By the preceding theorem, we can assume that σ contains at least three elements. By induction on $|\sigma|$, we assume that $\mathfrak{G}$ satisfies C_τ for every proper subset τ of σ.

Let $\sigma = \{p_1, \cdots, p_n\}$, $n \geqq 3$, and let $\sigma_i = \sigma - p_i$, $\sigma_{ij} = \sigma - p_i - p_j$, $1 \leqq i,\ j \leqq n$, $i \neq j$. Let $\mathfrak{S}_i$ be a S_{σ_i}-subgroup of $\mathfrak{G}$, $1 \leqq i \leqq n$. Then the $S_{\sigma_{ij}}$-subgroups of $\mathfrak{S}_i$ are conjugate to the $S_{\sigma_{ij}}$-subgroups of $\mathfrak{S}_j$.

For $i \neq j$, let $m_{ij} = |O_{p_i}(\mathfrak{S}_j)|$. Note that by C_{σ_j}, m_{ij} depends only on i and j and not on the particular S_{σ_j}-subgroup of $\mathfrak{G}$ we choose.

Fix i, j, k, $i \neq j \neq k \neq i$, let $\mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{G}$, let $\mathfrak{S}_j^*$ be a S_{σ_j}-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$ and $\mathfrak{S}_k^*$ be a S_{σ_k}-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$, chosen so that $\mathfrak{S}_j^*\cap\mathfrak{S}_k^*$ is a $S_{\sigma_{j,k}}$-subgroup of $\mathfrak{G}$ which is possible by $C_{\sigma_{j,k}}$, C_{σ_j} and C_{σ_k}.

Let $\mathfrak{P}_{ij} = O_{p_i}(\mathfrak{S}_j^*)$, $\mathfrak{P}_{ik} = O_{p_i}(\mathfrak{S}_k^*)$. Suppose that $\mathfrak{P}_{ij}\cap\mathfrak{P}_{ik} = \langle 1\rangle$. With this assumption, we will show that $m_{ij} \leqq m_{jk}$. We can assume that $i = 1$, $j = 2$, $k = 3$, that $\mathfrak{P}_{12}\cap\mathfrak{P}_{13} = \langle 1\rangle$, and try to show that $m_{12} \leqq m_{23}$.

Let $\mathfrak{P}_1, \mathfrak{R}_4, \mathfrak{R}_5, \cdots, \mathfrak{R}_n$ be a Sylow system for $\mathfrak{S}_2^*\cap\mathfrak{S}_3^*$, and let $\mathfrak{P}_1, \mathfrak{R}_3, \cdots, \mathfrak{R}_n$ and $\mathfrak{P}_1, \mathfrak{R}_2, \mathfrak{R}_4, \cdots, \mathfrak{R}_n$ be Sylow systems for $\mathfrak{S}_2^*$ and $\mathfrak{S}_3^*$ respectively. Here $\mathfrak{R}_i$ is a S_{p_i}-subgroup of $\mathfrak{G}$, $i = 2, \cdots, n$.

Since $\mathfrak{P}_{13}$ is the S_{p_1}-subgroup of $F(\mathfrak{S}_3^*)$, the condition $\mathfrak{P}_{12}\cap\mathfrak{P}_{13} = \langle 1\rangle$ says that $\mathfrak{P}_{12}$ is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)$. Now

$$F(\mathfrak{S}_3^*) = F(\mathfrak{S}_3^*)\cap\mathfrak{P}_1 \times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_2 \times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_4 \times \cdots \times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_n\,,$$

where $\mathfrak{P}_{13} = F(\mathfrak{S}_3^*)\cap\mathfrak{P}_1$. Since $\mathfrak{P}_{12}$ and $\mathfrak{P}_{13}$ are disjoint normal subgroups of $\mathfrak{P}_1$, $\mathfrak{P}_{12}$ centralizes $\mathfrak{P}_{13}$. If $4 \leqq s \leqq n$, then $\langle\mathfrak{P}_{12}, F(\mathfrak{S}_3^*)\cap\mathfrak{R}_s\rangle = \mathfrak{H}_s$ is clearly contained in $\mathfrak{S}_2^*\cap\mathfrak{S}_3^*$ and so $\mathfrak{P}_{12}$ and $F(\mathfrak{S}_3^*)\cap\mathfrak{R}_s$ are disjoint normal subgroups of $\mathfrak{H}_s$, and so commute elementwise. But $\mathfrak{P}_{12}$ is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)$, so is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)\cap\mathfrak{R}_2$. It follows from Lemma 5.2 that $m_{12} \leqq m_{23}$.

Returning to the general situation, if $O_{p_i}(\mathfrak{S}_j)\cap O_{p_i}(\mathfrak{S}_k) = \langle 1\rangle$, whenever $i \neq j \neq k \neq i$, and $\mathfrak{S}_j\cap\mathfrak{S}_k$ is a $S_{\sigma_{j,k}}$-subgroup of $\mathfrak{G}$, then

$m_{ij} \leqq m_{jk}$. Permuting i, j, k cyclically, we would have $m_{ij} \leqq m_{jk} \leqq m_{ki} \leqq m_{ij}$. The integers m_{ij}, m_{jk}, m_{ki} being pairwise relatively prime, we would find $m_{ij} = 1$ for all $i \neq j$. This is not possible since a S_{σ_i}-subgroup of $\mathfrak{G}$ is solvable.

Returning to the groups $\mathfrak{S}_2^*$ and $\mathfrak{S}_3^*$, we suppose without loss of generality that $\mathfrak{P}_{12} \cap \mathfrak{P}_{13} = \mathfrak{D}_{123} \neq \langle 1 \rangle$. Since $\mathfrak{D}_{123} \subseteq \mathfrak{P}_{12} \lhd \mathfrak{S}_2^*$, $\mathfrak{D}_{123}$ commutes elementwise with $O_{p_1'}(\mathfrak{S}_2^*)$. Similarly, $\mathfrak{D}_{123}$ commutes elementwise with $O_{p_1'}(\mathfrak{S}_3^*)$. Hence $\langle \mathfrak{P}_1, O_{p_1'}(\mathfrak{S}_2^*), O_{p_1'}(\mathfrak{S}_3^*) \rangle = \mathfrak{L}$ is a proper subgroup of $\mathfrak{G}$ normalizing $\mathfrak{D}_{123}$. By Lemma 7.5, both $O_{p_1'}(\mathfrak{S}_2^*)$ and $O_{p_1'}(\mathfrak{S}_3^*)$ are S-subgroups of $O_{p_1'}(\mathfrak{L})$; in particular, $\mathfrak{L}$ has a normal p_1-complement. Since $\mathfrak{L}$ has a normal p_1-complement, we can find an element C in $C_{\mathfrak{L}}(\mathfrak{P})$ such that $O_{p_1'}(\mathfrak{S}_2^*)$ is permutable with $C^{-1}O_{p_1'}(\mathfrak{S}_3^*)C$. For such an element C, let $\mathfrak{M} = \langle O_{p_1'}(\mathfrak{S}_2^*), C^{-1}O_{p_1'}(\mathfrak{S}_3^*)C \rangle$.

We will now show directly that for each q in σ, $N(\mathfrak{M})$ contains a S_q-subgroup of $\mathfrak{G}$. This is trivially true if $\mathfrak{M} = \langle 1 \rangle$, so suppose that $\mathfrak{M} \neq \langle 1 \rangle$. Let $\mathfrak{M}_2, \cdots, \mathfrak{M}_n$ be a Sylow system for $\mathfrak{M}$ which is normalized by $\mathfrak{P}_1$, where $\mathfrak{M}_i$ is an S_{p_i}-subgroup of $\mathfrak{M}$, $i = 2, \cdots, n$. We remark that by $C^*_{p_1, p_i}$, each $\mathfrak{M}_i$ is a maximal element of $\mathsf{M}(\mathfrak{P}_1; p_i)$.

Let $|\mathfrak{M}_i| = p_i^{e_i}$ and let $|\mathfrak{G}|_{p_i} = p_i^{f_i}$. By Lemma 17.5 and $C^*_{p_1, p_i}$ we see that $p_i^{f_i - e_i} = |N(\mathfrak{B}) : C(\mathfrak{B})|_{p_i}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, $\mathfrak{P} = \mathfrak{P}_1$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.

Let $\mathfrak{N}_1 = N(Z(\mathfrak{B}))$. Let $\mathfrak{C}$ be a coset of $O_{p'}(\mathfrak{N}_1)$ in $\mathfrak{N}_1$. Then $\mathfrak{C}$ contains an element N of $N(\mathfrak{B})$ by Lemma 17.5. Hence, $\mathfrak{M}_i^{N^{-1}} = \mathfrak{M}_i^{C_i}$, $i = 2, \cdots, n$ where $C_2, \cdots, C_n$ all lie in $C(\mathfrak{A})$. Let $\mathfrak{K} = \langle \mathfrak{P}, \mathfrak{M}_2, \cdots, \mathfrak{M}_n, C_2, \cdots, C_n \rangle$. Since $\mathfrak{D}_{123} \cap Z(\mathfrak{P}_1) \neq 1$, and since $\mathfrak{K}$ centralizes $\mathfrak{D}_{123} \cap Z(\mathfrak{P}_1)$, we have $\mathfrak{K} \subset \mathfrak{G}$. Let $\mathfrak{L} = O_{p'}(\mathfrak{K})$ $(p = p_1)$ so that $\mathfrak{L}\mathfrak{P} = \mathfrak{K}$ by Lemmas 7.3 and 7.4. Hence, $\mathfrak{L}$ contains both $\mathfrak{M}$ and $\mathfrak{M}^{N^{-1}}$, and since $\mathfrak{B}$ normalizes $\mathfrak{M}$, $\mathfrak{A}$ normalizes both $\mathfrak{M}$ and $\mathfrak{M}^{N^{-1}}$. By C^*_{p, p_i}, $i = 2, \cdots, n$, $\mathfrak{M}$ is a S-subgroup of $\mathfrak{L}$. By the conjugacy of Sylow systems in $\mathfrak{A}\mathfrak{L}$, there is an element C in $\mathfrak{L}\mathfrak{A}$ such that $\mathfrak{A}^C = \mathfrak{A}$, $\mathfrak{M}^C = \mathfrak{M}^{N^{-1}}$. Since $\mathfrak{L}\mathfrak{A}$ has a normal p-complement, $C \in C(\mathfrak{A}) \subseteq O_{p'}(\mathfrak{N}_1)$, so $\mathfrak{C}$ contains $CN \in N(\mathfrak{M})$.

Thus, if $\mathfrak{T} = \mathfrak{N}_1 \cap N(\mathfrak{M})$, we have $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\mathfrak{T}$. Since $\mathfrak{P} \subseteq \mathfrak{T}$, we have $O_{p'}(\mathfrak{T}) = \mathfrak{T} \cap O_{p'}(\mathfrak{N}_1)$. Hence $\mathfrak{T} = O_{p'}(\mathfrak{T})N_{\mathfrak{T}}(\mathfrak{B})$ by Lemma 17.5, so that $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)N_{\mathfrak{T}}(\mathfrak{B})$. Thus $N_{\mathfrak{T}}(\mathfrak{B})$ maps onto $N(\mathfrak{B})/C(\mathfrak{B})$. Since $N_{\mathfrak{T}}(\mathfrak{B}) \cap \mathfrak{M}$ centralizes $\mathfrak{B}$, it follows that $|\mathfrak{T} : \mathfrak{T} \cap \mathfrak{M}|_{p_i} = p_i^{f_i - e_i}$, $i = 2, \cdots, n$. Hence $|\mathfrak{T}\mathfrak{M}|_{p_i} = |\mathfrak{G}|_{p_i}$, as required.

If now $\mathfrak{M} \neq \langle 1 \rangle$, then $N(\mathfrak{M}) \subset \mathfrak{G}$ and so $\mathfrak{G}$ satisfies E_σ.

We now treat the possibility that $\mathfrak{M} = \langle 1 \rangle$. In this case, both $F(\mathfrak{P}_2^*)$ and $F(\mathfrak{P}_3^*)$ are p_1-groups. By (B), both groups contain $\mathfrak{A}$. By Lemma 17.4, both $\mathfrak{P}_2^*$ and $\mathfrak{P}_3^*$ are contained in $N(Z(\mathfrak{B}))$, so once again $\mathfrak{G}$ satisfies E_σ.

It remains to prove C_σ, given E_σ and C_τ for every proper subset

τ of σ.

Let $\mathfrak{H}$ and $\mathfrak{H}_1$ be two S_σ-subgroups of $\mathfrak{G}$ with Sylow systems $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ and $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$ respectively, $\mathfrak{P}_i$ and $\mathfrak{Q}_i$ being S_{p_i}-subgroups of $\mathfrak{G}$, $1 \leqq i \leqq n$.

If $F(\mathfrak{H})$ and $F(\mathfrak{H}_1)$ are p_1-groups, we apply Lemma 17.4 and conclude that $\mathfrak{H}$ and $\mathfrak{H}_1$ are conjugate in $N(Z(\mathfrak{B}))$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_1)$, $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P}_1)$ and we have normalized by taking $\mathfrak{P}_1 = \mathfrak{Q}_1$.

If $F(\mathfrak{H})$ is a p_1-group, then C_{p_1, p_i} for $i = 2, \cdots, n$ imply that $F(\mathfrak{H}_1)$ is a p_1-group. Thus, we can assume that neither $F(\mathfrak{H})$ nor $F(\mathfrak{H}_1)$ is a p-group for any prime p.

Let $m_i = |O_{p_i}(\mathfrak{H})|$, $m'_i = |O_{p_i}(\mathfrak{H}_1)|$, $1 \leqq i \leqq n$. For each i, we can choose G_i in $\mathfrak{G}$ so that $\mathfrak{Q}_j^{G_i} = \mathfrak{P}_j$, $1 \leqq j \leqq n$, $i \neq j$. Let $\mathfrak{K}_i = \mathfrak{H}_1^{G_i}$, $i = 1, \cdots, n$, so that $\mathfrak{H} \cap \mathfrak{K}_i$ contains a S_{σ_i}-subgroup of $\mathfrak{G}$.

Suppose $O_{p_j}(\mathfrak{K}_i) \cap O_{p_j}(\mathfrak{H}) = \langle 1 \rangle$ for some i, j, $i \neq j$. Then $O_{p_j}(\mathfrak{K}_i)$ is faithfully represented on $F(\mathfrak{H})$, since $O_{p_j}(\mathfrak{K}_i) \subseteqq \mathfrak{H}$. But in this case, $O_{p_j}(\mathfrak{K}_i)$ centralizes $O_{p_j}(\mathfrak{H})$ and also centralizes $O_{p_k}(\mathfrak{H})$ for $k \neq i$. Hence, $O_{p_j}(\mathfrak{K}_i)$ is faithfully represented on $O_{p_i}(\mathfrak{H})$, and so $m'_j \leqq m_i$ by Lemma 5.2. For the same reasons, $m_j \leqq m'_i$, since $O_{p_j}(\mathfrak{H})$ is faithfully represented on $F(\mathfrak{K}_i)$. If for all i, j, $1 \leqq i, j \leqq n$, $i \neq j$, $O_{p_j}(\mathfrak{K}_i) \cap O_{p_j}(\mathfrak{H}) = \langle 1 \rangle$, we find $m'_j \leqq m_i \leqq m'_j$, and so $m'_j = m_i = 1$. This is not possible since $\mathfrak{H}$ and $\mathfrak{H}_1$ are solvable.

Hence, we assume without loss of generality that $\mathfrak{D}_{12} = O_{p_1}(\mathfrak{K}_2) \cap O_{p_1}(\mathfrak{H}) \neq \langle 1 \rangle$. We will now show that $O_{p'_1}(\mathfrak{K}_2)$ is conjugate to $O_{p'_1}(\mathfrak{H})$. To see this, we first apply Lemma 7.4 and C_{p_1, p_i} to conclude that $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$ have the same order. Since $\mathfrak{D}_{12}$ centralizes both $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$, it follows that $\mathfrak{L} = \langle \mathfrak{P}_1, O_{p'_1}(\mathfrak{K}_2), O_{p'_1}(\mathfrak{H}) \rangle \subset \mathfrak{G}$. By Lemma 7.4, it follows that $\langle O_{p'_1}(\mathfrak{K}_2), O_{p'_1}(\mathfrak{H}) \rangle \subseteqq O_{p'_1}(\mathfrak{L})$. By Theorem 17.1 and $C^*_{p_1, p_i}$, $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$ are S-subgroups of $O_{p'_1}(\mathfrak{L})$, so are conjugate in $\mathfrak{L}$, being of the same order. Since $O_{p'_1}(\mathfrak{H}) \neq \langle 1 \rangle$, C_σ follows immediately.

22. Linking Theorems

One of the purposes of this section is to clarify the relationship between π_3 and π_4.

Hypothesis 22.1.

(i) $p \in \pi_3$, $q \in \pi(\mathfrak{G})$.

(ii) *A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ does not centralize every element of* $\mathscr{U}(\mathfrak{P}; q)$.

THEOREM 22.1. *Under Hypothesis* 22.1, *if $\mathfrak{Q}_1$ is a maximal element of $\mathscr{U}(\mathfrak{P}; q)$ and Q is an element of $\mathfrak{Q}_1$ of order q, then $C_{\mathfrak{Q}_1}(Q)$ contains an elementary subgroup of order q^3. In particular, $q \in \pi_3 \cup \pi_4$.*

Proof. Choose $\mathfrak{C}$ char $\mathfrak{Q}_1$ in accordance with Lemma 8.2, and set $\mathfrak{C}_1 = \Omega_1(\mathfrak{C})$. From 3.6 and Lemma 8.2, it follows that $\mathfrak{P}$ does not centralize $\mathfrak{C}_1$. Since $cl(\mathfrak{C}) \leqq 2$, $\mathfrak{C}_1$ is of exponent q.

Since $N(\mathfrak{C}_1) \supseteqq N(\mathfrak{Q}_1)$, Lemma 17.3 implies that $O^p(N(\mathfrak{C}_1)) = N(\mathfrak{C}_1)$. Since $N(\mathfrak{C}_1)$ has odd order, this in turn implies that $\mathfrak{C}_1$ is not generated by two elements. Consider the chain $\mathscr{C}: \mathfrak{C}_1 \supseteqq \gamma\mathfrak{C}_1\mathfrak{Q}_1 \supseteqq \gamma^2\mathfrak{C}_1\mathfrak{Q}_1^2 \supseteqq \cdots$. Since $\mathfrak{P}$ does not centralize $\mathfrak{C}_1$, $\mathfrak{P}$ does not stabilize $\mathscr{C}$, so we can find an integer n and subgroups $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that $\gamma^{n+1}\mathfrak{C}_1\mathfrak{Q}_1^{n+1} \subseteqq \mathfrak{A}_1 \subseteqq \mathfrak{A}_2 \subseteqq \gamma^n\mathfrak{C}_1\mathfrak{Q}_1^n$ and such that $\mathfrak{B} = \mathfrak{A}_2/\mathfrak{A}_1$ is a chief factor of $N(\mathfrak{Q}_1)$ and with the additional property that $\mathfrak{P}$ does not centralize $\mathfrak{B}$. Since $N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$, we also have $\bar{\mathfrak{N}} = O^p(\bar{\mathfrak{N}})$, where $\bar{\mathfrak{N}} = (N(\mathfrak{B}) \cap N(\mathfrak{Q}_1))/(C(\mathfrak{B}) \cap N(\mathfrak{Q}_1))$. Since $|N(\mathfrak{Q}_1)|$ is odd it follows that $|\mathfrak{B}| \geqq q^3$. Since $\gamma\mathfrak{A}_2\mathfrak{Q}_1 \subseteqq \mathfrak{A}_1$, it follows that $|C_{\mathfrak{A}_2}(Q)| \geqq q^3$. If $C_{\mathfrak{Q}_1}(Q)$ did not contain an elementary subgroup of order q^3, then we would necessarily have $Q \in \mathfrak{A}_2$ since $\mathfrak{A}_2$ is of exponent q. Since $|C_{\mathfrak{A}_2}(Q)| \geqq q^3$, the only possibility is that $C_{\mathfrak{A}_2}(Q)$ is the non abelian group of order q^3 and exponent q. But in this case $Q \in C_{\mathfrak{A}_2}(Q)' \subseteqq Z(\mathfrak{C}_1)$, and $C_{\mathfrak{Q}_1}(Q)$ contains an elementary subgroup of order q^3 since $\mathfrak{Q}_1$ does, by Lemma 8.13, Lemma 8.1, and the equation $N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$.

Hypothesis 22.2.

(i) *$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $p \in \pi_3$.*

(ii) *q, $r \in \pi_3 \cup \pi_4$; $\mathfrak{P}$ does not centralize every element of $\mathsf{M}(\mathfrak{P}; q)$ and $\mathfrak{P}$ does not centralize every element of $\mathsf{M}(\mathfrak{P}; r)$.*

THEOREM 22.2. *Under Hypothesis* 22.2, $q \sim r$.

The proof of this theorem is by contradiction. The following lemmas assume that $q \not\sim r$.

Since Hypothesis 22.2 is symmetric in q and r we can assume that $q > r$, thereby destroying the symmetry.

Let $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. Let $\mathfrak{Q}_1$, $\mathfrak{R}_1$ be maximal elements in $\mathsf{M}(\mathfrak{P}; q)$, $\mathsf{M}(\mathfrak{P}; r)$ respectively.

LEMMA 22.1. *If $\mathfrak{H}$ is an $\mathfrak{A}$-invariant q, r-subgroup of $\mathfrak{G}$, and if a S_q-subgroup $\mathfrak{H}_q$ of $\mathfrak{H}$ is non cyclic, then $\mathfrak{H}_q \lhd \mathfrak{H}$.*

Proof. Let $\mathfrak{H}_r$ be a S_r-subgroup of $\mathfrak{H}$ normalized by $\mathfrak{A}$. Since $q \not\sim r$, either $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ or $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_q)$ is empty. If $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ is empty, application of Lemma 8.5 to $\mathfrak{H}$ yields this lemma.

Suppose $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ is non empty. Then $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_q)$ is empty, so $\mathfrak{H}$ has q-length one. Thus, it suffices to show that $\mathfrak{H}_q$ centralizes $O_r(\mathfrak{H})$. We suppose without loss of generality that $\mathfrak{A}$ normalizes $\mathfrak{H}_q$.

Then by Corollary 17.2 $\mathfrak{H}_q$ is contained in a conjugate of $\mathfrak{Q}_1$, so $C(H)$ possesses an elementary subgroup of order q^3 for H in $\mathfrak{H}_q$, H of order q, by Theorem 22.1. We will show that $\Omega_1(\mathfrak{H}_q)$ centralizes $O_r(\mathfrak{H})$. Since $\mathfrak{H}_q$ is assumed non cyclic, $\Omega_1(\mathfrak{H}_q)$ is generated by its subgroups $\mathfrak{E}$ which are elementary of order q^2, so it suffices to show that each such $\mathfrak{E}$ centralizes $O_r(\mathfrak{H})$. If $\mathfrak{E}$ does not centralize $O_r(\mathfrak{H})$, then $\mathfrak{E}$ does not centralize $O_r(\mathfrak{H}) \cap C(E)$ for suitable E in $\mathfrak{E}^\#$. By Lemma 8.4, $\mathscr{SCN}_3(O_r(\mathfrak{H}) \cap C(E))$ is non empty for such an E, so $q \not\sim r$ is violated in $C(E)$.

Since $\Omega_1(\mathfrak{H}_q)$ centralizes $O_r(\mathfrak{H})$, it follows that $\mathscr{SCN}_3(O_r(\mathfrak{H}))$ is empty, since $q \not\sim r$. Hence, $\mathfrak{H}_q$ centralizes $O_r(\mathfrak{H})$ by Lemma 8.4, as required.

We define $\mathscr{X}$ as the set of q, r-subgroups of $\mathsf{N}(\mathfrak{A})$ which have the additional property that no S_q- or S_r-subgroup is centralized by $\mathfrak{A}$.

LEMMA 22.2. *$\mathscr{X}$ is non empty.*

Proof. Suppose that $\gamma\mathfrak{Q}_1\mathfrak{A} = \langle 1 \rangle$. If we also had $\gamma\mathfrak{R}_1\mathfrak{A} = \langle 1 \rangle$, then $q \not\sim r$ would be violated in $C(\mathfrak{A})$. Hence, $\gamma\mathfrak{R}_1\mathfrak{A} \neq \langle 1 \rangle$, and we can find $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$, $\mathfrak{R}_2 \neq \langle 1 \rangle$, such that $\mathfrak{R}_2 = \gamma\mathfrak{R}_2\mathfrak{A}$ and such that $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{R}_2) \neq \langle 1 \rangle$. Consider $C(\mathfrak{A}_1) \supseteq \langle \mathfrak{A}, \mathfrak{Q}_1, \mathfrak{R}_2 \rangle = \mathfrak{L}$. By Lemma 17.6, $\mathfrak{A} \supseteq O_{p', p}(\mathfrak{L})$ and it follows readily that $\mathfrak{L}$ possesses a normal complement $\mathfrak{H}_0$ to $\mathfrak{A}$. We can then find C in $C_{\mathfrak{L}}(\mathfrak{A})$ such that $\mathfrak{H} = \langle \mathfrak{Q}_1, \mathfrak{R}_2^C \rangle$ is a q, r-group. By Lemma 22.1 and the fact that $\mathfrak{Q}_1$ is a maximal element of $\mathsf{N}(\mathfrak{A}; q)$, we have $\mathfrak{Q}_1 \lhd \mathfrak{H}$. But now $\mathfrak{R}_2^C \subseteq N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$. Since $q \not\sim r$, if $\mathfrak{S}_r$ is a S_r-subgroup of $N(\mathfrak{Q}_1)$, then $\mathscr{SCN}_3(\mathfrak{S}_r)$ is empty. By Lemma 8.13 $N(\mathfrak{Q}_1)'$ centralizes every chief r-factor of $N(\mathfrak{Q}_1)$. It follows that $\mathfrak{A}$ centralizes $\mathfrak{R}_2^C$, contrary to construction, so we can assume that $\gamma\mathfrak{Q}_1\mathfrak{A} \neq \langle 1 \rangle$.

Suppose $\gamma\mathfrak{R}_1\mathfrak{A} = \langle 1 \rangle$. Since $\mathfrak{A}$ possesses an elementary subgroup of order p^3, we can find A in $\mathfrak{A}$ such that $C_{\mathfrak{Q}_1}(A)$ is non cyclic. Consider $C(A) \supseteq \langle \mathfrak{A}, C_{\mathfrak{Q}_1}(A), \mathfrak{R}_1 \rangle$. By Lemma 17.6 we can assume that $\mathfrak{H} = \langle C_{\mathfrak{Q}_1}(A), \mathfrak{R}_1 \rangle$ is a q, r-group. Then Lemma 22.1 implies that $\mathfrak{S}_q \lhd \mathfrak{H}$, $\mathfrak{S}_q$ being a S_q-subgroup of $\mathfrak{H}$. Enlarge $\mathfrak{H}$ to $\mathfrak{K}$, a maximal $\mathfrak{A}$-invariant q, r-subgroup with Sylow system $\mathfrak{K}_q, \mathfrak{R}_1$. Lemma 17.6, Lemma 22.1 and maximality of $\mathfrak{K}$ imply that $\mathfrak{K}_q$ is a maximal element of $\mathsf{N}(\mathfrak{A}; q)$, contrary to $q \not\sim r$.

We can now assume that $\gamma\mathfrak{Q}_1\mathfrak{A} \neq \langle 1 \rangle$ and $\gamma\mathfrak{R}_1\mathfrak{A} \neq \langle 1 \rangle$.

Let $\mathfrak{Q}_2$ be an $\mathfrak{A}$-invariant subgroup of $\mathfrak{Q}_1$ of minimal order subject to $\gamma\mathfrak{Q}_2\mathfrak{A} \neq \langle 1 \rangle$. Let $\mathfrak{R}_2$ be an $\mathfrak{A}$-invariant subgroup of $\mathfrak{R}_1$ of minimal order subject to $\gamma\mathfrak{R}_2\mathfrak{A} \neq \langle 1 \rangle$. Let $\mathfrak{A}_1 = \ker(\mathfrak{A} \to \operatorname{Aut} \mathfrak{Q}_2)$, $\mathfrak{A}_2 = \ker(\mathfrak{A} \to \operatorname{Aut} \mathfrak{R}_2)$. Since $\mathfrak{A}$ acts irreducibly on $\mathfrak{Q}_2/D(\mathfrak{Q}_2)$ and on $\mathfrak{R}_2/D(\mathfrak{R}_2)$, it follows that $\mathfrak{A}/\mathfrak{A}_i$ is cyclic, $i = 1, 2$. Since $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{A}_1 \cap \mathfrak{A}_2 =$

$\mathfrak{A}_3 \neq \langle 1 \rangle$. An $\mathfrak{A}$-invariant $S_{q,r}$-subgroup of $\langle \mathfrak{A}, \mathfrak{Q}_2, \mathfrak{R}_2 \rangle \subseteq C(\mathfrak{A}_3)$ satisfies the conditions defining $\mathscr{K}$, by Lemma 17.6 and $D_{p,q,r}$ in $\langle \mathfrak{A}, \mathfrak{Q}_2, \mathfrak{R}_2 \rangle$.

Let $\mathfrak{K}$ be a maximal element of $\mathscr{K}$, with Sylow system $\mathfrak{K}_q$, $\mathfrak{K}_r$, chosen so that $\mathfrak{A}$ normalizes both $\mathfrak{K}_q$ and $\mathfrak{K}_r$, $\mathfrak{K}_q$ being a S_q-subgroup of $\mathfrak{K}$.

LEMMA 22.3. *$\mathfrak{K}_q$ is cyclic and $O_q(\mathfrak{K}) = \langle 1 \rangle$.*

Proof. Suppose $\mathfrak{K}_q$ is non cyclic. Then Lemma 22.1 yields $\mathfrak{K}_q \lhd \mathfrak{K}$. The maximal nature of $\mathfrak{K}$, together with Lemma 17.6, imply that $\mathfrak{K}_q$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, so is conjugate to $\mathfrak{Q}_1$.

By Lemma 17.3, $N(\mathfrak{K}_q) = \mathfrak{N} = O^p(\mathfrak{N})$. Since $q \not\sim r$, $N(\mathfrak{K}_q)$ does not possess an elementary subgroup of order r^3. Now $\mathfrak{N} = O^p(\mathfrak{N})$ and Lemma 8.13 imply that $\gamma \mathfrak{K}_r \mathfrak{A} = \langle 1 \rangle$, contrary to construction. Hence, $\mathfrak{K}_q$ is cyclic.

If $O_q(\mathfrak{K}) \neq \langle 1 \rangle$, then $\Omega_1(O_q(\mathfrak{K})) = \Omega_1(\mathfrak{K}_q) \lhd \mathfrak{K}$. The maximal nature of $\mathfrak{K}$ now conflicts with Lemma 17.6 and Theorem 22.1 proving this lemma.

We choose C in $C(\mathfrak{A})$ so that $\mathfrak{K}_r^C \subseteq \mathfrak{R}_1$; since $\mathfrak{K}^C$ is also a maximal element of $\mathscr{K}$, we assume without loss of generality that $\mathfrak{K}_r \subseteq \mathfrak{R}_1$.

LEMMA 22.4.

(i) *$\mathfrak{K}_r$ is non abelian.*

(ii) *No non identity weakly closed subgroup of $\mathfrak{K}_r$ is contained in $O_r(\mathfrak{K})$.*

(iii) *$O_r(\mathfrak{K})$ contains an element of $\mathscr{U}(\mathfrak{R})$, $\mathfrak{R}$ being any S_r-subgroup of $\mathfrak{G}$ containing a S_r-subgroup $\mathfrak{R}^*$ of $N(\mathfrak{R}_1)$.*

Proof. We first prove (ii). Suppose $\mathfrak{T} \neq \langle 1 \rangle$, $\mathfrak{T}$ is weakly closed in $\mathfrak{K}_r$, and $\mathfrak{T} \subseteq O_r(\mathfrak{K})$. Then $\mathfrak{T} \lhd \mathfrak{K}\mathfrak{A}$, so the maximal nature of $\mathfrak{K}$ together with Lemma 17.6 imply that $\mathfrak{K}_r = \mathfrak{R}_1$.

Since $N(\mathfrak{R}_1) = O^p(N(\mathfrak{R}_1))$, so also $N(\mathfrak{T}) = O^p(N(\mathfrak{T}))$. Since $q \not\sim r$, Lemma 8.13 implies $\gamma \mathfrak{A} \mathfrak{K}_q = \langle 1 \rangle$, contrary to construction, proving (ii).

If $\mathfrak{K}_r$ were abelian, then $O_q(\mathfrak{K}) = \langle 1 \rangle$ and Lemma 1.2.3 of [21] imply that $\mathfrak{K}_r = O_r(\mathfrak{K})$, in violation of (ii). This proves (i).

Suppose $r \in \pi_3$. In this case, $C_{p,r}^*$ implies $\mathfrak{R}^* = \mathfrak{R}$, and since $\mathfrak{R}_1' \neq \langle 1 \rangle$, it is clear that $\mathfrak{R}_1$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{R})$. Since $\mathfrak{K}_r = N(O_r(\mathfrak{K})) \cap \mathfrak{R}_1$, it follows that $\mathfrak{U} \cap Z(\mathfrak{R}) \subseteq \mathfrak{K}_r$ and so by (B), $\mathfrak{U} \cap Z(\mathfrak{R}) \subseteq O_r(\mathfrak{K})$. It now follows that $\mathfrak{U} \subseteq \mathfrak{K}_r$, and so $\mathfrak{U} \subseteq O_r(\mathfrak{K})$, again by (B). Next, suppose that $r \in \pi_4$. In this case, since $\mathfrak{R}_1' \neq \langle 1 \rangle$, $\mathfrak{R}^*$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{R})$, $\mathfrak{R}^*$ being a S_r-subgroup of $N(\mathfrak{R}_1')$. Since $\mathfrak{B}$ centralizes $O_p(\mathfrak{P}\mathfrak{R}^*)$, by Lemma 19.1, we have $\mathfrak{B} \subseteq \mathfrak{R}_1$. Since $\mathfrak{B} \subseteq \mathfrak{R}_1$, $\mathfrak{B} \cap Z(\mathfrak{R}) \subseteq \mathfrak{K}_r$ and so by (B), $\mathfrak{B} \cap Z(\mathfrak{R}) \subseteq O_r(\mathfrak{K})$. It follows that $\mathfrak{B} \subseteq O_r(\mathfrak{K})$. This proves (iii).

To prove Theorem 22.2 we will now show that $\mathfrak{R}_q$ centralizes $Z(O_r(\mathfrak{R})) = \mathfrak{Z}$. Suppose by way of contradiction that this is not the case. We can choose $\mathfrak{E} \in \mathscr{U}(r)$ such that $\mathfrak{E} \subseteq \mathfrak{R}_r$ but $\mathfrak{E} \not\subseteq O_r(\mathfrak{R})$. Since $\mathfrak{R}_q$ is cyclic $\mathfrak{E}_1 = \mathfrak{E} \cap O_r(\mathfrak{R})$ is of order r. From (B), we then have $\gamma^{r-1}\mathfrak{Z}\mathfrak{E}^{r-1} \neq \langle 1 \rangle$.

If $r \geqq 5$, we apply Lemma 16.2 and conclude that $\gamma^4\mathfrak{Z}\mathfrak{E}^4 = \langle 1 \rangle$, contrary to the above statement. Hence $r = 3$, and by Lemma 16.3 we have $\gamma^2\mathfrak{Z}\mathfrak{E}^2 = \mathfrak{E}_1$; in particular, $\mathfrak{E}_1 \subseteq \mathfrak{Z}$. Now apply Lemma 16.3 again, this time with $O_3(\mathfrak{R})$ in the role of $\mathfrak{F}$, and conclude that $\gamma^2 O_3(\mathfrak{R})\mathfrak{E}^2 = \mathfrak{E}_1$.

Let $\mathfrak{T} = \gamma O_3(\mathfrak{R})\mathfrak{R}_q$. By Lemma 8.11, we have $\mathfrak{T} = \gamma\mathfrak{T}\mathfrak{R}_q$, and so $\mathfrak{E}_1 \subseteq \Omega_1(Z(\mathfrak{T}))$. Hence by (B), $\mathfrak{R}_q$ acts trivially on $\mathfrak{T}/\Omega_1(Z(\mathfrak{T}))$, and this implies that $\mathfrak{T} = \Omega_1(Z(\mathfrak{T}))$, so that $\mathfrak{T}$ is elementary.

The equality $\gamma^2\mathfrak{T}\mathfrak{E}^2 = \mathfrak{E}_1$ and (B) imply that an element of $\mathfrak{E} - \mathfrak{E}_1$ induces an automorphism of $\mathfrak{T}$ with matrix J_3. Since $|\mathfrak{R}_q|$ divides $3^3 - 1$, we have $|\mathfrak{R}_q| = 13$.

By definition of $\mathscr{K}$ we have $p/12 = |\operatorname{Aut}\mathfrak{R}_{13}|$. Since $p \neq r = 3$, we have a contradiction, completing the proof that $\mathfrak{R}_q$ centralizes $\mathfrak{Z}$ in all cases.

Now $Z(\mathfrak{N}_1)$ centralizes $O_r(\mathfrak{R})$, so by maximality of $\mathfrak{R}$, we have $Z(\mathfrak{N}_1) \subseteq \mathfrak{R}_r$ and (B) implies that $Z(\mathfrak{N}_1) \subseteq Z(O_r(\mathfrak{R}))$. Hence, $\mathfrak{R} \subseteq N(Z(\mathfrak{N}_1)) = \mathfrak{N}_1$. But $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$ and since $q \not\sim r$, $\mathfrak{N}_1$ does not possess an elementary subgroup of order q^3. Lemma 8.13 implies that $\gamma\mathfrak{R}_q\mathfrak{A} = \langle 1 \rangle$, contrary to construction, completing the proof of Theorem 22.2.

For p in $\pi_3 \cup \pi_4$, let $\mathscr{W}(p)$ be the set of all subgroups $\mathfrak{W}$ of $\mathfrak{G}$ of type (p, p) such that every element W of $\mathfrak{W}$ centralizes an element $\mathfrak{B}$ of $\mathscr{U}(p)$. We allow $\mathfrak{B}$ to depend on W.

Hypothesis 22.3.

(i) $p \in \pi_3$, $q \in \pi(\mathfrak{G})$.

(ii) $p \not\sim q$.

THEOREM 22.3. *Under Hypothesis* 22.3, *if* $\mathfrak{R}$ *is a* p, q*-subgroup of* $\mathfrak{G}$ *and if* $\mathfrak{R}$ *contains an element of* $\mathscr{W}(p)$, *then a* S_p*-subgroup of* $\mathfrak{R}$ *is normal in* $\mathfrak{R}$.

Proof. Let $\mathscr{K}$ be the set of subgroups of $\mathfrak{G}$ satisfying the hypotheses but not the conclusion of this theorem and let $\mathscr{K}_1$ be the subset of all $\mathfrak{R}$ in $\mathscr{K}$ which contain at least one element of $\mathscr{U}(p)$.

We first show that $\mathscr{K}_1$ is empty. Suppose false and $\mathfrak{R}$ in $\mathscr{K}_1$ is chosen to maximize $|\mathfrak{R}|_p$. Let $\mathfrak{R}_p$ be a S_p-subgroup of $\mathfrak{R}$, and let $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{R}_p)$ where $\mathfrak{B} \in \mathscr{U}(p)$ and $\mathfrak{B} \subseteq \mathfrak{R}_p$.

Since $p \not\sim q$, Hypothesis 22.1 does not hold. Hence, Hypothesis

19.1 holds. Apply Lemma 19.1 and conclude that $\mathfrak{B}$ centralizes $O_q(\mathfrak{K})$.

Suppose $\mathfrak{K}_p$ is a S_p-subgroup of $\mathfrak{G}$. Then $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$. By Lemma 17.5 and Hypothesis 22.3 (i), if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{K}_p)$, and $\mathfrak{B}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{K}_p)$, then $\mathfrak{B}_1 \subseteq O_{q,p}(\mathfrak{K})$. Since $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$, it follows that $\mathfrak{B}_1 \subseteq O_p(\mathfrak{K})$, and so $\mathfrak{B}_1 \lhd \mathfrak{K}$. By Lemma 17.2, $N(Z(\mathfrak{B}_1)) = O^p(N(Z(\mathfrak{B}_1)))$. Since $p \not\sim q$, $N(Z(\mathfrak{B}_1))$ does not possess an elementary subgroup of order q^3, so Lemma 8.13 implies that $\mathfrak{K}_p \lhd \mathfrak{K}$, contrary to the definition of $\mathscr{K}_1$. Hence, in showing that $\mathscr{K}_1$ is empty, we can suppose that $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$.

Since $\mathfrak{B}$ centralizes $O_q(\mathfrak{K})$, we have $\mathfrak{K}_p \cdot O_q(\mathfrak{K}) \subseteq N(\mathfrak{B})$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{K}_p$ and $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{K}_p$ is not a S_p-subgoup of $N(\mathfrak{B})$. Maximality of $|\mathfrak{K}|_p$ implies that $\mathfrak{K}_p \lhd \mathfrak{K}_p \cdot O_q(\mathfrak{K})$, and so $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$.

Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{K}_p$, and let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. Since $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$, it follows from (B) that $\mathfrak{A} \cap \mathfrak{K}_p = \mathfrak{A} \cap O_p(\mathfrak{K})$. By maximality of $|\mathfrak{K}|_p$, $\mathfrak{K}_p$ is a S_p-subgroup of $N(O_p(\mathfrak{K}))$ and it follows readily that $\mathfrak{A} \subseteq O_p(\mathfrak{K})$. But in this case, $\mathfrak{B}_2 = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{K}_p) \lhd \mathfrak{K}$, by Lemma 17.5. Since $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, it is not a S_p-subgroup of $N(\mathfrak{B}_2)$, and the maximality of $\mathfrak{K}_p$ is violated in a $S_{p,q}$-subgroup of $N(\mathfrak{B}_2)$. This contradiction shows that $\mathscr{K}_1$ is empty.

Now let $\mathfrak{K}$ be in $\mathscr{K}$ with $|\mathfrak{K}|_p$ maximal. Let $\mathfrak{W} \subseteq \mathfrak{K}_p$, $\mathfrak{W} \in \mathscr{W}(p)$. If $\gamma \mathfrak{W} O_q(\mathfrak{K}) \neq \langle 1 \rangle$, then $\mathfrak{W}$ does not centralize $C(W) \cap O_q(\mathfrak{K})$ for suitable W in $\mathfrak{W}^\#$. But in this case a $S_{p,q}$-subgroup of $C(W)$ contains an element of $\mathscr{U}(p)$ and also contains non normal S_p-subgroups, and $\mathscr{K}_1$ is non empty. Since this is not the case, $\mathfrak{W}$ centralizes $O_q(\mathfrak{K})$, and so $\mathfrak{W}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{W}); \mathfrak{K}_p)$ centralizes $O_q(\mathfrak{K})$, $\mathfrak{W}$ being an arbitrary element of $\mathscr{W}(p)$ contained in $\mathfrak{K}_p$. Since $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, it is not a S_p-subgroup of $N(\mathfrak{W}_1)$, so maximality of $|\mathfrak{K}|_p$ implies that $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$. Hence, $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$. Since $\mathfrak{K}_p$ is a S_p-subgroup of $N(O_p(\mathfrak{K}))$ in this case, $Z(\mathfrak{P}) \subseteq O_p(\mathfrak{K})$ for every S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ which contains $\mathfrak{K}_p$. It follows that $\mathfrak{K}_p$ contains an element of $\mathscr{U}(p)$. This contradiction completes the proof of this theorem.

If $p \in \pi_3 \cup \pi_4$, we define $\pi(p)$ to be the set of primes q such that $p \sim q$, and we set $\pi_3(p) = \pi(p) \cap \pi_3$.

THEOREM 22.4. *If $p, q \in \pi_3$ and $p \sim q$, then $\pi_3(p) = \pi_3(q)$.*

Proof. We only need to show that if $r \in \pi_3$ and $p \sim r$, then $r \sim q$.

Apply Theorem 21.1, let $\mathfrak{K}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{Q}$, and let $\mathfrak{L}$ be a $S_{p,r}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{R}$.

If Hypothesis 22.2 is satisfied, Theorem 22.2 applies and yields this theorem. Hence, we suppose without loss of generality that $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$.

Let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$. Apply Lemma 17.5 and conclude that $\mathfrak{B} \lhd \mathfrak{K}$.

If $\mathfrak{P}$ also centralizes $O_r(\mathfrak{L})$, then we also have $\mathfrak{B} \lhd \mathfrak{L}$, and $q \sim r$ follows from consideration of $N(\mathfrak{B})$. We can suppose that $\mathfrak{P}$ does not centralize $O_r(\mathfrak{L})$.

Suppose we are able to show that $N(O_r(\mathfrak{L}))$ contains a S_q-subgroup of $C(\mathfrak{P})$. Apply Lemma 17.3 and conclude that $N(Z(\mathfrak{B})) = \mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot \mathfrak{N}_1 \cap \mathfrak{N}$, where $\mathfrak{N} = N(O_r(\mathfrak{L}))$. Let $\mathfrak{Q}_1$ be a S_q-subgroup of $C(\mathfrak{P})$ which is contained in $\mathfrak{N}$. Since $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$, it follows that $\mathfrak{Q}_1$ is a S_q-subgroup of $O_{p'}(\mathfrak{N}_1)$. Let $\mathfrak{N}_1^*$ be a $S_{q'}$-subgroup of $O_{p'}(\mathfrak{N}_1)$, so that $O_{p'}(\mathfrak{N}_1) = \mathfrak{N}_1^*\mathfrak{Q}_1$. Hence,

$$\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot \mathfrak{N}_1 \cap \mathfrak{N} = \mathfrak{N}_1^*\mathfrak{Q}_1 \cdot \mathfrak{N}_1 \cap \mathfrak{N} = \mathfrak{N}_1^* \cdot \mathfrak{N}_1 \cap \mathfrak{N} ,$$

since $\mathfrak{Q}_1 \subseteq \mathfrak{N}_1 \cap \mathfrak{N}$. Since $\mathfrak{N}_1$ contains a S_q-subgroup of $\mathfrak{G}$, so does $\mathfrak{N}_1 \cap \mathfrak{N}$. But $\mathfrak{N}$ contains a S_r-subgroup of $\mathfrak{G}$ as well, and so $q \sim r$.

Thus, in proving this theorem, it suffices to show that $N(O_r(\mathfrak{L}))$ contains a S_q-subgroup of $C(\mathfrak{P})$.

We wish to show first that some element A of $\mathfrak{A}^\#$ centralizes a subgroup $\mathfrak{W}$ of $\mathscr{W}(r)$ with $\mathfrak{W} \subseteq O_r(\mathfrak{L})$. If $D(O_r(\mathfrak{L})) = \mathfrak{D}$ is non cyclic, then every subgroup of $\mathfrak{D}$ of type (r, r) is in $\mathscr{W}(r)$ and since $\mathfrak{A}$ possesses an elementary subgroup of order p^3, an element A is available. Suppose then that $\mathfrak{D}$ is cyclic. If $\mathfrak{D} = \langle 1 \rangle$, then of course $\mathfrak{P}$ centralizes $\mathfrak{D}$. If $\mathfrak{D} \neq \langle 1 \rangle$, then $N(\mathfrak{D}) = O^p(N(\mathfrak{D}))$ and once again $\mathfrak{P}$ centralizes $\mathfrak{D}$. It now follows that $\mathfrak{A}^\#$ contains an element A whose fixed-point set on $\Omega_1(O_r(\mathfrak{L}))/\Omega_1(\mathfrak{D})$ is non cyclic, and this implies that $C(\mathfrak{A}) \cap O_r(\mathfrak{L})$ contains an element of $\mathscr{W}(r)$.

For such an element A, let $\mathfrak{H}$ be a $S_{q,r}$-subgroup of $O_{p'}(C(A))$ which is $\mathfrak{A}$-invariant and contains $O_q(\mathfrak{K})$. Then Lemma 17.5 implies that $\mathfrak{H}$ contains an element of $\mathscr{W}(r)$. Apply Theorem 22.3 and conclude that $\mathfrak{H}_r \lhd \mathfrak{H}$, $\mathfrak{H}_r$ being a S_r-subgroup of $\mathfrak{H}$. If $\mathfrak{H}^*$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q, r)$ containing $\mathfrak{H}$, then Theorem 22.3 implies that $\mathfrak{H}_r^* \lhd \mathfrak{H}^*$, $\mathfrak{H}_r^*$ being a S_r-subgroup of $\mathfrak{H}^*$. By maximality of $\mathfrak{H}^*$, $\mathfrak{H}_r^*$ is a maximal element of $\mathsf{M}(\mathfrak{A}; r)$. Since $\mathfrak{H}$ contains a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, namely, $O_q(\mathfrak{K})$, so does $\mathfrak{H}^*$. It follows that $N(O_r(\mathfrak{L}))$ contains a maximal element $\mathfrak{Q}^*$ of $\mathsf{M}(\mathfrak{P}^*; q)$ where $\mathfrak{P}^*$ is a suitable S_p-subgroup of $N(O_r(\mathfrak{L}))$. But $\mathfrak{P} \subseteq N(O_r(\mathfrak{L}))$, and so $\mathfrak{P} = \mathfrak{P}^{*N}$ for some N in $N(O_r(\mathfrak{L}))$, and so $\mathfrak{Q}^{*N} = \mathfrak{Q}_2$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$ normalizing $O_r(\mathfrak{L})$. By Lemma 17.4, $\mathfrak{Q}_2$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$.

Now $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$, and $O_q(\mathfrak{K})$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$. It follows that $N(O_q(\mathfrak{K}))/C(O_q(\mathfrak{K}))$ is a p'-group. Since $\mathfrak{Q}_2$ and $O_q(\mathfrak{K})$

are conjugate by Theorem 17.1, it follows that $N(\mathfrak{Q}_2)/C(\mathfrak{Q}_2)$ is a p'-group, and so $\mathfrak{P}$ centralizes $\mathfrak{Q}_2$. By $C^*_{p,q}$, it follows that $\mathfrak{Q}_2$ is a S_q-subgroup of $C(\mathfrak{P})$, completing the proof of this theorem.

THEOREM 22.5. *If* $p \in \pi_3$, *then* $\mathfrak{G}$ *satisfies* $C_{\pi_3(p)}$.

Proof. By Theorem 22.4, if $q, r \in \pi_3(p)$, then $q \sim r$. By Theorem 20.1, $\mathfrak{G}$ satisfies $E_{q,r}$ for $q, r \in \pi_3(p)$. By Theorem 21.2, $\mathfrak{G}$ satisfies $C_{\pi_3(p)}$.

Hypothesis 22.4.
(i) $p \in \pi_3$, $q \in \pi_3 \cup \pi_4$.
(ii) *If* $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{G}$, *then* $\mathfrak{P}$ *contains a normal subgroup* $\mathfrak{E}$ *of type* (p, p) *which centralizes at least one maximal element of* $\mathsf{M}(\mathfrak{P}; q)$.

LEMMA 22.5. *Under Hypothesis* 22.4, $\mathfrak{E}$ *centralizes every element of* $\mathsf{M}(\mathfrak{E}; q)$.

Proof. Suppose false and $\mathfrak{Q}$ is an element of $\mathsf{M}(\mathfrak{E}; q)$ minimal with respect to $\gamma\mathfrak{Q}\mathfrak{E} \neq \langle 1 \rangle$. Then $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{E}$ and $\mathfrak{E}_0 = C_{\mathfrak{E}}(\mathfrak{Q}) \neq \langle 1 \rangle$. Let $\mathfrak{H} = C(\mathfrak{E}_0)$. Then $\mathfrak{H}$ contains an element $\mathfrak{A}$ of $\mathscr{SCN}_3(\mathfrak{P})$ with $\mathfrak{E} \subseteq \mathfrak{A}$. By Lemma 17.5, $\mathfrak{A} \subseteq O_{p',p}(\mathfrak{H})$, and so $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{E}$ is contained in $O_{p'}(\mathfrak{H})$. If $\mathfrak{Q}^*$ is an $\mathfrak{A}$-invariant S_q-subgroup of $O_{p'}(\mathfrak{H})$, it follows readily that $\gamma\mathfrak{Q}^*\mathfrak{E} \neq \langle 1 \rangle$. If $\tilde{\mathfrak{Q}}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ containing $\mathfrak{Q}^*$, then $\mathfrak{E}$ does not centralize $\tilde{\mathfrak{Q}}$. Let $\mathfrak{Q}_0$ be a maximal element of $\mathsf{M}(\mathfrak{P}; q)$ centralizing $\mathfrak{E}$. Since $\mathfrak{Q}_0$ is also a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, we have $\mathfrak{Q}_0 = \tilde{\mathfrak{Q}}^C$ for suitable C in $C(\mathfrak{A}) \subseteq C(\mathfrak{E})$. Since $\mathfrak{E}$ does not centralize $\tilde{\mathfrak{Q}}$, $\mathfrak{E}^C = \mathfrak{E}$ does not centralize $\mathfrak{Q}_0$. This contradiction completes the proof of this lemma.

The next theorem is fairly delicate and brings π_4 into play explicitly for the first time.

Hypothesis 22.5.
(i) $p \in \pi_3$, $q \in \pi_4$.
(ii) $p \sim q$.

THEOREM 22.6. *Under Hypothesis* 22.5, *if* $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{G}$ *and* $\mathfrak{Q}_1$ *is a maximal element of* $\mathsf{M}(\mathfrak{P}; q)$, *then* $\mathfrak{Q}_1 \neq \langle 1 \rangle$. *If* $\mathfrak{Q}_2$ *is a* S_q*-subgroup of* $N(\mathfrak{Q}_1)$ *permutable with* $\mathfrak{P}$ *and* $\mathfrak{Q}_3$ *is a* S_q*-subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{Q}_2$, *then* $\mathfrak{Q}_1$ *contains every element of* $\mathscr{SCN}(\mathfrak{Q}_3)$. *Furthermore,* $O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$.

Proof. By Theorem 19.1, $\mathfrak{P}$ does not centralize $\mathfrak{Q}_1$, so in particular $\mathfrak{Q}_1 \neq \langle 1 \rangle$.

Suppose that $\mathfrak{Q}_2$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{Q}_3)$. By Lemma 19.1, $\mathfrak{B}$ centralizes $O_p(\mathfrak{P}\mathfrak{Q}_2)$ and since $\mathfrak{B}$ is a normal abelian subgroup of $\mathfrak{Q}_2$, (B) implies that $\mathfrak{B} \subseteq O_q(\mathfrak{P}\mathfrak{Q}_2)$. Let $\mathfrak{A}$ be an element of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q}_3)$ containing $\mathfrak{B}$. Let $\mathfrak{H} = N(\mathfrak{B}) \supseteq \langle \mathfrak{Q}_3, O_p(\mathfrak{P}\mathfrak{Q}_2)\rangle$. Since $q \in \pi_4$, $O_{q'}(\mathfrak{H}) = \langle 1 \rangle$, and (B) implies that $\mathfrak{A} \subseteq O_q(\mathfrak{H})$. Hence, $[\mathfrak{A} \cap \mathfrak{Q}_2, O_p(\mathfrak{P}\mathfrak{Q}_2)] \subseteq O_q(\mathfrak{H}) \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$, and by (B), $\mathfrak{A} \cap \mathfrak{Q}_2 \subseteq O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$, and so $\mathfrak{A} \cap \mathfrak{Q}_2 \subseteq O_q(\mathfrak{P}\mathfrak{Q}_2)$, that is, $\mathfrak{A} \cap \mathfrak{Q}_2 = \mathfrak{A} \cap \mathfrak{Q}_1$. If $\mathfrak{A} \cap \mathfrak{Q}_1 \subset \mathfrak{A}$, then $\mathfrak{A} \cap \mathfrak{Q}_1 \subset N_{\mathfrak{A}}(\mathfrak{Q}_1)$, contrary to $N_{\mathfrak{A}}(\mathfrak{Q}_1) \subseteq \mathfrak{A} \cap \mathfrak{Q}_2 = \mathfrak{A} \cap \mathfrak{Q}_1$. Hence, $\mathfrak{A} \subseteq \mathfrak{Q}_1$. Since $q \in \pi_4$, Corollary 17.3 implies that $\mathsf{M}(\mathfrak{A}; p)$ is trivial, so $O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$. By Lemma 7.9, it follows that $\mathfrak{Q}_1$ contains every element of $\mathscr{S}\mathscr{C}\mathscr{N}(\mathfrak{Q}_3)$, and not merely $\mathfrak{A}$. This proves the theorem in this case.

We can now assume that $\mathfrak{Q}_2$ does not contain any element of $\mathscr{U}(\mathfrak{Q}_3)$, and try to derive a contradiction.

Since $\mathfrak{Q}_2$ is a S_q-subgroup of the normalizer of every non identity normal subgroup of $\mathfrak{P}\mathfrak{Q}_2$, if $D(\mathfrak{Q}_1) \neq \langle 1 \rangle$, then $N_{\mathfrak{Q}_3}(D(\mathfrak{Q}_1))$ contains an element of $\mathscr{U}(\mathfrak{Q}_3)$, and since $N_{\mathfrak{Q}_3}(D(\mathfrak{Q}_1)) = \mathfrak{Q}_2$ in this case, $\mathfrak{Q}_2$ contains an element of $\mathscr{U}(\mathfrak{Q}_3)$, contrary to assumption. Hence, $D(\mathfrak{Q}_1) = \langle 1 \rangle$.

Let $\mathfrak{Q}_1^* = O_{p,q}(\mathfrak{P}\mathfrak{Q}_2) \cap \mathfrak{Q}_2$. Since $[\mathfrak{Q}_1^*, \mathfrak{Q}_1] = [O_{p,q}(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{Q}_1] \lhd \mathfrak{P}\mathfrak{Q}_2$, and since every element of $\mathscr{U}(\mathfrak{Q}_3)$ normalizes $[\mathfrak{Q}_1^*, \mathfrak{Q}_1]$, we conclude that $\mathfrak{Q}_1 \subseteq Z(\mathfrak{Q}_1^*)$. Since $D(\mathfrak{Q}_1^*) \cap \mathfrak{Q}_1$ is normalized by every element of $\mathscr{U}(\mathfrak{Q}_3)$ and also by $\langle O_p(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{P}\mathfrak{Q}_2 \cap N(\mathfrak{Q}_1^*)\rangle = \mathfrak{P}\mathfrak{Q}_2$, we have $D(\mathfrak{Q}_1^*) \cap \mathfrak{Q}_1 = \langle 1 \rangle$. This implies that $\mathfrak{Q}_1^* = \mathfrak{Q}_1 \times \mathfrak{F}$ for a suitable subgroup $\mathfrak{F}$ of $\mathfrak{Q}_1^*$.

Since $Z(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_2$, we have $Z(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_1^*$, by (B). Since $\mathfrak{Q}_2$ contains no element of $\mathscr{U}(\mathfrak{Q}_3)$, $Z(\mathfrak{Q}_3)$ is cyclic. For the same reason, $Z(\mathfrak{Q}_3) \cap \mathfrak{Q}_1 = \langle 1 \rangle$, since otherwise, $\Omega_1(Z(\mathfrak{Q}_3)) \subseteq \mathfrak{Q}_1$ and every element of $\mathscr{U}(\mathfrak{Q}_3)$ normalizes $\mathfrak{Q}_1$. In particular, $\mathfrak{Q}_1$ is a proper subgroup of $\mathfrak{Q}_1^*$. This implies that $O_p(\mathfrak{P}\mathfrak{Q}_2) \neq \langle 1 \rangle$. More exactly, $\mathfrak{Q}_1 = C_{\mathfrak{Q}_1^*}(O_p(\mathfrak{P}\mathfrak{Q}_2))$.

Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q}_3)$ and let $\tilde{\mathfrak{Q}}_1 = C_{\mathfrak{Q}_1}(\mathfrak{B})$, so that $|\mathfrak{Q}_1 : \tilde{\mathfrak{Q}}_1| = q$.

Suppose $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is non cyclic. In this case, a basic property of p-groups implies that $O_p(\mathfrak{P}\mathfrak{Q}_2)$ contains a subgroup $\mathfrak{E}$ of type (p, p) which is normal in $\mathfrak{P}$. Since $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$, Hypothesis 22.4 is satisfied. Since $\tilde{\mathfrak{Q}}_1$ is of index q in $\mathfrak{Q}_1$, Theorem 22.1 implies that $\tilde{\mathfrak{Q}}_1 \neq \langle 1 \rangle$. Hence, $\langle \mathfrak{B}, \mathfrak{E}\rangle$ is a proper subgroup of $\mathfrak{G}$ centralizing $\tilde{\mathfrak{Q}}_1$. Choose $\mathfrak{B}_1 \in ccl_{\mathfrak{G}}(\mathfrak{B})$ and $\mathfrak{E}_1 \in ccl_{\mathfrak{G}}(\mathfrak{E})$ so that $\mathfrak{K} = \langle \mathfrak{B}_1, \mathfrak{E}_1\rangle$ is minimal. By $D_{p,q}$ in $\mathfrak{K}$, it follows that $\mathfrak{K}$ is a p, q-group. By Lemma 19.1, $\mathfrak{B}_1^{\mathfrak{K}}$ centralizes $O_p(\mathfrak{K})$ and by Lemma 22.5, $\mathfrak{E}_1^{\mathfrak{K}}$ centralizes $O_q(\mathfrak{K})$. It follows that $\mathfrak{K} = \mathfrak{B}_1 \times \mathfrak{E}_1$. Let $\mathfrak{N} = N(\mathfrak{B}_1)$. Since $q \in \pi_4$, $F(\mathfrak{N})$ is a q-group. By Lemma 22.5, $\mathfrak{E}_1$ centralizes $F(\mathfrak{N})$ so 3.3 is violated. This contradiction shows that $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is cyclic.

Since $\mathfrak{Q}_1 = C_{\mathfrak{Q}_1^*}(O_p(\mathfrak{P}\mathfrak{Q}_2))$, it follows that $\mathfrak{F}$ is cyclic of an order dividing $p - 1$.

Let $\mathfrak{B} = \Omega_1(\mathfrak{Q}_1^*) = \mathfrak{Q}_1 \times \Omega_1(\mathfrak{F})$, and let $\mathfrak{H}_1 = N_{\mathfrak{P}\mathfrak{Q}_2}(\mathfrak{B})$. We see that $\mathfrak{P}\mathfrak{Q}_2 = \mathfrak{H}_1 O_p(\mathfrak{P}\mathfrak{Q}_2)$, $\mathfrak{H}_1 \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$. Let $\mathfrak{M} = N(\mathfrak{B})$, $\mathfrak{M}_1 = C(\mathfrak{B})$. It is clear that $\mathfrak{M}_1 \cap \mathfrak{P}\mathfrak{Q}_2 = \mathfrak{Q}_1^*$, since $\mathfrak{M}_1 \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$, and since $\mathfrak{Q}_1^*$ is a S_q-subgroup of $O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$.

Let $\mathfrak{L} = O_q(\mathfrak{M} \bmod \mathfrak{M}_1)$. We see that $\mathfrak{L} \cap \mathfrak{P}\mathfrak{Q}_2 = \mathfrak{Q}_1^*$, again since $\mathfrak{Q}_1^*$ is a S_q-subgroup of $O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$. We observe that since $\mathfrak{Q}_1^*$ contains $Z(\mathfrak{Q}_3)$, $\mathfrak{M}$ contains every element of $\mathscr{U}(\mathfrak{Q}_3)$, and so contains $\mathfrak{B}$. By Lemma 7.1, $\mathfrak{L}$ contains $\mathfrak{B}$. Hence, $\mathfrak{L} \supset \mathfrak{M}_1$.

We next show that $\mathfrak{L}/\mathfrak{M}_1 = \bar{\mathfrak{L}}$ is elementary. If $D(\bar{\mathfrak{L}}) \neq \langle 1 \rangle$, then by a basic property of q-groups, $C_{\mathfrak{B}}(D(\bar{\mathfrak{L}}))$ is of order at least q^2. Hence, $\hat{\mathfrak{Q}}_1 = C_{\mathfrak{B}}(D(\bar{\mathfrak{L}})) \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$. But in this case, $\hat{\mathfrak{Q}}_1$ is normalized by $\langle O_p(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{H}_1 \rangle = \mathfrak{P}\mathfrak{Q}_2$, and is centralized by $D(\mathfrak{L} \bmod \mathfrak{M}_1)$, and so $\mathfrak{Q}_2$ is not a S_q-subgroup of $N(\mathfrak{Q}_1)$. This is not possible, so $D(\bar{\mathfrak{L}}) = \langle 1 \rangle$. We have in fact shown that if $\mathfrak{L}_1 \lhd \mathfrak{M}$, and $\mathfrak{M}_1 \subset \mathfrak{L}_1 \subseteq \mathfrak{L}$, then $C_{\mathfrak{B}}(\mathfrak{L}_1)$ is of order q.

Since $\bar{\mathfrak{L}}$ is abelian, $\mathfrak{L}$ normalizes $[\mathfrak{B}, \mathfrak{B}] = \Omega_1(Z(\mathfrak{Q}_3))$. It follows that $C_{\mathfrak{B}}(\mathfrak{L}) = \Omega_1(Z(\mathfrak{Q}_3))$.

Let $\mathfrak{L}_1 = \langle \mathfrak{B}^{\mathfrak{M}}, \mathfrak{M}_1 \rangle$, and let $\mathfrak{B}_1 = \mathfrak{B}^M$, $M \in \mathfrak{M}$. Since $\mathfrak{B}$ and $\mathfrak{B}_1$ are conjugate in $\mathfrak{M}$, $[\mathfrak{B}, \mathfrak{B}_1]$ is of order q and is centralized by $\mathfrak{L}$. It follows that $[\mathfrak{B}, \mathfrak{B}_1] = \Omega_1(Z(\mathfrak{Q}_3))$. Since $\bar{\mathfrak{L}}$ is abelian, and since $\mathfrak{B}^{\mathfrak{M}}$ covers $\mathfrak{L}_1/\mathfrak{M}_1 = \bar{\mathfrak{L}}_1$, it follows that $[\mathfrak{B}, \mathfrak{L}_1] = \Omega_1(Z(\mathfrak{Q}_3))$. Let $|\mathfrak{Q}_1| = q^n$, and $|\mathfrak{L}_1 : \mathfrak{M}_1| = q^m$. Since each element of $\bar{\mathfrak{L}}_1^\#$ determines a non trivial homomorphism of $\mathfrak{B}/\Omega_1(Z(\mathfrak{Q}_3))$ into $\Omega_1(Z(\mathfrak{Q}_3))$, it follows that $m \leqq n$. Since $C_{\mathfrak{B}}(\mathfrak{L}_1) = \Omega_1(Z(\mathfrak{Q}_3))$, it also follows that $m \geqq n$. Hence, $m = n$. This implies that $\mathfrak{L}_1 = \mathfrak{L}$, since any q-element of $\operatorname{Aut}\mathfrak{B}$ which centralizes $\bar{\mathfrak{L}}_1$ is in $\bar{\mathfrak{L}}_1$, by 3.10. Here we are invoking the well known fact that $\bar{\mathfrak{L}}_1$ is normal in a S_q-subgroup $\mathfrak{Q}$ of $\operatorname{Aut}\mathfrak{B}$ and is in fact in $\mathscr{SCN}(\mathfrak{Q})$. (This appeal to the "enormous" group $\operatorname{Aut}\mathfrak{B}$ is somewhat curious.)

Returning to $\mathfrak{L}$, let $\mathfrak{B}^*$ be a S_q-subgroup of $\mathfrak{L}$, and let $\mathfrak{W} = \Omega_1(\mathfrak{B}^*)$. Since $\mho^1(\mathfrak{B}^*) \subseteq Z(\mathfrak{B}^*)$, and $Z(\mathfrak{B}^*)$ is cyclic, it is easy to see that $\Omega_1(Z(\mathfrak{W})) = \Omega_1(Z(\mathfrak{Q}_3))$, and that $\mathfrak{W}/\Omega_1(Z(\mathfrak{W}))$ is abelian. Hence, $\mathfrak{W}$ is an extra special group of order q^{2n+1} and exponent q.

We next show that $\mathfrak{M}_1$ is a p'-group. Since $\mathfrak{M}_1 \subseteq C(\mathfrak{Q}_1)$, it suffices to show that no non identity p-element of $N(\mathfrak{Q}_1)$ centralizes $\mathfrak{B}$. This is clear by $D_{p,q}$ in $N(\mathfrak{Q}_1)$, together with the fact that no non identity p-element of $\mathfrak{P}\mathfrak{Q}_2$ centralizes $\mathfrak{B}$.

Since $\mathfrak{M}_1$ is a p'-group, so is $\mathfrak{L}$. Since $\mathfrak{L} \lhd \mathfrak{M}$, we assume without loss of generality that $N_{\mathfrak{P}}(\mathfrak{B})$ normalizes $\mathfrak{B}^*$.

Let $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{P})$, and set $\mathfrak{C}_1 = \mathfrak{C} \cap N_{\mathfrak{P}}(\mathfrak{B})$. Since $\mathfrak{P} = O_p(\mathfrak{P}\mathfrak{Q}_2) \cdot \mathfrak{N}_{\mathfrak{P}}(\mathfrak{B})$ and $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is cyclic, $\mathfrak{C}_1$ is non cyclic. Since $\mathfrak{C}_1$ is faithfully represented on $\mathfrak{B}$, it is faithfully represented on $\mathfrak{W} = \Omega_1(\mathfrak{B}^*)$. Since $p > q$, $\mathfrak{C}_1$ centralizes $\Omega_1(Z(\mathfrak{W}))$.

We can now choose C in $\mathfrak{C}_1^\sharp$ so that $\mathfrak{C}_1$ does not centralize $\mathfrak{W}_1 = C_{\mathfrak{W}}(C)$. Let $\mathfrak{W}_2 = [\mathfrak{W}_1, \mathfrak{C}_1]$. We will show that $\mathfrak{W}_2$ is non abelian. To do this, we first show that $\mathfrak{W}_1$ is extra special. Let $W \in \mathfrak{W}_1 - \Omega_1(\boldsymbol{Z}(\mathfrak{W}))$. Since C centralizes W, C normalizes $C_{\mathfrak{W}}(W)$. Since $p > q$, C acts trivially on $\mathfrak{W}/C_{\mathfrak{W}}(W)$, and so C centralizes some element of $\mathfrak{W} - C_{\mathfrak{W}}(W)$. It follows that $\boldsymbol{Z}(\mathfrak{W}_1) = \boldsymbol{Z}(\mathfrak{W})$, so that $\mathfrak{W}_1$ is extra special. We can now find $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$ so that $\mathfrak{W}_1 = \mathfrak{W}_2\mathfrak{W}_3$ and $\mathfrak{W}_2 \cap \mathfrak{W}_3 \subseteq \boldsymbol{Z}(\mathfrak{W})$; in fact, we take $\mathfrak{W}_3 = C_{\mathfrak{W}_1}(\mathfrak{W}_2)$. By the argument just given, $\mathfrak{W}_3$ is extra special. Since $\mathfrak{W}_1$ is, too, it follows that $\mathfrak{W}_2$ is extra special, hence is non abelian.

For such an element C, let $\mathfrak{X} = C(C) \supseteq \langle \mathfrak{C}, \mathfrak{W}_2 \rangle$. By Lemma 17.5, $\mathfrak{C} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{X})$. Since $\mathfrak{W}_2 = [\mathfrak{W}_2, \mathfrak{C}_1]$, by Lemma 8.11, it follows that $\mathfrak{W}_2 \subseteq \boldsymbol{O}_{p'}(\mathfrak{X})$. It follows now that $\mathsf{M}(\mathfrak{C}; q)$ contains a non abelian group. But now Theorem 17.1 implies that the maximal elements of $\mathsf{M}(\mathfrak{C}; q)$ are non abelian. Since $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{C}; q)$ and $\mathfrak{Q}_1$ is elementary, we have a contradiction, completing the proof of this theorem.

THEOREM 22.7. *If p, $q \in \pi_3$, and $p \sim q$, then $\pi(p) = \pi(q)$.*

Proof. Suppose $p \sim r$. By Theorem 22.4, we can suppose that $r \in \pi_4$. Proceeding by way of contradiction, we can assume that a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ centralizes every element of $\mathsf{M}(\mathfrak{Q}; r)$, by Theorem 22.1. By Theorem 19.1, a S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ does not centralize every element of $\mathsf{M}(\mathfrak{P}; r)$. Applying Theorem 22.2, we can suppose that $\mathfrak{P}$ centralizes every element of $\mathsf{M}(\mathfrak{P}; q)$.

Let $\mathfrak{Q}_1$ be a maximal element of $\mathsf{M}(\mathfrak{P}; q)$ and let $\mathfrak{R}_1$ be a maximal element of $\mathsf{M}(\mathfrak{P}; r)$. Let $\mathfrak{R}_2$ be a S_r-subgroup of $N(\mathfrak{R}_1)$ permutable with $\mathfrak{P}$ and let $\mathfrak{R}_3$ be a S_r-subgroup of $\mathfrak{G}$ containing $\mathfrak{R}_2$. Let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. By Theorem 22.6, $\boldsymbol{O}_p(\mathfrak{P}\mathfrak{R}_2) = \langle 1 \rangle$, so $\mathfrak{A}$ does not centralize $\mathfrak{R}_1$. We can then find A in $\mathfrak{A}^\sharp$ such that $\mathfrak{R}_1^* = [C_{\mathfrak{R}_1}(A), \mathfrak{A}] \neq \langle 1 \rangle$.

Suppose $\mathfrak{Q}_1$ is non cyclic. Then by $C_{p,q}^*$, $\mathfrak{Q}_1$ contains an element of $\mathscr{U}(q)$. Let $\mathfrak{H} = C(A) \supseteq \langle \mathfrak{A}, \mathfrak{R}_1^*, \mathfrak{Q}_1 \rangle = \mathfrak{X}$, and let $\mathfrak{K}$ be an $\mathfrak{A}$-invariant $S_{q,r}$-subgroup of $\boldsymbol{O}_{p'}(\mathfrak{X})$ with Sylow system $\mathfrak{K}_r$, $\mathfrak{Q}_1$. By Theorem 22.3, $\mathfrak{Q}_1 \lhd \mathfrak{K}$. Since $N(\mathfrak{Q}_1) = \boldsymbol{O}^p(N(\mathfrak{Q}_1))$, it follows that $\gamma\mathfrak{A}\mathfrak{K}_r = \langle 1 \rangle$ by Lemma 8.11 and the fact that $N(\mathfrak{Q}_1)$ does not contain an elementary subgroup of order r^3. This violates the fact that $\mathfrak{R}_1^* = \gamma\mathfrak{R}_1^*\mathfrak{A} \neq \langle 1 \rangle$, by $D_{p,r}$ in $\mathfrak{H}$. Hence, $\mathfrak{Q}_1$ is cyclic.

Since $\gamma\mathfrak{Q}_1\mathfrak{P} = \langle 1 \rangle$, $\mathfrak{P}_1 = \boldsymbol{O}_p(\mathfrak{P}\mathfrak{Q}_2) \neq \langle 1 \rangle$, where $\mathfrak{Q}_2$ is a S_q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}$ and containing $\mathfrak{Q}_1$, which exists by $C_{p,q}^*$. Since $N(\mathfrak{P}_1) = \boldsymbol{O}^q(N(\mathfrak{P}_1))$, it follows that $\mathfrak{Q}_1 \subseteq \boldsymbol{Z}(\mathfrak{Q}_2)$, $\mathfrak{Q}_1$ being a S_q-subgroup of $\boldsymbol{O}_{p'}(N(\mathfrak{P}_1))$.

Let $\mathfrak{B} = \boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, and $\mathfrak{N}_1 = N(\boldsymbol{Z}(\mathfrak{B}))$. By Lemma 17.3, $\mathfrak{N}_1 =$

$O_{p'}(\mathfrak{N}_1)\cdot\mathfrak{N}_1 \cap N(\mathfrak{N}_1)$. Since $\mathfrak{Q}_1$ is a S_q-subgroup of $O_{p'}(\mathfrak{N}_1)$, it follows readily that $\mathfrak{N}_1 \cap N(\mathfrak{N}_1)$ contains an element of $\mathscr{W}(q)$. In particular, $N(\mathfrak{N}_1)$ contains an element of $\mathscr{W}(q)$. If $\mathfrak{L}$ is a $S_{q,r}$-subgroup of $N(\mathfrak{N}_1)$ with Sylow system $\mathfrak{L}_q$, $\mathfrak{L}_r$, then $\mathfrak{L}_q \lhd \mathfrak{L}$, by Theorem 22.3. By Theorem 22.6, $\mathfrak{N}_1$ contains an element $\mathfrak{C}$ of $\mathscr{SCN}_3(\mathfrak{N}_3)$. By Corollary 17.3, $\mathsf{M}(\mathfrak{C})$ is trivial. Since $\mathfrak{L}_q \in \mathsf{M}(\mathfrak{C})$, we have a contradiction, completing the proof of this theorem.

23. Preliminary Results about the Maximal Subgroups of $\mathfrak{G}$

Hypothesis 23.1.

(i) *ϖ is a non empty subset of π_3.*

(ii) *For at least one p in ϖ, $\varpi = \pi(p)$.*

We remark that by Theorem 22.7, Hypothesis 23.1 (ii) is equivalent to

(ii)′ *$\pi(p) = \varpi$ for all p in ϖ.*

Under Hypothesis 23.1, Theorem 22.5 implies that $\mathfrak{G}$ contains a S_ϖ-subgroup $\mathfrak{H}$. Since $\mathfrak{H}$ also satisfies E_{ϖ_1} for all subsets ϖ_1 of ϖ, $\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$ by P. Hall's characterization of solvable groups [15]. This section is devoted to a study of $\mathfrak{H}$ and its normalizer $\mathfrak{M} = N(\mathfrak{H})$. All results of this section assume that Hypothesis 23.1 holds. Let $\varpi = \{p_1, \cdots, p_n\}$, $n \geqq 1$, and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be a Sylow system for $\mathfrak{H}$.

LEMMA 23.1. *$\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$ and is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$.*

Proof. Let $\mathfrak{K}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. We must show that $\mathfrak{K} \subseteq \mathfrak{M}$. Since $\mathfrak{K}$ is solvable we assume without loss of generality that $\mathfrak{K}$ is a ϖ, q-group for some $q \notin \varpi$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$, $\mathfrak{Q}$ be a Sylow system for $\mathfrak{K}$. It suffices to show that $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$.

Since $q \notin \varpi$, $p_1 \not\sim q$. Theorem 22.1 implies that $\mathfrak{P}_1$ centralizes $O_q(\mathfrak{P}_1\mathfrak{Q})$. By Lemma 17.5, $\mathfrak{B} \lhd \mathfrak{P}_1\mathfrak{Q}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_1)$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P}_1)$. By Lemma 17.2, $\mathfrak{N}_1 = N(Z(\mathfrak{B})) = O^{p_1}(\mathfrak{N}_1)$. Since $\mathfrak{N}_1$ does not contain an elementary subgroup of order q^3, Lemma 8.13 implies that $\mathfrak{P}_1$ centralizes every q-factor of $\mathfrak{P}_1\mathfrak{Q}$ and so $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$, completing the proof of this lemma.

LEMMA 23.2. *If $p_i \in \pi(F(\mathfrak{H}))$, and $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, then $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$.*

Proof. We can assume that $i = 1$. By $C^*_{p_1,p_j}$ $\mathfrak{H}$ contains a S_{p_j}-subgroup of $C(\mathfrak{A}_1)$ for each $j = 2, \cdots, n$. Thus, it suffices to show that if $q \notin \varpi$, and $\mathfrak{Q}$ is a S_q-subgroup of $C(\mathfrak{A}_1)$ permutable with $\mathfrak{P}_1$, then $\mathfrak{Q} \subseteq \mathfrak{M}$.

By the preceding argument, $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$. Since $\mathfrak{P}_1$ normalizes $C(\mathfrak{A}_1) = \mathfrak{A}_1 \times \mathfrak{D}$, $\mathfrak{D}$ being a p_1'-group, it follows that $\mathfrak{P}_1\mathfrak{Q} = \mathfrak{P}_1 \times \mathfrak{Q}$.

Since $F(\mathfrak{H}) \cap \mathfrak{P}_1 \neq \langle 1 \rangle$, it follows that $\mathfrak{M} = N(F(\mathfrak{H}) \cap \mathfrak{P}_1)$, since $F(\mathfrak{H}) \cap \mathfrak{P}_1$ char $\mathfrak{H} \lhd \mathfrak{M}$ and $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. The lemma follows since $N(F(\mathfrak{H}) \cap \mathfrak{P}_1) \supseteq C(F(\mathfrak{H}) \cap \mathfrak{P}_1) \supseteq \mathfrak{Q}$.

LEMMA 23.3. *Let* $1 \leqq i \leqq n$, *and let* $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$. *If* $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, *then* $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$.

Proof. We can assume that $i = 1$. If $F(\mathfrak{H})$ is a p_1-group, then Lemma 17.5 implies that $\mathfrak{B}_1 \lhd \mathfrak{H}$ and so $\mathfrak{B}_1 \lhd \mathfrak{M}$, since $\mathfrak{B}_1$ is weakly closed in $F(\mathfrak{H}) \cap \mathfrak{P}_1$. In this case, $N(\mathfrak{B}_1) = \mathfrak{M}$ and we are done.

We can suppose that $F(\mathfrak{H})$ is not a p_1-group, and so $\mathfrak{T} = O_{p'}(\mathfrak{H}) \neq \langle 1 \rangle$. Let $\mathfrak{T}_2, \cdots, \mathfrak{T}_n$ be a $\mathfrak{P}_1$-invariant Sylow system for $\mathfrak{T}$, where $\mathfrak{T}_i$ is a S_{p_i}-subgroup of $\mathfrak{T}$ and we allow $\mathfrak{T}_i = \langle 1 \rangle$. By $C^*_{p_1, p_i}$, $\mathfrak{T}_i$ is a maximal element of $\mathsf{M}(\mathfrak{P}_1; p_i)$.

Let $N \in N(\mathfrak{B}_1)$. Then by Theorem 17.1, $\mathfrak{T}_i^N = \mathfrak{T}_i^{C_i}$ where $C_2, \cdots, C_n$ are in $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$. Since $\mathfrak{T}$ char $\mathfrak{H} \lhd \mathfrak{M}$, each $\mathfrak{T}_i^{C_i}$ is contained in $\mathfrak{T}$ and so $\mathfrak{T}^N = \mathfrak{T}$. Since $\mathfrak{T} \neq \langle 1 \rangle$, $\mathfrak{M} = N(\mathfrak{T}) \supseteq N(\mathfrak{B}_1)$, as required.

LEMMA 23.4. *Let* $1 \leqq i \leqq n$, $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$. *If* $\langle C(\mathfrak{A}_i), N(\mathfrak{B}_i) \rangle \subseteq \mathfrak{M}$, *then* $\mathfrak{M}$ *is the unique maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. We can assume that $i = 1$. Let $\mathfrak{Q}$ be a q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}_1$. It suffices to show that $\mathfrak{Q} \subseteq \mathfrak{M}$.

Since $\mathfrak{Q} = O_q(\mathfrak{P}_1\mathfrak{Q}) \cdot N_{\mathfrak{Q}}(\mathfrak{B}_1)$, it suffices to show that $\mathfrak{Q}_1 = O_q(\mathfrak{P}_1\mathfrak{Q}) \subseteq \mathfrak{M}$. If $\mathfrak{Q}$ is centralized by $\mathfrak{P}_1$, then by hypothesis $\mathfrak{Q} \subseteq \mathfrak{M}$. Otherwise we apply Theorem 22.1 and conclude that $q \in \varpi$. By Theorem 17.1, $\mathfrak{Q}_1^C \subseteq \mathfrak{H}$ for suitable $C \in C(\mathfrak{A}_1) \subseteq \mathfrak{M}$, and the lemma follows.

LEMMA 23.5. *For each* $i = 1, \cdots, n$, *if* $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, *then* $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, *and* $\mathfrak{M}$ *is the unique maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. First, suppose $p_i \in \pi(F(\mathfrak{H}))$. Then $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, by Lemma 23.2. Then by Lemma 23.3, $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$, and then by Lemma 23.4, this lemma follows. We can suppose that $p_i \notin \pi(F(\mathfrak{H}))$.

We assume that $i = 1$. Let $C(\mathfrak{A}_1) = \mathfrak{A}_1 \times \mathfrak{D}$, where $\mathfrak{D}$ is a p_1'-group. It suffices to show that for each q in $\pi(\mathfrak{D})$, $\mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{D}$. If $q \in \varpi$, this is the case by $C^*_{p_1, q}$, so we can suppose that $q \notin \varpi$.

Since $p_1 \notin \pi(F(\mathfrak{H}))$, $\mathfrak{A}_1$ does not centralize $F(\mathfrak{H})$. If $F(\mathfrak{H})$ were cyclic, and $p = \max\{p_1, \cdots, p_n\}$, then a S_p-subgroup of $\mathfrak{H}$ would be contained in $F(\mathfrak{H})$ and so be cyclic. Since this is not the case, $F(\mathfrak{H})$ is non cyclic, so we can assume that $\boldsymbol{F}(\mathfrak{H}) \cap \mathfrak{P}_2$ is non cyclic. We can then find A in $\mathfrak{A}_1^*$ so that $\boldsymbol{C}(A) \cap \boldsymbol{F}(\mathfrak{H}) \cap \mathfrak{P}_2$ contains an element of $\mathscr{W}(p_2)$, say $\mathfrak{W}$.

Let $\mathfrak{L}^* = \langle \mathfrak{D}, \mathfrak{W}, \mathfrak{A}_1 \rangle \subseteq \boldsymbol{C}(A)'$, and let $\mathfrak{L}$ be a $S_{p_1,p_2,q}$-subgroup of $\mathfrak{L}^*$ with Sylow system $\mathfrak{L}_{p_1}, \mathfrak{L}_{p_2}, \mathfrak{L}_q$, where $\mathfrak{A}_1 \subseteq \mathfrak{L}_{p_1}$ and $\mathfrak{Q} \subseteq \mathfrak{L}_q$. Since $\mathfrak{A}_1 \subseteq \boldsymbol{O}_{p_1',p_1}(\mathfrak{L}^*)$ by Lemma 17.5, it follows that $\mathfrak{A}_1\boldsymbol{O}_{p_1'}(\mathfrak{L}^*)/\boldsymbol{O}_{p_1'}(\mathfrak{L}^*)$ is a central factor of $\mathfrak{L}^*$. Hence, $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{L}^*$ and so $\mathfrak{L}^* = \mathfrak{A}_1 \cdot \boldsymbol{O}_{p_1'}(\mathfrak{L}^*)$.

We apply Theorem 22.3 and conclude that $\mathfrak{L}_{p_2} \lhd \mathfrak{L}$. If $\tilde{\mathfrak{L}}$ is a maximal element of $\mathsf{M}(\mathfrak{A}_1; p_2, q)$ containing $\mathfrak{L}_{p_2} \cdot \mathfrak{L}_q$, it follows that $\tilde{\mathfrak{L}}_{p_2} \lhd \tilde{\mathfrak{L}}$, where $\tilde{\mathfrak{L}}_{p_2}$ is a maximal element of $\mathsf{M}(\mathfrak{A}_1; p_2)$. By construction, $\tilde{\mathfrak{L}}$ contains $\mathfrak{Q}$. By Theorem 17.1, there is an element C in $\boldsymbol{C}(\mathfrak{A}_1)$ such that $\tilde{\mathfrak{L}}_{p_2}^C = \boldsymbol{O}_{p_2}(\mathfrak{P}_1\mathfrak{P}_2)$. Since $\mathfrak{Q}^C$ normalizes $\tilde{\mathfrak{L}}_{p_2}^C$, it follow that $\boldsymbol{N}(\boldsymbol{O}_{p_2}(\mathfrak{P}_1\mathfrak{P}_2))$ contains a S_q-subgroup of $\boldsymbol{C}(\mathfrak{A}_1)$. But $p_2 \in \pi(\boldsymbol{F}(\mathfrak{H}))$, so by what is already proved, we have $\boldsymbol{N}(\boldsymbol{O}_{p_2}(\mathfrak{P}_1\mathfrak{P}_2)) \subseteq \mathfrak{M}$, and so $\mathfrak{M}$ contains $\boldsymbol{C}(\mathfrak{A}_1)$. We apply Lemmas 23.3 and 23.4 and complete the proof of this lemma.

24. Further Linking Theorems

LEMMA 24.1. *If* $p \in \pi_4$, $q \in \pi_3 \cup \pi_4$ *and* $q \sim p$, *then* $\pi(q) \subseteq \pi(p)$.

Proof. If $q = p$, there is nothing to prove, so suppose $q \neq p$. Corollary 19.1 implies that $q \in \pi_3$. Let $r \sim q$, $r \neq q$, $r \neq p$. We must show that $r \sim p$.

If $r \in \pi_4$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ does not centralize every element of $\mathsf{M}(\mathfrak{Q}; p)$ and $\mathfrak{Q}$ does not centralize every element of $\mathsf{M}(\mathfrak{Q}; r)$. By Theorem 22.2, we have $p \sim r$.

If $r \in \pi_3$, then since also $q \in \pi_3$, we have $r \sim p$, by Theorem 22.7. This completes the proof of this lemma.

If $p \in \pi_4$ and $p_1 \in \pi(p)$, $p_1 \neq p$, let $\pi(p_1) = \{p, p_1, \cdots, p_n\}$. By Theorem 22.7 and Lemma 24.1, $\pi(p_i) = \pi(p_j)$, $1 \leq i, j \leq n$. It follows from $p \in \pi_4$ that $p_i \in \pi_3$, $1 \leq i \leq n$. By Theorem 22.5, $\mathfrak{G}$ satisfies $C_{\pi_3(p_1)}$. Let $\mathfrak{H}$ be a $S_{\pi_3(p_1)}$-subgroup of $\mathfrak{G}$. Clearly, $\mathfrak{H} \subset \mathfrak{G}$ since $p \notin \pi_3(p_1)$.

It is easy to see that $\boldsymbol{F}(\mathfrak{H})$ is non cyclic. Choose i so that the S_{p_i}-subgroup of $\boldsymbol{F}(\mathfrak{H})$ is non cyclic. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be a Sylow system for $\mathfrak{H}$, $\mathfrak{P}_j$ being a S_{p_j}-subgroup of $\mathfrak{H}$. Thus, $\mathfrak{P}_i \cap \boldsymbol{F}(\mathfrak{H})$ is non cyclic, so that $\mathfrak{P}_i \cap \boldsymbol{F}(\mathfrak{H})$ contains a subgroup $\mathfrak{B}$ of type (p, p) which is normal in $\mathfrak{P}_i$. Let $\mathfrak{A}$ be an element of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P}_i)$ which contains $\mathfrak{B}$. Let $\mathfrak{P}_0$ be a maximal element of $\mathsf{M}(\mathfrak{A}; p)$. By Lemma 24.1 and Theorem 22.6, $\mathfrak{P}_0 \neq \langle 1 \rangle$. Let $\boldsymbol{C}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, $\mathfrak{D}$ being a p_i'-group.

THEOREM 24.1. $\langle \mathfrak{P}_0, \boldsymbol{O}_{p_i'}(\mathfrak{H}), \mathfrak{D} \rangle$ *is a* p_i'-*group.*

Proof. Let $\mathscr{P}$ be the set of $\mathfrak{A}$-invariant subgroups $\tilde{\mathfrak{P}}_0$ of $\mathfrak{P}_0$ such that $\langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subset \mathfrak{G}$. Since $\langle O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq C(\mathfrak{B})$, it follows that $\langle 1\rangle \in \mathscr{P}$.

Suppose $\tilde{\mathfrak{P}}_0 \in \mathscr{P}$, and $\mathfrak{T} = \langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle$. Since $\mathfrak{A}$ normalizes $\mathfrak{T}$, $\langle\mathfrak{A}, \mathfrak{T}\rangle = \mathfrak{A}\mathfrak{T} = \mathfrak{L} \subset \mathfrak{G}$. By Lemma 17.6, $\mathfrak{A} \subseteq O_{p_i',p_i}(\mathfrak{L})$.

Let $\bar{\mathfrak{A}}$ be the image of $\mathfrak{A}$ under the projection of $O_{p_i',p_i}(\mathfrak{L})$ onto $O_{p_i',p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$. Since $\bar{\mathfrak{A}} \cong \mathfrak{A}$, we see that $\bar{\mathfrak{A}}$ is a self centralizing subgroup of $O_{p_i',p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$, and it follows readily that $O_{p_i',p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$ is centralized by $\tilde{\mathfrak{P}}_0$, $O_{p_i'}(\mathfrak{H})$ and $\mathfrak{D}$. By Lemma 1.2.3 of [21], we have $\langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq O_{p_i'}(\mathfrak{L})$ and hence $\mathfrak{T} = O_{p_i'}(\mathfrak{L})$ is a p_i'-group.

Let $\mathfrak{T}_1, \cdots, \mathfrak{T}_m$ be an $\mathfrak{A}$-invariant Sylow system of $\mathfrak{T}$, $\mathfrak{T}_j$ being a S_{q_j}-subgroup of $\mathfrak{T}$. If $q_j \in \{p_1, \cdots, p_n\}$, it follows from $C^*_{p_i,q_j}$ that $\mathfrak{T}_j$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q_j)$. Since $\mathfrak{D} \subseteq \mathfrak{T}$, this implies that $O_{p_i'}(\mathfrak{H})$ is a S-subgroup of $\mathfrak{T}$. If $q_j \neq p$, $q_j \notin \{p_1, \cdots, p_n\}$, then Theorem 22.1 implies that $\mathfrak{A}$ centralizes $\mathfrak{T}_j$, so that $\mathfrak{T}_j \subseteq \mathfrak{D}$. Finally, if $q_j = p$, then there is an element D of $\mathfrak{D}$ such that $\mathfrak{T}_j^D \subseteq \mathfrak{P}_0$, by Theorem 17.1.

Let $\mathfrak{K}$ be a fixed $S_{p'}$-subgroup of $\langle\mathfrak{D}, O_{p_i'}(\mathfrak{H})\rangle$. By the preceding paragraph, $\mathfrak{K}$ is a $S_{p'}$-subgroup of $\mathfrak{T}$, and $\mathfrak{P}_0 \cap \mathfrak{T}$ is a S_p-subgroup of $\mathfrak{T}$. Since $\tilde{\mathfrak{P}}_0 \subseteq \mathfrak{P}_0 \cap \mathfrak{T}$, it follows that $\langle\tilde{\mathfrak{P}}_0 \mid \tilde{\mathfrak{P}}_0 \in \mathscr{P}\rangle = \mathfrak{P}^*$ is permutable with $\mathfrak{K}$ so that $\mathfrak{P}^*\mathfrak{K}$ is a proper p_i'-subgroup of $\mathfrak{G}$. This means that $\mathscr{P}$ contains a unique maximal element. Since $C_{\mathfrak{P}_0}(B)$ is $\mathfrak{A}$-invariant for each $B \in \mathfrak{B}^\sharp$, since $\mathfrak{P}_0 = \langle C_{\mathfrak{P}_0}(B) \mid B \in \mathfrak{B}^\sharp\rangle$, and since $\langle C_{\mathfrak{P}_0}(B), O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq C(B) \subset \mathfrak{G}$, the theorem follows.

THEOREM 24.2. *Let* $\mathfrak{N} = \langle\mathfrak{P}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle$, *and* $\mathfrak{M} = N(\mathfrak{N})$. *Then* $\mathfrak{M}$ *contains* $\mathfrak{H}$, $\mathfrak{M}$ *is a maximal subgroup of* $\mathfrak{G}$ *and* $\mathfrak{M}$ *is the only maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. Since $\mathfrak{P}_0 \neq \langle 1\rangle$, $\mathfrak{M}$ is a proper subgroup of $\mathfrak{G}$. We first show that $\mathfrak{M}$ contains $\mathfrak{P}_i$. Let $\mathfrak{Q}$ be an $\mathfrak{A}$-invariant S_q-subgroup of $\mathfrak{N}$, so that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, either by virtue of $q \in \pi(p_i)$, or by virtue of $q \notin \pi(p_i)$ so that $\mathfrak{A}$ centralizes $\mathfrak{Q}$. For P in $\mathfrak{P}_i$, $\mathfrak{Q}^P = \mathfrak{Q}^D$ for some D in $\mathfrak{D}$ by Theorem 17.1 together with $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$. Since $\mathfrak{D} \subseteq \mathfrak{N}$, $\mathfrak{Q}^P$ is a S_q-subgroup of $\mathfrak{N}$. Hence, $\mathfrak{N}^P \subseteq \mathfrak{N}$, and so $\mathfrak{N}^P = \mathfrak{N}$. Thus, $\mathfrak{P}_i \subseteq \mathfrak{M}$.

To show that $\mathfrak{H} \subseteq \mathfrak{M}$, we use the fact that $\mathfrak{H} = O_{p_i'}(\mathfrak{H}) \cdot N_{\mathfrak{H}}(\mathfrak{B})$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_i)$. Since $O_{p_i'}(\mathfrak{H}) \subseteq \mathfrak{N}$, it suffices to show that $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{M}$. We will in fact show that $N(\mathfrak{B}) \subseteq \mathfrak{M}$. Let $\mathfrak{Q}$ be a $\mathfrak{B}$-invariant S_q-subgroup of $\mathfrak{N}$. If $N \in N(\mathfrak{B})$, then $\mathfrak{A}^{N^{-1}} \subseteq \mathfrak{B}$, so that $\mathfrak{A}^{N^{-1}}$ normalizes $\mathfrak{Q}$. Hence, $\mathfrak{A}$ normalizes $\mathfrak{Q}^N = \mathfrak{Q}^D$, $D \in \mathfrak{D}$, and we see that $\mathfrak{N}^N = \mathfrak{N}$. Thus, $\mathfrak{M}$ contains $\mathfrak{H}$ and $N(\mathfrak{B})$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{M}$. It is easy to see that $\mathfrak{N} = O_{p_i'}(\mathfrak{M}_1)$ by Lemma 7.3, so that $\mathfrak{M}_1 \subseteq \mathfrak{M}$, and $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$.

Let $\mathfrak{K}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$. To show that $\mathfrak{K} \subseteq \mathfrak{M}$, it suffices to treat the case that $\mathfrak{K}$ is a q, p_i-group. Let $\mathfrak{K}_q$ be a S_q-subgroup of $\mathfrak{K}$ permutable with $\mathfrak{P}_i$. Since $N(\mathfrak{B}) \subseteq \mathfrak{M}$, it suffices to show that $O_{p_i'}(\mathfrak{K}) \subseteq \mathfrak{M}$. This is clear by $C^*_{p_i,p}$ if $q \in \{p_1, \cdots, p_n\}$. If $q = p$, this is also clear, by Theorem 17.1, since $C(\mathfrak{A}) \subseteq \mathfrak{M}$ and $\mathfrak{P}_0 \subseteq \mathfrak{M}$. If $q \notin \{p, p_1, \cdots, p_n\}$, then $\mathfrak{P}_i$ centralizes $O_q(\mathfrak{P}_i\mathfrak{K}_q)$ by Theorem 22.1, and we are done, since $C(\mathfrak{A}) \subseteq \mathfrak{M}$.

If $q \in \pi_3 \cup \pi_4$, and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, we define

$$\mathscr{A}_1(\mathfrak{Q}) = \{\mathfrak{Q}_0 \mid \mathfrak{Q}_0 \subseteq \mathfrak{Q}, \mathfrak{Q}_0 \text{ contains some element of } \mathscr{SCN}_3(\mathfrak{Q})\}\,,$$

$$\mathscr{A}_i(\mathfrak{Q}) = \{\mathfrak{Q}_0 \mid \mathfrak{Q}_0 \subseteq \mathfrak{Q}, \mathfrak{Q}_0 \text{ contains a subgroup } \mathfrak{Q}_1 \text{ of type } (q, q) \text{ such that } C_{\mathfrak{Q}}(Q) \in \mathscr{A}_{i-1}(\mathfrak{Q}) \text{ for each } Q \text{ in } \mathfrak{Q}_1\},\ i = 2, 3, 4\,.$$

LEMMA 24.2. *If $q \in \pi_3 \cup \pi_4$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then every subgroup $\mathfrak{Q}_0$ of $\mathfrak{Q}$ which contains a subgroup of type (q, q, q) is in $\mathscr{A}_3(\mathfrak{Q})$.*

Proof. Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q})$, $\mathfrak{Q}_0^* = C_{\mathfrak{Q}_0}(\mathfrak{B})$, so that $\mathfrak{Q}_0^*$ is non cyclic. Let $\mathfrak{Q}_1$ be a subgroup of $\mathfrak{Q}_0^*$ of type (q, q). If $Q \in \mathfrak{Q}_1$, then $C_{\mathfrak{Q}}(Q) \supseteq \mathfrak{B}$. Since $\mathfrak{B}$ is contained in an element of $\mathscr{SCN}_3(\mathfrak{Q})$, it follows that $C_{\mathfrak{Q}}(Q)$ is in $\mathscr{A}_2(\mathfrak{Q})$.

THEOREM 24.3. *If $q \in \pi_3$, $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{Q}$ is contained in a unique maximal subgroup of $\mathfrak{G}$, then each element of $\mathscr{A}_1(\mathfrak{Q})$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. We remark that if this theorem is proved for the pair $(\mathfrak{Q}, \mathfrak{M})$, then it will also be proved for all pairs $(\mathfrak{Q}^M, \mathfrak{M})$ where $M \in \mathfrak{M}$. This prompts the following definition: $\mathscr{A}_1{}^*(\mathfrak{Q})$ is the set of all subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ such that $\mathfrak{Q}_0$ contains $\mathfrak{C}^M$ for some $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{Q})$ and some $M \in \mathfrak{M}$. Clearly $\mathscr{A}_1(\mathfrak{Q}) \subseteq \mathscr{A}_1{}^*(\mathfrak{Q})$.

Suppose some element of $\mathscr{A}_1{}^*(\mathfrak{Q})$ is contained in a maximal subgroup of $\mathfrak{G}$ different from $\mathfrak{M}$. Among all such elements $\mathfrak{Q}_0$ of $\mathscr{A}_1{}^*(\mathfrak{Q})$, let $|\mathfrak{Q}_0|$ be maximal. By hypothesis, $\mathfrak{Q}_0 \subset \mathfrak{Q}$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ different from $\mathfrak{M}$ which contains $\mathfrak{Q}_0$ and let $\mathfrak{Q}_0^*$ be a S_q-subgroup of $\mathfrak{M}_1$ which contains $\mathfrak{Q}_0$. If $\mathfrak{Q}_0 \subset \mathfrak{Q}_0^*$, then $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0)$. Since $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}}(\mathfrak{Q}_0)$, maximality of $|\mathfrak{Q}_0|$ implies that $N_{\mathfrak{G}}(\mathfrak{Q}_0) \subseteq \mathfrak{M}$, so that $N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0) \subseteq \mathfrak{Q}^M$ for some M in $\mathfrak{M}$. Since $\mathfrak{M}_1 \neq \mathfrak{M}$, so also $\mathfrak{M}_1^{M^{-1}} \neq \mathfrak{M}$. But $N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0)^{M^{-1}} \in \mathscr{A}_1{}^*(\mathfrak{Q})$, and maximality of $|\mathfrak{Q}_0|$ is violated. Hence, $\mathfrak{Q}_0$ is a S_q-subgroup of $\mathfrak{M}_1$.

Let $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{Q})$ be chosen so that $\mathfrak{C}^M \subseteq \mathfrak{Q}_0$ for some $M \in \mathfrak{M}$. Since every element of $\mathsf{M}(\mathfrak{C})$ is contained in $\mathfrak{M}$, every element of

$\mathsf{U}(\mathfrak{C}^M)$ is contained in $\mathfrak{M}^M = \mathfrak{M}$. Hence $O_{q'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$. If $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{Q}_0)$, then $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}}(\mathfrak{B})$, so $N_{\mathfrak{M}_1}(\mathfrak{B}) \subseteq \mathfrak{M}$, by maximality of $|\mathfrak{Q}_0|$. Since $\mathfrak{M}_1 = O_{q'}(\mathfrak{M}_1) \cdot N_{\mathfrak{M}_1}(\mathfrak{B})$ by Lemma 17.6, we find that $\mathfrak{M}_1 \subseteq \mathfrak{M}$, contrary to assumption. The theorem is proved.

THEOREM 24.4. *Let $q \in \pi_3 \cup \pi_4$, and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$. If each element of $\mathscr{A}_1(\mathfrak{Q})$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, then for each $i = 2, 3, 4$, and each element $\mathfrak{Q}_0$ of $\mathscr{A}_i(\mathfrak{Q})$, $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_0$.*

Proof. For $i = 2, 3, 4$, let $\mathscr{A}_i{}^*(\mathfrak{Q})$ be the set of subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ such that $\mathfrak{Q}_0$ contains a subgroup $\mathfrak{Q}_1$ of type (q, q) such that $C_{\mathfrak{M}}(Q)$ contains an element of $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ for some $M \in \mathfrak{M}$ and all $Q \in \mathfrak{Q}_1$. Here $\mathscr{A}_i{}^*(\mathfrak{Q}^M)$ denotes the set of $\mathfrak{Q}_0^M$, $\mathfrak{Q}_0 \in \mathscr{A}_i{}^*(\mathfrak{Q})$. Suppose $i = 2, 3$, or 4 is minimal with the property that some element of $\mathscr{A}_i{}^*(\mathfrak{Q})$ is contained in at least two maximal subgroups of $\mathfrak{G}$. This implies that $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ does not contain any elements which are contained in two maximal subgroups of $\mathfrak{G}$, M being an arbitrary element of $\mathfrak{M}$. Choose $\mathfrak{Q}_0$ in $\mathscr{A}_i{}^*(\mathfrak{Q})$ with $|\mathfrak{Q}_0|$ maximal subject to the condition that $\mathfrak{Q}_0$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ with $\mathfrak{M}_1 \neq \mathfrak{M}$. We see that $\mathfrak{Q}_0$ is a S_q-subgroup of $\mathfrak{M}_1$. Let $\mathfrak{Q}_1$ be a subgroup of $\mathfrak{Q}_0$ of type (q, q) with the property that $C_{\mathfrak{M}}(Q)$ contains an element of $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ for suitable M in $\mathfrak{M}$, and each Q in $\mathfrak{Q}_1$. (We allow M to depend on Q.) Since $O_{q'}(\mathfrak{M}_1)$ is generated by its subgroups $O_{q'}(\mathfrak{M}_1) \cap C(Q)$, $Q \in \mathfrak{Q}_1^\sharp$, it follows that $O_{q'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$.

Let $\mathfrak{C}$ be an element of $\mathscr{SCN}_3(\mathfrak{Q})$. Then $\mathfrak{C} \not\subseteq \mathfrak{Q}_0$, or we are done. Let $\tilde{\mathfrak{Q}}_0 = \mathfrak{Q}_0 \cap O_{q',q}(\mathfrak{M}_1)$. Since $\tilde{\mathfrak{Q}}_0 \cap \mathfrak{C} = \mathfrak{Q}_0 \cap \mathfrak{C}$ by (B), it follows that $N_{\mathfrak{Q}}(\tilde{\mathfrak{Q}}_0) \supset \mathfrak{Q}_0$. Hence, $N(\tilde{\mathfrak{Q}}_0) \subseteq \mathfrak{M}$, by maximality of $|\mathfrak{Q}_0|$. Since $\mathfrak{M}_1 = O_{q'}(\mathfrak{M}_1) \cdot N_{\mathfrak{M}_1}(\tilde{\mathfrak{Q}}_0)$, we have $\mathfrak{M}_1 \subseteq \mathfrak{M}$, completing the proof of this theorem, since $\mathscr{A}_i(\mathfrak{Q}) \subseteq \mathscr{A}_i{}^*(\mathfrak{Q})$, $i = 2, 3, 4$.

THEOREM 24.5. *If $q \in \pi_3$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. If $\pi(q) \subseteq \pi_3$, this theorem follows from Lemma 23.5. Suppose $p \in \pi(q) \cap \pi_4$. Let $\pi(q) = \{p, p_1, \cdots, p_n\}$, where $q = p_1$, and let $\mathfrak{H}$ be a $S_{\pi_3(p_1)}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. If $\mathfrak{Q} \cap F(\mathfrak{H})$ is non cyclic, we are done by Theorem 24.2, so we suppose that $\mathfrak{Q} \cap F(\mathfrak{H})$ is cyclic.

Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. Suppose we are able to show that $C(\mathfrak{C}) \subseteq \mathfrak{M}$ for some $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{Q})$. Since $F(\mathfrak{H}) \cap \mathfrak{Q}$ is cyclic, $F(\mathfrak{M}) \cap \mathfrak{Q}$ is also cyclic. Hence, $O_{q'}(\mathfrak{M}) \neq 1$. If $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{Q})$, then $N(\mathfrak{B})$ normalizes $O_{q'}(\mathfrak{M})$, by Theorem 17.1, together with $C(\mathfrak{C}) \subseteq \mathfrak{M}$. Since $\mathfrak{K} = O_{q'}(\mathfrak{K}) \cdot N_{\mathfrak{K}}(\mathfrak{B})$ for every proper subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which contains $\mathfrak{Q}$, it suffices to show that every element of $\mathscr{N}(\mathfrak{Q})$ is contained in $\mathfrak{M}$. This follows readily by C^*_{q,p_j}, Theorem 22.1 and $C(\mathfrak{C}) \subseteq \mathfrak{M}$.

Thus, it suffices to show that $C(\mathfrak{C}) \subseteq \mathfrak{M}$. Choose i such that $\mathfrak{F}_i$, a S_{p_i}-subgroup of $F(\mathfrak{H})$, is non cyclic, and let $\mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{H}$ permutable with $\mathfrak{Q}$. It suffices to show that $C_{\mathfrak{F}_i}(C) \in \mathscr{SCN}_4(\mathfrak{P}_i)$ for some $C \in \mathfrak{C}^{\#}$, by Theorems 24.3 and 24.4 together with the fact that $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$.

Let $\mathfrak{F}_i^* = O_{p_i}(\mathfrak{Q}\mathfrak{P}_i)$, so that $\mathfrak{F}_i^*$ is a maximal element of $\mathscr{N}(\mathfrak{Q}; p_i)$. By Lemma 17.3, $\mathfrak{Q} \subseteq N(\mathfrak{F}_i^*)'$. Since $\mathfrak{P}_i$ is contained in $\mathfrak{M}$ and no other maximal subgroup, $\mathfrak{Q} \subseteq \mathfrak{M}'$. Thus, if $\Omega_1(Z_2(\mathfrak{F}_i))$ is generated by two elements, then $\mathfrak{Q}$ centralizes $Z_2(\mathfrak{F}_i)$ and we are done. If $Z(\mathfrak{F}_i)$ is non cyclic, then every subgroup of $Z(\mathfrak{F}_i)$ of type (p_i, p_i) is contained in $\mathscr{SCN}_4(\mathfrak{P}_i)$. Since $\mathfrak{C}$ contains a subgroup of type (q, q, q), $C(C) \cap Z(\mathfrak{F}_i)$ is non cyclic for some C in $\mathfrak{C}^{\#}$, and we are done in this case. There remains the possibility that $Z(\mathfrak{F}_i)$ is cyclic, while $\Omega_1(Z_2(\mathfrak{F}_i))$ is not generated by two elements. Since every subgroup of $\Omega_1(Z_2(\mathfrak{F}_i))$ of type (p_i, p_i) which contains $\Omega_1(Z(\mathfrak{F}_i))$ is contained in $\mathscr{SCN}_4(\mathfrak{P}_i)$, by Lemma 24.2, and since $C(C)$ contains such a subgroup for some C in $\mathfrak{C}^{\#}$, we are done.

The preceding theorems give precise information regarding the S_q-subgroups of the maximal subgroups of $\mathfrak{G}$ for q in π_3.

THEOREM 24.6. *Let $q \in \pi_3$ and let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$. If $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{M}$ and $\mathfrak{Q}$ is not a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ contains a cyclic subgroup of index at most q.*

Proof. Let $\mathfrak{Q}^*$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q}^*)$ and let $\mathfrak{Q}_0 = C_{\mathfrak{Q}}(\mathfrak{B})$ so that $|\mathfrak{Q}:\mathfrak{Q}_0| = 1$ or q. If $\mathfrak{Q}_0$ is non cyclic, then $\mathfrak{Q}_0 \in \mathscr{SCN}_4(\mathfrak{Q}^*)$, and so $\mathfrak{Q}_0$ is contained in a unique maximal subgroup of $\mathfrak{G}$, which must be $\mathfrak{M}$, since $\mathfrak{Q}_0 \subseteq \mathfrak{M}$. But $\mathfrak{Q}^* \not\subseteq \mathfrak{M}$, a contradiction, so $\mathfrak{Q}_0$ is cyclic, as required.

Theorem 24.6 is of interest in its own right, and plays an important role in the study of π_4, to which all the preceding results are now turned.

Hypothesis 24.1.

1. $3 \in \pi_4$.
2. *$\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{G}$.*
3. *$\mathfrak{K}$ is a proper subgroup of $\mathfrak{G}$ such that*
   (i) $\mathfrak{P} \subseteq \mathfrak{K}$.
   (ii) *If $\mathfrak{H} = O_3(\mathfrak{K})$, there is a subgroup $\mathfrak{C}$ of $\mathfrak{H}$ chosen in*

accordance with Lemma 8.2 *such that* $Z(\mathfrak{C})$ *is generated by two elements.*

THEOREM 24.7. *Under Hypothesis* 24.1, $\mathfrak{P}$ *is contained in a unique maximal subgroup* $\mathfrak{M}$ *of* $\mathfrak{G}$, *and* $\mathfrak{M}$ *centralizes* $Z(\mathfrak{P})$.

Proof. Let $\mathfrak{L}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. We must show that $\mathfrak{L}$ centralizes $Z(\mathfrak{P})$.

By Lemma 8.2, $\ker(\mathfrak{K} \rightarrow \mathrm{Aut}\,\mathfrak{C})$ is a 3-group, so is contained in $\mathfrak{H}$. It follows that $C_{\mathfrak{K}}(\mathfrak{C}) = Z(\mathfrak{C})$ and in particular $C_{\mathfrak{P}}(\mathfrak{C}) = Z(\mathfrak{C})$.

Suppose $\mathfrak{C} \subseteq O_3(\mathfrak{L})$. Then $Z(O_3(\mathfrak{L})) \subseteq C_{\mathfrak{P}}(\mathfrak{C}) \subseteq Z(\mathfrak{C})$, so $Z(O_3(\mathfrak{L}))$ is generated by two elements. Since $|\mathfrak{L}|$ is odd, a $S_{3'}$-subgroup of $\mathfrak{L}$ centralizes $Z(O_3(\mathfrak{L}))$, so centralizes $Z(\mathfrak{P})$. Since $\mathfrak{P}$ also centralizes $Z(\mathfrak{P})$, we have $\mathfrak{L} \subseteq C(Z(\mathfrak{P}))$.

Suppose $\mathfrak{C} \not\subseteq O_3(\mathfrak{L})$. Since $Z(\mathfrak{C})$ is a normal abelian subgroup of $\mathfrak{P}$ we have $Z(\mathfrak{C}) \subseteq O_3(\mathfrak{L})$. Since $\gamma^2\mathfrak{P}\mathfrak{C}^2 \subseteq Z(\mathfrak{C})$, we conclude that $\mathfrak{C} \subseteq O_{3,3',3}(\mathfrak{L})$. By the preceding paragraph, $N(\mathfrak{P} \cap O_{3,3',3}(\mathfrak{L}))$ centralizes $Z(\mathfrak{P})$. Thus, it suffices to show that $\mathfrak{P}O_{3,3'}(\mathfrak{L}) = \mathfrak{L}_1$ centralizes $Z(\mathfrak{P})$. Since $\mathfrak{L}_1 = N_{\mathfrak{L}_1}(O_3(\mathfrak{L})\mathfrak{C}) \cdot [O_{3,3'}(\mathfrak{L}), \mathfrak{C}]$, and since $\mathfrak{P}$ normalizes $O_3(\mathfrak{L})\mathfrak{C}$, it suffices to show that $[O_{3,3'}(\mathfrak{L}), \mathfrak{C}]$ centralizes $Z(\mathfrak{P})$. Let $\mathfrak{Z} = Z(O_3(\mathfrak{L}))$, so that $\mathfrak{Z}$ contains $Z(\mathfrak{P})$. Since $\mathfrak{Z}$ is a normal abelian subgroup of $\mathfrak{P}$, (B) implies that $\mathfrak{Z} \subseteq O_3(\mathfrak{K})$. Hence, $\gamma^2\mathfrak{Z}\mathfrak{C}^2 = 1$, which implies that $[O_{3,3'}(\mathfrak{L}), \mathfrak{C}]$ induces only 3-automorphisms on $\mathfrak{Z}$, and suffices to complete the proof.

Hypothesis 24.2.

1. $3 \in \pi_4$.
2. $\mathfrak{P}$ *is a* S_3*-subgroup of* $\mathfrak{G}$.
3. *If* $\mathfrak{K}$ *is any proper subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}$, *and if* $\mathfrak{H} = O_3(\mathfrak{K})$, *then every subgroup* $\mathfrak{C}$ *of* $\mathfrak{H}$ *chosen in accordance with Lemma* 8.2 *satisfies* $m(Z(\mathfrak{C})) \geq 3$.

REMARK. If $3 \in \pi_4$, then Hypothesis 24.1 and Hypothesis 24.2 exhaust all possibilities.

LEMMA 24.3. *Under Hypothesis* 24.2, $\mathfrak{P}$ *contains an element* $\mathfrak{B}$ *of* $\mathscr{U}(\mathfrak{P})$ *such that the normal closure of* $\mathfrak{B}$ *in* $C(\Omega_1(Z(\mathfrak{P})))$ *is abelian.*

Proof. If $Z(\mathfrak{P})$ is non cyclic, every element of $\mathscr{U}(\mathfrak{P})$ satisfies this lemma. Otherwise, set $\mathfrak{K} = C(\Omega_1(Z(\mathfrak{P})))$, and let $\mathfrak{A}$ be a non cyclic normal abelian subgroup of $\mathfrak{K}$. Since $\mathfrak{A} \lhd \mathfrak{P}$, $\mathfrak{A}$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{P})$ which meets the demands of this lemma.

THEOREM 24.8. *Let* $p \in \pi_4$ *and let* $\mathfrak{P}$ *be a* S_p*-subgroup of* $\mathfrak{G}$. *If* $p = 3$, *assume that* $\mathscr{U}(\mathfrak{P})$ *contains an element* $\mathfrak{B}$ *such that the normal* *closure of* $\mathfrak{B}$ *in* $C(\Omega_1(Z(\mathfrak{P})))$ *is abelian. If* $p \geq 5$, *let* $\mathfrak{B}$ *be any element of* $\mathscr{U}(\mathfrak{P})$. *If* $\mathfrak{K}$ *is any proper subgroup of* $\mathfrak{G}$ *such that* $O_{p'}(\mathfrak{K}) = 1$ *and if* $\mathfrak{K}_p$ *is a* S_p*-subgroup of* $\mathfrak{K}$, *then* $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, *where* $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{K}_p)$, *and* $\mathfrak{L}$ *is the largest normal subgroup of* $\mathfrak{K}$ *which centralizes* $Z(\mathfrak{K}_p)$.

Proof. Observe that $\mathfrak{L}$ contains $O_p(\mathfrak{K})$.

Since $O_p(\mathfrak{K} \bmod \mathfrak{L}) = \mathfrak{L} \cdot (\mathfrak{K}_p \cap O_p(\mathfrak{K} \bmod \mathfrak{L}))$, maximality of $\mathfrak{L}$ guarantees that $\mathfrak{L} = O_p(\mathfrak{K} \bmod \mathfrak{L})$. If $\mathfrak{B} \subseteq \mathfrak{L}$, then Sylow's theorem yields this theorem since $\mathfrak{B}$ is weakly closed in $\mathfrak{L} \cap \mathfrak{K}_p$.

Suppose by way of contradiction that $\mathfrak{B} \not\subseteq \mathfrak{L}$. Let $\mathfrak{L}_1 = O_{p'}(\mathfrak{K} \bmod \mathfrak{L})$. By Lemma 1.2.3 of [21], $\gamma\mathfrak{B}\mathfrak{L}_1 \not\subseteq \mathfrak{L}$.

Let $\mathfrak{P}_1 = \mathfrak{K}_p \cap \mathfrak{L}$, and let $\tilde{\mathfrak{L}}_1 = \mathfrak{L}_1 \cap N_{\mathfrak{K}}(\mathfrak{P}_1)$. Let $\mathfrak{B}_1$ be the normal closure of $\mathfrak{B}$ in $N_{\mathfrak{K}}(\mathfrak{P}_1)$. Suppose $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \subseteq C(Z(\mathfrak{K}_p))$. Since $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \lhd N_{\mathfrak{K}}(\mathfrak{P}_1)$, and since $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{P}_1)$ by Sylow's theorem, we see that $\mathfrak{L} \cdot \gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \lhd \mathfrak{K}$. Maximality of $\mathfrak{L}$ implies that $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \subseteq \mathfrak{L}$. In particular, $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B} \subseteq \mathfrak{L}$. Since $\mathfrak{L}_1 = \mathfrak{L} \cdot \tilde{\mathfrak{L}}_1$, by Sylow's theorem we have $\mathfrak{B} \subseteq O_p(\mathfrak{K} \bmod \mathfrak{L})$, which is not the case. Hence, $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \not\subseteq C(Z(\mathfrak{K}_p))$. Since $Z(\mathfrak{P}_1) \supseteq Z(\mathfrak{K}_p)$, we also have $\gamma\tilde{\mathfrak{L}}_1\mathfrak{B}_1 \not\subseteq C(Z(\mathfrak{P}_1))$. Since $\langle \mathfrak{B}_1, \tilde{\mathfrak{L}}_1 \rangle \subseteq N(Z(\mathfrak{P}_1))$, the identity $[X, YZ] = [X, Z][X, Y]^Z$ implies that $\mathfrak{B}_1$ contains a conjugate $\mathfrak{B}_1 = \mathfrak{B}^G$ of $\mathfrak{B}$ such that $\gamma\mathfrak{L}_1\mathfrak{B}_1 \not\subseteq C(Z(\mathfrak{P}_1))$. Since $\tilde{\mathfrak{L}}_1 \subseteq N_{\mathfrak{K}}(\mathfrak{P}_1)$, application of Theorem C of [21] to $\mathfrak{B}_1\tilde{\mathfrak{L}}_1/\tilde{\mathfrak{L}}_1 \cap C(Z(\mathfrak{P}_1))$ yields a special q-group $\bar{\mathfrak{Q}} = \mathfrak{Q}/\mathfrak{L}_1 \cap C(Z(\mathfrak{P}_1))$ such that $\mathfrak{B}_1$ acts irreducibly and non trivially on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$. Since $\bar{\mathfrak{Q}}$ is a p'-group, and $\bar{\mathfrak{Q}}$ does not centralize $Z(\mathfrak{P}_1)$, $\bar{\mathfrak{Q}}$ does not centralize $\mathfrak{W} = \Omega_1(Z(\mathfrak{P}_1))$. Furthermore, $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$, where $\mathfrak{W}_1 = C_{\mathfrak{W}}(\mathfrak{Q})$ and $\mathfrak{W}_2 = \gamma\mathfrak{W}\mathfrak{Q}$, and $\mathfrak{W}_i$ is invariant under $\mathfrak{B}_1\mathfrak{Q}$, $i = 1, 2$.

Since $\mathfrak{W}_2$ is a p-group and $\mathfrak{W}_2 \neq 1$, we have $\mathfrak{W}_3 \neq 1$, where $\mathfrak{W}_3 = C_{\mathfrak{W}_2}(\mathfrak{B}_0)$ and $\mathfrak{B}_0 = \ker(\mathfrak{B}_1 \rightarrow \operatorname{Aut} \tilde{\mathfrak{Q}}) \neq 1$. If $p \geq 5$, Lemma 18.1 gives an immediate contradiction. If $p = 3$, and $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 = 1$, we also have a contradiction with (B), since $\gamma\mathfrak{W}_3\mathfrak{Q} \neq 1$. If $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 \neq 1$, Lemma 16.3 implies that $Z(\mathfrak{P})$ is cyclic, and that $\mathfrak{B}_0 = \Omega_1(Z(\mathfrak{P}^G))$. However, the normal closure of $\mathfrak{B}_1$ in $C(\Omega_1(Z(\mathfrak{P}^G)))$ is abelian, and so $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 = 1$, the desired contradiction, completing the proof of this theorem.

REMARK. Except for the case $p = 3$, and the side conditions $O_{p'}(\mathfrak{K}) = 1$ and $\mathfrak{L} \lhd \mathfrak{K}$, Theorem 24.8 is a repetition of Lemma 18.1.

Hypothesis 24.3.

1. $p \in \pi_4$, $q \in \pi(p)$, $q \neq p$.
2. $\mathfrak{Q}$ *is a* S_q*-subgroup of* $\mathfrak{G}$, $\mathfrak{P}_0$ *is a maximal element of* $\mathsf{M}(\mathfrak{Q}; p)$, *and* $\mathfrak{P}_1$ *is a* S_p*-subgroup of* $N(\mathfrak{P}_0)$ *permutable with* $\mathfrak{Q}$.
3. $\mathfrak{P}$ *is a* S_p*-subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_1$, *and* $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, *where*

for $p = 3$, *the normal closure of* $\mathfrak{B}$ *in* $C(\Omega_1(Z(\mathfrak{P})))$ *is abelian.*

4. $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P}_1)$.

THEOREM 24.9. *Under Hypothesis* 24.3, *either* $N_{\mathfrak{Q}}(\mathfrak{B})$ *contains an element of* $\mathscr{A}_4(\mathfrak{Q})$ *or* $C_{\mathfrak{Q}}(Z(\mathfrak{P}_0))$ *contains an element of* $\mathscr{A}_4(\mathfrak{Q})$. *Furthermore,* $\mathfrak{P}_1 = \mathfrak{P}$ *and* $\mathfrak{G}$ *satisfies* $C_{\pi(q)}$.

Proof. Let $\mathfrak{L}$ be the largest normal subgroup of $\mathfrak{K} = N(\mathfrak{P}_0)$ which centralizes $Z(\mathfrak{P}_0)$. Then $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, by Theorem 24.8. Since $\mathfrak{L} \lhd \mathfrak{K}$, $\mathfrak{L} \cap \mathfrak{Q} \lhd \mathfrak{Q}$. If $\mathfrak{L} \cap \mathfrak{Q}$ is non cyclic, then $\mathfrak{L} \cap \mathfrak{Q} \in \mathscr{A}_4(\mathfrak{Q})$.

Suppose $\mathfrak{L} \cap \mathfrak{Q}$ is a non identity cyclic group. By Lemma 17.6, $\mathfrak{Q} \subseteq \mathfrak{K}'$. Since a Sylow q-subgroup of $\mathfrak{L}$ is cyclic, it follows that $\mathfrak{K}'$ centralizes $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, where $\mathfrak{L}_0 = O_{q'}(\mathfrak{L})$, and so $\mathfrak{Q} \cap \mathfrak{L} \subseteq Z(\mathfrak{Q})$. If $\mathfrak{B}$ centralizes $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, then $N_{\mathfrak{K}}(\mathfrak{B})$ contains a S_q-subgroup of $\mathfrak{K}$. In this case, $\mathfrak{Q}$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$. Let $\langle \mathfrak{Q}, \mathfrak{P}_1^* \rangle$ be a $S_{q,p}$-subgroup of $\mathfrak{K}$ containing $\mathfrak{Q}\mathfrak{B}^K$, with $\mathfrak{B}^K \subseteq \mathfrak{P}_1^*$. By the conjugacy of Sylow systems in $\mathfrak{K}$, we have $\mathfrak{P}_1^{*K_1} = \mathfrak{P}_1$, $\mathfrak{Q}^{K_1} = \mathfrak{Q}$ for suitable K_1 in $\mathfrak{K}$. Hence, $\mathfrak{Q}$ normalizes $\mathfrak{B}^{KK_1}$ and $\mathfrak{B}^{KK_1} \subseteq \mathfrak{P}_1$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, $\mathfrak{B} = \mathfrak{B}^{KK_1}$ and we are done. If $\mathfrak{B}$ does not centralize $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, then $N(\mathfrak{B}) \cap \mathfrak{L}$ is a q'-group, since $\mathfrak{Q} \cap \mathfrak{L}$ is cyclic. In this case the factorization, $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, together with $\mathfrak{Q} \cap \mathfrak{L} \subseteq Z(\mathfrak{Q})$, yields that $\mathfrak{Q} = \mathfrak{Q} \cap \mathfrak{L} \times \mathfrak{Q}_1$, for some subgroup $\mathfrak{Q}_1$ of $\mathfrak{Q}$. This in turn implies that every non cyclic subgroup of $\mathfrak{Q}$ is in $\mathscr{A}_4(\mathfrak{Q})$.

Since $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$ and $\mathfrak{L} \cap \mathfrak{Q}$ is cyclic, the S_q-subgroups of $N_{\mathfrak{K}}(\mathfrak{B})$ are non cyclic. Hence, $\mathfrak{Q}$ contains a non cyclic subgroup $\mathfrak{Q}_0$ such that $\mathfrak{Q}_0$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$. By the conjugacy of Sylow systems, we can find K_1 in $\mathfrak{K}$ such that $\mathfrak{B}^{KK_1} \subseteq \mathfrak{P}_1$ and $\mathfrak{Q}_0^{K_1} \subseteq \mathfrak{Q}$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, $\mathfrak{B} = \mathfrak{B}^{KK_1}$, and we are done, since every non cyclic subgroup of $\mathfrak{Q}$ is contained in $\mathscr{A}_4(\mathfrak{Q})$.

Suppose $\mathfrak{L} \cap \mathfrak{Q} = \langle 1 \rangle$. Then $\mathfrak{L}$ is a q'-group. From $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, we conclude that $\mathfrak{Q}$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$ and the conjugacy of Sylow systems, together with the fact that $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, imply that $\mathfrak{Q}$ normalizes $\mathfrak{B}$. This completes the proof of the first assertion of the theorem.

If $\mathfrak{P}_1 \subset \mathfrak{P}$, then $\mathfrak{P}_1 \subset N_{\mathfrak{P}}(\mathfrak{B})$. Since every element of $\mathscr{A}_4(\mathfrak{Q})$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, by Theorem 24.3, if $N(\mathfrak{B})$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$, then $\mathfrak{P}_1$ is not a S_p-subgroup of $\mathfrak{M}$. But $\mathfrak{P}_1\mathfrak{Q}$ is a maximal p, q-subgroup of $\mathfrak{G}$, by Lemma 7.3. If $C(Z(\mathfrak{P}_0))$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$, then since $Z(\mathfrak{P}_0) \supseteq Z(\mathfrak{P})$ by (B) and Theorem 22.7, we see that $C(Z(\mathfrak{P}))$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$. Hence, $\mathfrak{P} \subseteq \mathfrak{M}$. Thus, in all cases, $\mathfrak{P} \subseteq \mathfrak{M}$. Since $\mathfrak{M}$ also contains a $S_{\pi_3(q)}$-subgroup of $\mathfrak{G}$, $\mathfrak{G}$ satisfies $E_{\pi(q)}$. Since $\mathfrak{Q}$ is contained in $\mathfrak{M}$ and no other maximal subgroup of $\mathfrak{G}$, $\mathfrak{G}$ satisfies $C_{\pi(q)}$ as required.

Hypothesis 24.4.

1. $3 \in \pi_4$.
2. $\mathfrak{P}$ *is a* S_3*-subgroup of* $\mathfrak{G}$.
3. $\mathfrak{P}$ *contains a subgroup* $\mathfrak{A}$ *which is elementary of order* 27 *with the property that* $\gamma^2 C(A)\mathfrak{A}^2 = 1$ *for all* $A \in \mathfrak{A}^\#$.

Hypothesis 24.5.

1. $p \in \pi_4$.
2. *A* S_p*-subgroup* $\mathfrak{P}$ *of* $\mathfrak{G}$ *is contained in at least two maximal subgroups of* $\mathfrak{G}$.

LEMMA 24.4. *Assume that Hypothesis* 24.5 *is satisfied and that if* $p = 3$, *Hypothesis* 24.4 *is also satisfied. If* $p \geqq 5$, *let* $\mathfrak{A}$ *be an arbitrary element of* $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. *If* $p = 3$, *let* $\mathfrak{A}$ *be the subgroup given in Hypothesis* 24.4. *Let* $\mathfrak{W}$ *be the weak closure of* $\mathfrak{A}$ *in* $\mathfrak{P}$, *and let* $\mathfrak{W}^*$ *be the subgroup of* $\mathfrak{P}$ *generated by its subgroups* $\mathfrak{B}$ *such that* $\mathfrak{B} \subseteq \mathfrak{A}^G$ *and* $\mathfrak{A}^G/\mathfrak{B}$ *is cyclic for suitable* G *in* $\mathfrak{G}$. *Let* $\mathfrak{M}$ *be a proper subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}$, *with the properties that* $\mathfrak{M}$ *is a* p, q*-group for some prime* q *and* $\mathfrak{M}$ *has* p*-length at most two. Let* $(\mathfrak{X}, \mathfrak{Y})$ *be any one of the pairs* $(\boldsymbol{Z}(\mathfrak{P}), \mathfrak{W})$, $(\boldsymbol{Z}(\mathfrak{W}^*), \mathfrak{W})$, $(\boldsymbol{Z}(\mathfrak{P}), \mathfrak{W}^*)$. *Then* $\mathfrak{M} = \mathfrak{M}_1\mathfrak{M}_2$, *where* $\mathfrak{M}_1$ *normalizes* $\mathfrak{X}$ *and* $\mathfrak{M}_1/C_{\mathfrak{M}_1}(\mathfrak{X})$ *is a* p*-group, and* $\mathfrak{M}_2$ *normalizes* $\mathfrak{Y}$.

Proof. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$, and let $\mathfrak{H} = \boldsymbol{O}_p(\mathfrak{M})$. Then $\mathfrak{H}\mathfrak{Q} \lhd \mathfrak{M}$. The lemma will follow immediately if we can show that $\gamma\mathfrak{Q}\mathfrak{Y}$ normalizes $\mathfrak{X}$ and induces only p-automorphisms on $\mathfrak{X}$.

Suppose by way of contradiction that either some element of $\gamma\mathfrak{Q}\mathfrak{Y}$ induces a non trivial q-automorphism on $\mathfrak{X}$, or $\gamma\mathfrak{Q}\mathfrak{Y}$ does not normalize $\mathfrak{X}$. If $\mathfrak{Y} = \mathfrak{W}$, we can find $\mathfrak{B} = \mathfrak{A}^G \subseteq \mathfrak{Y}$ such that either some element of $\gamma\mathfrak{Q}\mathfrak{B}$ induces a non trivial q-automorphism of $\mathfrak{X}$ or else $\gamma\mathfrak{Q}\mathfrak{B}$ does not normalize $\mathfrak{X}$. Similarly, if $\mathfrak{Y} = \mathfrak{W}^*$, we can find $\mathfrak{B} \subseteq \mathfrak{Y}$ and G in $\mathfrak{G}$ such that $\mathfrak{B} \subseteq \mathfrak{A}^G$, $\mathfrak{A}^G/\mathfrak{B}$ is cyclic and such that either some element of $\gamma\mathfrak{Q}\mathfrak{B}$ induces a non trivial q-automorphism of $\boldsymbol{Z}(\mathfrak{P})$ or else $\gamma\mathfrak{Q}\mathfrak{B}$ does not normalize $\boldsymbol{Z}(\mathfrak{P})$.

Let $\bar{\mathfrak{Q}} = \mathfrak{Q}\mathfrak{H}/\mathfrak{H}$, so that $\gamma\bar{\mathfrak{Q}}\mathfrak{B} = (\gamma\mathfrak{Q}\mathfrak{B})\mathfrak{H}/\mathfrak{H}$. Since $\gamma\bar{\mathfrak{Q}}\mathfrak{B}$ is generated by the subgroups $\gamma\bar{\mathfrak{Q}}_1\mathfrak{B}$ which have the property that $\mathfrak{B}$ acts irreducibly and non trivially on $\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1)$, we can find $\bar{\mathfrak{Q}}_1 = \mathfrak{Q}_1\mathfrak{H}/\mathfrak{H}$ such that $\gamma\mathfrak{Q}_1\mathfrak{B}$ either does not normalize $\mathfrak{X}$ or some element of $\gamma\mathfrak{Q}_1\mathfrak{B}$ induces a non trivial q-automorphism on $\mathfrak{X}$, and with the additional property that $\mathfrak{B}$ acts irreducibly on $\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1)$.

Let $\mathfrak{B}_0 = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\bar{\mathfrak{Q}}_1) = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1))$, so that $\mathfrak{B}/\mathfrak{B}_0$ is cyclic. Let $\mathfrak{M}_1 = \mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$, and $\mathfrak{H}_1 = \boldsymbol{O}_p(\mathfrak{M}_1)$. Since $\mathfrak{H}\mathfrak{B} \subseteq \mathfrak{P}$, and since

$\boldsymbol{Z}(\mathfrak{P}) \subseteq \mathfrak{H}$, it follows that $\boldsymbol{Z}(\mathfrak{P}) \subseteq \boldsymbol{Z}(\mathfrak{H}_1)$. Also, since $\boldsymbol{Z}(\mathfrak{W}^*)$ is a normal abelian subgroup of $\mathfrak{P}$, we have $\boldsymbol{Z}(\mathfrak{W}^*) \subseteq \mathfrak{H}$.

Suppose that $\mathfrak{X} = \boldsymbol{Z}(\mathfrak{P})$. If $p = 3$, then since $\mathfrak{A}^G/\mathfrak{B}_0$ is generated by two elements, it follows that $\mathfrak{B}_0 \neq \langle 1 \rangle$. Hence, $\boldsymbol{Z}(\mathfrak{H}_1) \subseteq \boldsymbol{C}(\mathfrak{B}_0)$. Since the normal closure of $\mathfrak{A}^G$ in $\boldsymbol{C}(\mathfrak{B}_0)$ is abelian, we have $\gamma^2\boldsymbol{Z}(\mathfrak{H}_1)\mathfrak{B}^2 = \langle 1 \rangle$, and (B) implies that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $\boldsymbol{Z}(\mathfrak{H}_1)$, so centralizes $\boldsymbol{Z}(\mathfrak{P})$.

Suppose $p \geqq 5$. We first treat the case that $\mathfrak{H}_1 \cap \mathfrak{U} \neq \langle 1 \rangle$ for some $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{A}^G$. Then $\langle \boldsymbol{Z}(\mathfrak{H}_1), \mathfrak{A}^G \rangle \subseteq \boldsymbol{C}(\mathfrak{H}_1 \cap \mathfrak{U}) = \mathfrak{C}$ and $\mathfrak{P}^G \cap \mathfrak{C}$ is of index at most p in $\mathfrak{P}^G$. If $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}^G \cap \mathfrak{C}$, then $\mathfrak{P}^G \cap \mathfrak{C} \lhd \mathfrak{P}^*$. Hence, $\gamma\mathfrak{P}^*\mathfrak{B} \subseteq \gamma\mathfrak{P}^*\mathfrak{A}^G \subseteq \mathfrak{P}^G \cap \mathfrak{C}$, and so $\gamma\mathfrak{P}^*\mathfrak{B}^3 = \langle 1 \rangle$. It follows that $\mathfrak{B} \subseteq \boldsymbol{O}_p(\mathfrak{C})$. (Note that $\boldsymbol{O}_{p'}(\mathfrak{C}) = \langle 1 \rangle$ since $\mathfrak{A}^G \subseteq \mathfrak{C}$.) Hence, $\gamma\boldsymbol{Z}(\mathfrak{H}_1)\mathfrak{B} \subseteq \boldsymbol{O}_p(\mathfrak{C})$, and so $\gamma^4\boldsymbol{Z}(\mathfrak{H}_1)\mathfrak{B}^4 = \langle 1 \rangle$, so that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $\boldsymbol{Z}(\mathfrak{H}_1)$ and so centralizes $\boldsymbol{Z}(\mathfrak{P})$.

We can now suppose that $\mathfrak{H}_1 \cap \mathfrak{U} = \langle 1 \rangle$ for all $\mathfrak{U}$ such that $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{A}^G$. In this case, since $\mathfrak{A}^G/\mathfrak{B}_0$ is generated by two elements, there is a normal elementary subgroup $\mathfrak{E}$ of $\mathfrak{P}^G$ of order p^3 such that $\mathfrak{E} \subseteq \mathfrak{A}^G$. Hence, $\mathfrak{E} \cap \mathfrak{B}_0 \neq \langle 1 \rangle$. Since $\mathfrak{E} \cap \mathfrak{B}_0 \subseteq \mathfrak{E} \cap \mathfrak{H}_1$, we can find E in $\mathfrak{E} \cap \mathfrak{H}_1^\#$. Consider $\boldsymbol{C}(E) \supseteq \langle \boldsymbol{Z}(\mathfrak{H}_1), \boldsymbol{C}_{\mathfrak{P}^G}(E) \rangle$. Since $\mathfrak{A}^G/\mathfrak{B}$ is cyclic, if $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$ and $\mathfrak{U} \subseteq \mathfrak{A}^G$, then $\mathfrak{B} \cap \mathfrak{U} = \mathfrak{U}_0 \neq \langle 1 \rangle$. Let $U \in \mathfrak{U}_0^\#$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\boldsymbol{C}(E)$ containing $\boldsymbol{C}_{\mathfrak{P}^G}(E)$, so that $| \mathfrak{P}^* : \boldsymbol{C}_{\mathfrak{P}^G}(E) | = 1$, p or p^2. We have $\gamma^2\mathfrak{P}^*\mathfrak{B}^2 \subseteq \boldsymbol{C}_{\mathfrak{P}^G}(E)$, and so $\gamma^4\mathfrak{P}^*\mathfrak{B}^4 = \langle 1 \rangle$. This implies that $\mathfrak{B} \subseteq \boldsymbol{O}_p(\boldsymbol{C}(E))$. Let $Z \in \boldsymbol{Z}(\mathfrak{H}_1)$; then $[Z, U] \in \boldsymbol{O}_p(\boldsymbol{C}(E))$, so that $[Z, U, U, U] \in \boldsymbol{C}_{\mathfrak{P}^G}(E)$. Since $U \in \mathfrak{U}_0 \subseteq \mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, it follows that $[Z, U, U, U, U] \in \boldsymbol{Z}(\mathfrak{P}^G)$. Since $\mathfrak{H}_1 \cap \mathfrak{U} = \langle 1 \rangle$, and since $[Z, U, U, U, U] \in \boldsymbol{Z}(\mathfrak{P}^G) \cap \mathfrak{H}_1$, we have $[Z, U, U, U, U] = \langle 1 \rangle$. This shows that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $\boldsymbol{Z}(\mathfrak{H}_1)$ and so centralizes $\boldsymbol{Z}(\mathfrak{P})$.

Suppose now that $\mathfrak{X} = \boldsymbol{Z}(\mathfrak{W}^*)$, so that $\mathfrak{Y} = \mathfrak{W}$. In this case, $\mathfrak{B} = \mathfrak{A}^G$. Hence, $\mathfrak{B}_0 \subseteq \mathfrak{W}^*$, since $\mathfrak{B}/\mathfrak{B}_0$ is cyclic. Since $\boldsymbol{Z}(\mathfrak{W}^*)$ is contained in $\mathfrak{H}_1$, if $\mathfrak{B}^*$ denotes the normal closure of $\mathfrak{B}_0$ in $\mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$, then $\boldsymbol{Z}(\mathfrak{W}^*)$ centralizes $\mathfrak{B}^*$, $\mathfrak{B}^*$ being a subgroup of $\mathfrak{W}^*$.

Let $\mathfrak{C}^* = \boldsymbol{C}(\mathfrak{B}^*) \cap \mathfrak{H}_1$ so that $\mathfrak{C}^*$ is normal in $\mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$. If $p = 3$, we have $\gamma^2\mathfrak{C}^*\mathfrak{B}^2 = \langle 1 \rangle$, since $\mathfrak{B}_0 \neq \langle 1 \rangle$, and it follows that a S_q-subgroup of $\mathfrak{M}_1$ centralizes $\mathfrak{C}^*$. Namely, if $\mathfrak{C}^* = \mathfrak{C}_1^* \supset \mathfrak{C}_2^* \supset \cdots$ is part of a chief series for $\mathfrak{M}_1$, then $\mathfrak{H}_1$ centralizes each $\mathfrak{C}_i^*/\mathfrak{C}_{i+1}^*$, so that a S_q-subgroup of $\mathfrak{M}_1$ centralizes each $\mathfrak{C}_i^*/\mathfrak{C}_{i+1}^*$, so centralizes $\mathfrak{C}^*$. If $p \geqq 5$, then $\mathfrak{B}_0 \cap \mathfrak{U} \neq \langle 1 \rangle$ for some $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{B}$, and we have $\gamma^4\mathfrak{C}^*\mathfrak{B}^4 = \langle 1 \rangle$, and we are done.

THEOREM 24.10. *Under Hypothesis* 24.5, $p = 3$ *and* $\pi(3) = \{3\}$. *Furthermore, Hypothesis* 24.4 *is not satisfied.*

Proof. Suppose that either $p \geqq 5$ or Hypothesis 24.4 is satisfied. Let $\mathfrak{A}$ be any element of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$ in case $p \geqq 5$ and let $\mathfrak{A}$ be the subgroup given by Hypothesis 24.4 in case $p = 3$. Let $\mathfrak{W}, \mathfrak{W}^*$ be as in Lemma 24.4. Let $\mathfrak{N}_1 = N(Z(\mathfrak{P})), \mathfrak{N}_2 = N(\mathfrak{W}), \mathfrak{N}_3 = N(Z(\mathfrak{W}^*))$, and let $\mathfrak{H}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Then by Lemma 24.4 and Lemma 7.7, we have $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{N}_1)(\mathfrak{H} \cap \mathfrak{N}_2) = (\mathfrak{H} \cap \mathfrak{N}_1)(\mathfrak{H} \cap \mathfrak{N}_3) = (\mathfrak{H} \cap \mathfrak{N}_2)(\mathfrak{H} \cap \mathfrak{N}_3)$. Taking $\mathfrak{H} = \mathfrak{N}_1$, we get $\mathfrak{N}_1 \subseteq \mathfrak{N}_2\mathfrak{N}_3, \mathfrak{N}_1 \subseteq \mathfrak{N}_3\mathfrak{N}_2$. Taking $\mathfrak{H} = \mathfrak{N}_2$, we get $\mathfrak{N}_2 \subseteq \mathfrak{N}_1\mathfrak{N}_3, \mathfrak{N}_2 \subseteq \mathfrak{N}_3\mathfrak{N}_1$. Taking $\mathfrak{H} = \mathfrak{N}_3$, we get $\mathfrak{N}_3 \subseteq \mathfrak{N}_1\mathfrak{N}_2, \mathfrak{N}_3 \subseteq \mathfrak{N}_2\mathfrak{N}_1$. By Lemma 8.6, we conclude that $\mathfrak{N}_1\mathfrak{N}_2$ is a group and so $\mathfrak{H} \subseteq \mathfrak{N}_1\mathfrak{N}_2$ for every proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ containing $\mathfrak{P}$. If $\mathfrak{N}_1\mathfrak{N}_2 = \mathfrak{G}$, then $O_p(\mathfrak{N}_1)$ is contained in every conjugate of $\mathfrak{N}_2$, against the simplicity of $\mathfrak{G}$. Hence, $\mathfrak{N}_1\mathfrak{N}_2$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$.

We can now suppose that $p = 3$ and that Hypothesis 24.4 is not satisfied. Suppose $q \in \pi(3), q \neq 3$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}$ and let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. If $\mathfrak{H} = O_3(\mathfrak{M})$ and $\mathfrak{C}$ is a subgroup of $\mathfrak{H}$ chosen in accordance with Lemma 8.2, then Theorem 24.7 yields that $m(Z(\mathfrak{C})) \geqq 3$. Let $\mathfrak{F}$ be a subgroup of $\mathfrak{Q}$ of type (q, q, q) and let $\Omega_1(Z(\mathfrak{C})) = \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_r$, each $\mathfrak{C}_i$ being a minimal $\mathfrak{F}$-invariant subgroup. If $\mathfrak{F}$ centralizes $Z(\mathfrak{C})$, then any subgroup of $Z(\mathfrak{C})$ of type $(3, 3, 3)$ will serve as $\mathfrak{A}$. This is so, since in this case, $C(A) \subseteq \mathfrak{M}$ for all A in $\mathfrak{A}^\sharp$. Otherwise, $|\mathfrak{C}_i| \geqq 27$ for some i, and since $\mathfrak{F}/C_{\mathfrak{F}}(\mathfrak{C}_i)$ is cyclic, $C_{\mathfrak{F}}(\mathfrak{C}_i) \in \mathscr{A}_4(\mathfrak{Q})$, so we let $\mathfrak{A}$ be any subgroup of $\mathfrak{C}_i$ of type $(3, 3, 3)$. The proof is complete.

25. The Isolated Prime

Hypothesis 25.1.

1. $3 \in \pi_4$.

2. *A S_3-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is contained in at least two maximal subgroups of $\mathfrak{G}$.*

THEOREM 25.1. *Under Hypothesis* 25.1, *there is a q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ permutable with $\mathfrak{P}$ such that if $\mathfrak{H} = \mathfrak{P}\mathfrak{Q}$ and if $\bar{\mathfrak{P}}, \bar{\mathfrak{Q}}$ are the images of $\mathfrak{P}, \mathfrak{Q}$ respectively in $\mathfrak{H}/O_3(\mathfrak{H})$, then $\bar{\mathfrak{P}} \neq 1$ is cyclic, $\bar{\mathfrak{P}}$ is faithfully and irreducibly represented on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$, and $\mathfrak{Q}$ does not centralize $\mathfrak{V} = \Omega_1(Z(O_3(\mathfrak{H})))$.*

Proof. There is at least one proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and not normalizing $Z(\mathfrak{P})$, since otherwise $N(Z(\mathfrak{P}))$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathfrak{H}$ be minimal with these two properties. Then $\mathfrak{H} = \mathfrak{P}\mathfrak{Q}$ for some q-group $\mathfrak{Q}$. Since $3 \in \pi_4$, $O_q(\mathfrak{H}) = 1$. Since $\mathscr{SCN}_3(\mathfrak{Q})$ is empty, $\mathfrak{H}$ has q-length 1. Hence,

$O_3(\mathfrak{H})\mathfrak{Q} \lhd \mathfrak{H}$. By Lemma 8.13, $\bar{\mathfrak{P}}$ is abelian. By minimality of $\mathfrak{H}$, $\bar{\mathfrak{P}}$ acts faithfully and irreducibly on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$. If $\bar{\mathfrak{P}} = 1$, then $\mathfrak{P} \lhd \mathfrak{H}$, and $\mathfrak{Q}$ normalizes $Z(\mathfrak{P})$, which is not the case.

Since $\mathfrak{Q}$ does not normalize $Z(\mathfrak{P})$, $\mathfrak{Q}$ does not centralize $Z(O_3(\mathfrak{H}))$ so does not centralize $\Omega_1(Z(O_3(\mathfrak{H})))$. The proof is complete.

We will now show that Hypothesis 24.4 is satisfied. $\bar{\mathfrak{P}}\bar{\mathfrak{Q}}$ is represented on $\mathfrak{V} = \Omega_1(Z(O_3(\mathfrak{H})))$, and it follows from (B) that the minimal polynomial of a generator of $\bar{\mathfrak{P}}$ is $(x-1)^{|\bar{\mathfrak{P}}|}$. Hence, there is an elementary subgroup $\mathfrak{A}$ of $\mathfrak{V}$ of order 27 on which $\mathfrak{P}$ acts indecomposably. Let $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{A})$ and let $\mathfrak{E} = \Omega_1(Z(\mathfrak{P}_0))$ so that $\mathfrak{A} \subseteq \mathfrak{E}$. Choose $A \in \mathfrak{A}^\#$. and set $\mathfrak{C} = C(A)$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}_0$. (It may occur that $\mathfrak{P} = \mathfrak{P}^*$ but this makes no difference in the following argument.) If $\mathfrak{P}_0 = \mathfrak{P}^*$, then $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Suppose $|\mathfrak{P}^* : \mathfrak{P}_0| = 3$. Then $\langle \mathfrak{P}, \mathfrak{P}^* \rangle \subseteq N(\mathfrak{P}_0)$, so that $\langle \mathfrak{P}, \mathfrak{P}^* \rangle$ normalizes $\mathfrak{E}$. Since $\mathfrak{P}$ and $\mathfrak{P}^*$ are conjugate in $N(\mathfrak{P}_0)$, any element of $\mathfrak{P}^* - \mathfrak{P}_0$ has minimal polynomial $(x-1)^3$ on $\mathfrak{E}$.

Let $\mathfrak{R} = O_3(\mathfrak{C})$. Then $|\mathfrak{R} : \mathfrak{R} \cap \mathfrak{P}_0| = 1$ or 3, so that $\gamma\mathfrak{R}\mathfrak{E} \subseteq \mathfrak{P}_0$, and $\gamma^2\mathfrak{R}\mathfrak{E}^2 = 1$. By (B), $\mathfrak{E} \subseteq \mathfrak{R}$. If $\mathfrak{R} \subseteq \mathfrak{P}_0$, then $\mathfrak{E} \subseteq Z(\mathfrak{R})$, and $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Suppose $|\mathfrak{R} : \mathfrak{R} \cap \mathfrak{P}_0| = 3$. Then $D(\mathfrak{R}) \subseteq \mathfrak{P}_0$, so that $\mathfrak{E} \subseteq C_{\mathfrak{R}}(D(\mathfrak{R}))$. If $C_{\mathfrak{R}}(D(\mathfrak{R})) \subseteq \mathfrak{P}_0$, then $\mathfrak{E} \subseteq Z(C_{\mathfrak{R}}(D(\mathfrak{R})))$, and once again $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Hence we can suppose that $C_{\mathfrak{R}}(D(\mathfrak{R}))$ contains an element K of $\mathfrak{R} - \mathfrak{R} \cap \mathfrak{P}_0$. Since $\mathfrak{R} \subseteq \mathfrak{P}^*$, it follows from the preceding paragraph that the class of $C_{\mathfrak{R}}(D(\mathfrak{R}))$ is at least three. On the other hand, if X and Y are in $C_{\mathfrak{R}}(D(\mathfrak{R}))$, then $[X, Y] \in C_{\mathfrak{R}}(D(\mathfrak{R})) \cap \mathfrak{R}'$. Since $\mathfrak{R}' \subseteq D(\mathfrak{R})$, we have $[X, Y, Z] = 1$ for all X, Y, Z in $C_{\mathfrak{R}}(D(\mathfrak{R}))$. This contradiction shows that $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$ for all A in $\mathfrak{A}^\#$. Combining this result with the results of Section 24 yields the following theorem.

THEOREM 25.2. *If $p \in \pi_4$, and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

THEOREM 25.3. *Let $p \in \pi_4$ and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$. Then each element of $\mathscr{M}_1(\mathfrak{P})$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. First, assume that if $p = 3$, then $\mathscr{U}(\mathfrak{P})$ contains an element $\mathfrak{B}$ whose normal closure in $C(Z(\mathfrak{P}))$ is abelian, while if $p \geq 5$, $\mathfrak{B}$ is an arbitrary element of $\mathscr{U}(\mathfrak{P})$.

Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathscr{M}_1^*(\mathfrak{P})$ be the set of subgroups $\mathfrak{P}_0$ of $\mathfrak{P}$ such that $\mathfrak{P}_0$ contains $\mathfrak{E}^M$ for suitable $\mathfrak{E}$ in $\mathscr{SCN}_3(\mathfrak{P})$, M in $\mathfrak{M}$. Suppose by way of contradiction that some element $\mathfrak{P}_0$ of $\mathscr{M}_1^*(\mathfrak{P})$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ different from $\mathfrak{M}$, and that $|\mathfrak{P}_0|$ is maximal. It follows readily that $\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{M}_1$. Since $\mathfrak{P}_0$ contains $\mathfrak{E}^M$ for suitable $\mathfrak{E}$ in $\mathscr{SCN}_3(\mathfrak{P})$, M in $\mathfrak{M}$, $O_{p'}(\mathfrak{M}_1) = 1$. Thus the hypotheses of Theorem 24.8 are satisfied, $\mathfrak{M}_1$ playing the role of $\mathfrak{K}$ and $\mathfrak{P}_0$ the role of $\mathfrak{K}_p$, $\mathfrak{B} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P}_0)$. Since $N_{\mathfrak{P}}(\mathfrak{B}) \supset \mathfrak{P}_0$, and since $\mathfrak{P}_0 \supseteq \mathfrak{E}^M \supseteq Z(\mathfrak{P})$ ($\mathfrak{E}^M$ being self centralizing), we conclude from the factorization given in Theorem 24.8 and from the maximality of $\mathfrak{P}_0$ that $\mathfrak{M}_1 \subseteq \mathfrak{M}$.

There remains the possibility that for every $\mathfrak{B}$ in $\mathscr{U}(\mathfrak{P})$, the normal closure of $\mathfrak{B}$ in $\mathfrak{M} = C(Z(\mathfrak{P}))$ is non abelian, and $p = 3$.

Let $\mathfrak{H} = O_3(\mathfrak{M})$. If $\mathfrak{H}$ contains a non cyclic characteristic abelian subgroup $\mathfrak{A}$, then $\mathfrak{A}$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{P})$, and $\mathfrak{B}^{\mathfrak{M}}$ is abelian. Since we are assuming there are no such elements, every characteristic abelian subgroup of $\mathfrak{H}$ is cyclic. The structure of $\mathfrak{H}$ is given by 3.5. If $\mathfrak{E}$ is any element of $\mathscr{SCN}_3(\mathfrak{P})$, then $\mathfrak{E} \subseteq \mathfrak{H}$, by (B), so $\mathfrak{E} \in \mathscr{SCN}_3(\mathfrak{H})$.

As before, let $\mathfrak{P}_0 \in \mathscr{A}_1{}^*(\mathfrak{P})$ be chosen so that $\mathfrak{P}_0$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ different from $\mathfrak{M}$, with $|\mathfrak{P}_0|$ maximal. Then $\mathfrak{P}_0$ is a S_3-subgroup of $\mathfrak{M}_1$ and $O_{3'}(\mathfrak{M}_1) = 1$.

Let $\mathfrak{T} = O_3(\mathfrak{M}_1)$. Since $\gamma^2\mathfrak{T}\mathfrak{E}^2 = 1$, (B) implies that $\mathfrak{T} \cap \mathfrak{E} = \mathfrak{P}_0 \cap \mathfrak{E}$. Since $\mathfrak{P}_0 = N_{\mathfrak{P}}(\mathfrak{T})$ by maximality of $\mathfrak{P}_0$, we conclude that $\mathfrak{E} \subseteq \mathfrak{T}$. We need to show that $\mathfrak{H} \subseteq \mathfrak{P}_0$. Consider $\mathfrak{H} \cap \mathfrak{P}_0 = \mathfrak{H}_0$. Since $\gamma^2\mathfrak{P}_0\mathfrak{H}_0^2 \subseteq \Omega_1(Z(\mathfrak{P}))$, we conclude that $\mathfrak{H}_0 \subseteq O_{3\,3',3}(\mathfrak{M}_1)$, and maximality of $|\mathfrak{P}_0|$ implies that $N(\mathfrak{P}_0 \cap O_{3,3'\,3}(\mathfrak{M}_1)) \subseteq \mathfrak{M}$ so it suffices to show that $\mathfrak{T}_0 = \mathfrak{P}_0 O_{3,3'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$, and it follows readily from $\mathfrak{T}_0 = N_{\mathfrak{T}_0}(\mathfrak{T}\mathfrak{H}_0)\cdot\gamma\mathfrak{H}_0 O_{3,3'}(\mathfrak{M}_1)$ that it suffices to show that $\gamma\mathfrak{H}_0 O_{3,3'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$. Since $\mathfrak{E} \subseteq \mathfrak{T}$, we have $Z(\mathfrak{T}) \subseteq \mathfrak{E}$, so that $\gamma^2 Z(\mathfrak{T})\mathfrak{H}_0^2 = 1$, and $\gamma\mathfrak{H}_0 O_{3,3'}(\mathfrak{M}_1)$ induces only 3-automorphisms on $Z(\mathfrak{T})$, so centralizes $Z(\mathfrak{P})$, and $\mathfrak{M}_1 \subseteq \mathfrak{M}$ follows in case $\mathfrak{H}_0 \subset \mathfrak{H}$.

Suppose $\mathfrak{H} \subseteq \mathfrak{P}_0$. If $\mathfrak{H} \cap \mathfrak{T} \supset \mathfrak{E}$, then $\Omega_1(Z(\mathfrak{P})) \subseteq \mathfrak{T}'$, and since $\gamma^2\mathfrak{T}\mathfrak{H}^2 \subseteq \Omega_1(Z(\mathfrak{P})) \subseteq \mathfrak{T}' \subseteq D(\mathfrak{T})$, (B) implies that $\mathfrak{H} \subseteq \mathfrak{T}$. In this case, $\Omega_1(Z(\mathfrak{T})) = \Omega_1(Z(\mathfrak{P})) \lhd \mathfrak{M}_1$, so $\mathfrak{M}_1 \subseteq \mathfrak{M}$. There remains the possibility that $\mathfrak{H} \cap \mathfrak{T} = \mathfrak{E}$.

If $\mathfrak{T} = \mathfrak{E}$, then $\gamma^2\mathfrak{T}\mathfrak{H}^2 = 1$ and (B) is violated. Hence, $\mathfrak{T} \supset \mathfrak{E}$, so that $\mathfrak{T}' \neq 1$. Hence, $\mathfrak{T}' \cap Z(\mathfrak{T}) \neq 1$. If $\Omega_1(Z(\mathfrak{P})) \subseteq \mathfrak{T}'$, then $\gamma^2\mathfrak{T}\mathfrak{H}^2 \subseteq \mathfrak{T}'$ and we are done. If $\Omega_1(Z(\mathfrak{P})) \not\subseteq \mathfrak{T}'$, we conclude that $\mathfrak{H}$ centralizes $\mathfrak{T}' \cap Z(\mathfrak{T})$, since $\mathfrak{T}' \cap Z(\mathfrak{H}) \subseteq \mathfrak{E}$. This is absurd, since $\Omega_1(C_{\mathfrak{P}}(\mathfrak{H})) = \Omega_1(Z(\mathfrak{P}))$ by (B) applied to $\mathfrak{M}$, completing the proof of this theorem.

Before combining all these results, we require an additional result about π_4.

THEOREM 25.4. *Let $p \in \pi_4$, let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ and let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Then $\mathfrak{P} \subseteq \mathfrak{M}'$.*

Proof. Let $\mathfrak{E} \in \mathscr{SCN}_3(\mathfrak{P})$, and suppose G in $\mathfrak{G}$ has the property that $\mathfrak{E}^G \subseteq \mathfrak{P}$. Then $\mathfrak{E} \subseteq \mathfrak{M}^{G^{-1}}$. By Theorem 25.3, we have $\mathfrak{M}^{G^{-1}} = \mathfrak{M}$, so that $G \in \mathfrak{M}$. Hence $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{P}) = V(ccl_{\mathfrak{M}}(\mathfrak{C}); \mathfrak{P})$. By (B) and $p \in \pi_4$, $\mathfrak{C}^M \subseteq O_p(\mathfrak{M})$ for each M in $\mathfrak{M}$. Hence, $\mathfrak{B} \lhd \mathfrak{M}$, so maximality of $\mathfrak{M}$ implies $\mathfrak{M} = N(\mathfrak{B})$. By uniqueness of $\mathfrak{M}$ (or because $\mathfrak{B}$ is weakly closed in $\mathfrak{P}$), we have $\mathfrak{M} \supseteq N(\mathfrak{P})$. Furthermore, by Theorem 25.3, if $\mathfrak{M}^G \neq \mathfrak{M}$, then $\mathfrak{C} \not\subseteq \mathfrak{M}^G$. Thus, $\mathfrak{C}$ is not in the kernel $\mathfrak{K}(G)$ of the permutation representation of $\mathfrak{P}$ on the cosets of $\mathfrak{P}$ in $\mathfrak{M}G\mathfrak{P}$. We can then find C in $\mathfrak{C}$ such that $\mathfrak{K}(G)C$ has order p in $Z(\mathfrak{P}/\mathfrak{K}(G))$, so Theorem 14.4.1 in [12] yields this theorem.

We are now in a position to let π_3 and π_4 coalesce, that is, we set $\pi_0 = \pi_3 \cup \pi_4$.

THEOREM 25.5. *Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$. If $p \in \pi_0$ and $\mathfrak{M}_p$ is a S_p-subgroup of $\mathfrak{M}$, then either $\mathfrak{M}_p$ is a S_p-subgroup of $\mathfrak{G}$ or $\mathfrak{M}_p$ has a cyclic subgroup of index at most p, and $\mathfrak{M}_p \notin \mathscr{SCN}_4(\mathfrak{P})$ for every S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. If ϖ is the largest subset of π_0 with the property that $\mathfrak{M}$ contains a S_ϖ-subgroup $\mathfrak{S}$ of $\mathfrak{G}$, then $\mathfrak{S} \lhd \mathfrak{M}$, and $\mathfrak{S} \subseteq \mathfrak{M}'$.*

Proof. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{M}_p$. Suppose $\mathfrak{M}_p \subset \mathfrak{P}$. Then $\mathfrak{M}_p \notin \mathscr{SCN}_4(\mathfrak{P})$, by Theorems 24.3, 24.5, and 25.3. Thus, if $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, then $C(\mathfrak{B}) \cap \mathfrak{M}_p$ is cyclic. Since $|\mathfrak{M}_p : C(\mathfrak{B}) \cap \mathfrak{M}_p| = 1$ or p the first assertion follows.

Let $\mathfrak{S}_q$ be a S_q-subgroup of $\mathfrak{S}$ for q in ϖ. (If ϖ is empty there is no more to prove.) If $q \in \pi_3$, then $\mathfrak{S}_q \subseteq \mathfrak{M}'$ by uniqueness of $\mathfrak{M}$ and Lemma 17.2. If $q \in \pi_4$, then $\mathfrak{S}_q \subseteq \mathfrak{M}'$ by uniqueness of $\mathfrak{M}$ and Theorem 25.5. Hence, $\mathfrak{S} \subseteq \mathfrak{M}'$. If $r \in \pi(\mathfrak{M})$, $r \notin \varpi$, then $\mathfrak{M}'$ centralizes every chief r-factor of $\mathfrak{M}$, by Lemma 8.13. Since $\mathfrak{S} \subseteq \mathfrak{M}'$, we conclude that $\mathfrak{S} \lhd \mathfrak{M}$.

THEOREM 25.6. *π_0 is partitioned into non empty subsets $\sigma_1, \cdots, \sigma_n$, $n \geqq 1$, with the following properties:*

(i) If $\tau \subseteq \pi_0$, then $\mathfrak{G}$ satisfies E_τ if and only if $\tau \subseteq \sigma_i$ for some $i = 1, \cdots, n$.

(ii) If $\mathfrak{H}_i$ is a S_{σ_i}-subgroup of $\mathfrak{G}$, then $\mathfrak{N}_i = N(\mathfrak{H}_i)$ is a maximal subgroup of $\mathfrak{G}$, $\mathfrak{H}_i \subseteq \mathfrak{N}_i'$, and $\mathfrak{H}_i \cap \mathfrak{H}_i^G$ is of square free order for each $G \in \mathfrak{G} - \mathfrak{N}_i$, $i = 1, \cdots, n$.

(iii) If $p_i \in \sigma_i$ and $\mathfrak{P}_i$ is a S_{p_i}-subgroup of $\mathfrak{H}_i$, and if $\mathfrak{P}_i \cap \mathfrak{P}_i^G = \mathfrak{D}_i \neq 1$ for some $G \in \mathfrak{G} - \mathfrak{N}_i$, then $\mathfrak{D}_i$ is of order p_i and $C_{\mathfrak{P}_i}(\mathfrak{D}_i) = \mathfrak{D}_i \times \mathfrak{E}_i$, where $\mathfrak{E}_i$ is cyclic, $i = 1, 2, \cdots, n$.

Proof. By Lemma 8.5, π_0 is non empty. By Corollary 19.1, Theorems 24.3, 24.4, 24.5, 25.2 and 25.3 $\sim$ is an equivalence relation on π_0 and if $\sigma_1, \cdots, \sigma_n$ are the equivalence classes of π_0 under $\sim$, then (i) holds.

Let $\mathfrak{H} = \mathfrak{H}_i$ be a S_{σ_i}-subgroup of $\mathfrak{G}$ and let $\mathfrak{P} = \mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{H}$ for $p = p_i \in \sigma_i$. By Theorem 25.5, $N(\mathfrak{H}) = \mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$, and $\mathfrak{H} \subseteq \mathfrak{M}'$.

Suppose $G \in \mathfrak{G} - \mathfrak{M}$ and $\mathfrak{P} \cap \mathfrak{P}^G = \mathfrak{D} \neq 1$. If $\mathfrak{D}_1$ is any non identity characteristic subgroup of $\mathfrak{D}$, then either $N(\mathfrak{D}_1) \cap \mathfrak{P} \in \mathscr{M}_4(\mathfrak{P})$ or $N(\mathfrak{D}_1) \cap \mathfrak{P}^G \notin \mathscr{M}_4(\mathfrak{P}^G)$, by Theorems 24.3, 24.4, 24.5, 25.5 and 25.3. Since $N(D(\mathfrak{D}))$ contains every element of both $\mathscr{U}(\mathfrak{P})$ and $\mathscr{U}(\mathfrak{P}^G)$, we conclude that $\mathfrak{D}$ is elementary of order p or p^2. Suppose $|\mathfrak{D}| = p^2$. If $\mathfrak{D}$ contains $\Omega_1(Z(\mathfrak{P}))$ then $N(\mathfrak{D})$ contains an element of $\mathscr{U}(\mathfrak{P})$, so that $N(\mathfrak{D}) \cap \mathfrak{P} \in \mathscr{M}_4(\mathfrak{P})$. If $\mathfrak{D}$ does not contain $\Omega_1(Z(\mathfrak{P}))$ then $N(\mathfrak{D}) \cap \mathfrak{P}$ contains an elementary subgroup of order p^3, so once again $N(\mathfrak{D}) \cap \mathfrak{P} \in \mathscr{M}_4(\mathfrak{P})$. The same argument applies to $\mathfrak{P}^G$, so that $\mathfrak{P}^G \subseteq \mathfrak{M}$. Hence $\mathfrak{P}^G = \mathfrak{P}^N$ for some N in $\mathfrak{M}$. Hence $GN^{-1} \in N(\mathfrak{P}) \subseteq \mathfrak{M}$, so $G \in \mathfrak{M}$, contrary to hypothesis. Hence, $\mathfrak{D}$ is of order p.

If $C_{\mathfrak{P}}(\mathfrak{D}) \in \mathscr{M}_4(\mathfrak{P})$, then $N(\mathfrak{D}) \subseteq \mathfrak{M}$, so that $\mathfrak{P}^G \cap \mathfrak{M} \supset \mathfrak{D}$, contrary to the fact that $\mathfrak{P}^G \cap \mathfrak{P}^N$ has order 1 or p for all N in $\mathfrak{M}$, by the preceding paragraph. Hence, $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{M}_4(\mathfrak{P})$. If $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, and $C_{\mathfrak{P}}(\mathfrak{D}) \cap C_{\mathfrak{P}}(\mathfrak{B}) = \mathfrak{E}$, then $\mathfrak{E}$ is of index at most p in $C_{\mathfrak{P}}(\mathfrak{D})$ and $\mathfrak{E}$ is disjoint from $\mathfrak{D}$, since $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{M}_4(\mathfrak{P})$. Hence, $C_{\mathfrak{P}}(\mathfrak{D}) = \mathfrak{D} \times \mathfrak{E}$. This proves (iii), the cyclicity of $\mathfrak{E}$ following from $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{M}_4(\mathfrak{P})$. The proof is complete.

26. The Maximal Subgroups of $\mathfrak{G}$

The purpose of this section is to use the preceding results, notably Theorems 25.5 and 25.6, to complete the proofs of the results stated in Section 14.

LEMMA 26.1. *If $p \in \pi_1 \cup \pi_2$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P} \subseteq N(\mathfrak{P})'$.*

Proof. If $\mathfrak{P}$ is abelian, the lemma follows from Grün's theorem and the simplicity of $\mathfrak{G}$. If $\mathfrak{P}$ is non abelian, $\mathfrak{P}$ is not metacyclic, by 3.8. Also, $p \geqq 5$, as already observed several times. Thus, from 3.4 we see that $\Omega_1(\mathfrak{P})$ is a non abelian group of order p^3. The hypotheses of Lemma 8.10 are satisfied, so $\mathfrak{P} \subseteq N(\Omega_1(Z(\mathfrak{P})))'$ by Theorem 14.4.2 in [12] and the simplicity of $\mathfrak{G}$. Since $N(\mathfrak{P}) \subseteq N(\Omega_1(Z(\mathfrak{P})))$, and since $N(\Omega_1(Z(\mathfrak{P})))$ has p-length one, the lemma follows.

LEMMA 26.2. *If $p \in \pi_2$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ is abelian or is a central product of a cyclic group and a non abelian group of order p^3 and exponent p.*

Proof. We only need to show that $\mathfrak{P}$ is not isomorphic to (iii) in 3.4. Suppose false. Let $\mathfrak{P}_1 = \Omega_1(\mathfrak{P})$, and let $\mathfrak{R}$ be a fixed $S_{p'}$-subgroup of $N(\mathfrak{P})$. Set $\mathfrak{R}_1 = \mathfrak{R}/C_{\mathfrak{R}}(\mathfrak{P})$. The oddness of $|N(\mathfrak{P})|$ guarantees that $\mathfrak{R}_1$ is abelian.

Let $\mathscr{C}$ be a chief series for $\mathfrak{P}$, one of whose terms is $\mathfrak{P}_1$ and which is $\mathfrak{R}$-admissible. Let α_i be the character of $\mathfrak{R}_1$ on the ith term of $\mathscr{C}$ modulo the $(i+1)$st, where $i = 1, \cdots, \ell + 3$, and $|\mathfrak{P} : \mathfrak{P}_1| = p^{\ell}$. Since $\mathfrak{P}/\mathfrak{P}_1$ is cyclic, $\alpha_1 = \cdots = \alpha_l$. From 3.4, we see that $\alpha_1 = \alpha_{\ell+3}$. Furthermore, $\alpha_{\ell+2} = \alpha_1\alpha_{\ell+1}$, and $\alpha_{\ell+3} = \alpha_{\ell+1}\alpha_{\ell+2}$. Combining these equalities yields $\alpha_{\ell+1}^2 = 1$, so $\alpha_{\ell+1} = 1$, and Lemma 26.1 is violated.

If $\mathfrak{B}$ normalizes $\mathfrak{A}$ we say that $\mathfrak{B}$ *is prime on* $\mathfrak{A}$ provided any two elements of $\mathfrak{B}^{\sharp}$ have the same fixed points on $\mathfrak{A}$. If $|\mathfrak{B}|$ is a prime, $\mathfrak{B}$ is necessarily prime on $\mathfrak{A}$. If $\mathfrak{A}$ is solvable, then $\mathfrak{B}$ is prime on $\mathfrak{A}$ if and only if for each prime p, there is a S_p-subgroup $\mathfrak{A}_p$ of $\mathfrak{A}$ which is normalized by $\mathfrak{B}$ and such that $\mathfrak{B}$ is prime on $\mathfrak{A}_p$.

The next two lemmas are restatements of Lemma 13.12.

LEMMA 26.3. *Suppose $\mathfrak{A}$ is a solvable π-group, and $\mathfrak{B}$ is a cyclic π'-subgroup of* Aut$(\mathfrak{A})$ *which is prime on $\mathfrak{A}$. Assume also that $|\mathfrak{A}|\cdot|\mathfrak{B}|$ is odd. If $|\mathfrak{B}|$ is not a prime, if the centralizer of $\mathfrak{B}$ in $\mathfrak{A}$ is a Z-group, and if $\mathfrak{B}$ has no fixed points on $\mathfrak{A}/\mathfrak{A}'$, then $\mathfrak{A}$ is nilpotent.*

LEMMA 26.4. *Suppose $\mathfrak{A}$ is a solvable π-group and $\mathfrak{B}$ is a π'-subgroup of* Aut$(\mathfrak{A})$ *of prime order. Assume also that $|\mathfrak{A}|\cdot|\mathfrak{B}|$ is odd. If the centralizer of $\mathfrak{B}$ in $\mathfrak{A}$ is a Z-group, and if $\mathfrak{B}$ has no fixed points on $\mathfrak{A}/\mathfrak{A}'$, then $\mathfrak{A}/F(\mathfrak{A})$ is nilpotent.*

$\mathscr{X}$ denotes the set of all proper subgroups of $\mathfrak{G}$, $\mathscr{X}_0$ denotes those subgroups $\mathfrak{A}$ of $\mathfrak{G}$ such that, for all $p \in \pi_0$, $\mathfrak{A}$ does not contain an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P})$ for any S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$; $\mathscr{X}_1 = \mathscr{X} - \mathscr{X}_0$. $\mathscr{M}$ denotes the set of maximal subgroups of $\mathfrak{G}$, $\mathscr{M}_i = \mathscr{M} \cap \mathscr{X}_i$, $i = 0, 1$.

If $\mathfrak{R} \in \mathscr{X}_0$, then $\mathfrak{R}$ does not contain an elementary subgroup of order p^3 for any prime p, so $\mathfrak{R}'$ is nilpotent. Furthermore, if $\pi(\mathfrak{R}) = \{p_1, \cdots, p_n\}$, $p_1 > p_2 > \cdots > p_n$, then $\mathfrak{R}$ has a Sylow series of complexion $(p_1, \cdots, p_n)$.

Suppose $p \in \pi_0$ and $\mathfrak{P}_0$ is a subgroup of type (p, p) with $\mathfrak{P}_0 \in \mathscr{X}_0$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the distinct S_p-subgroups of $\mathfrak{G}$ which contain $\mathfrak{P}_0$. Since $\mathfrak{P}_0 \notin \mathscr{S}\!\mathscr{C}_4(\mathfrak{P}_i)$, $1 \leq i \leq n$, it follows that $\mathfrak{P}_0 \supseteq \Omega_1(Z(\mathfrak{P}_i))$, and that $N(\mathfrak{P}_0) - C(\mathfrak{P}_0)$ contains an element of order p centralizing $\Omega_1(Z(\mathfrak{P}_i))$. Since $N(\mathfrak{P}_0)/C(\mathfrak{P}_0)$ is p-closed, this implies that $\Omega_1(Z(\mathfrak{P}_i)) = \Omega_1(Z(\mathfrak{P}_j))$, $1 \leq i, j \leq n$. This fact is very important, since it shows that the $p+1$ subgroups of $\mathfrak{P}_0$ of order p are contained in two conjugate classes in $\mathfrak{G}$, one class containing $\Omega_1(Z(\mathfrak{P}_1))$, the remaining p subgroups lying in a single conjugate class.

If $\mathfrak{M} \in \mathscr{M}_0$, $\boldsymbol{H}(\mathfrak{M})$ denotes the largest normal nilpotent S-subgroup of $\mathfrak{M}$. Note that by Lemma 8.5, $\boldsymbol{H}(\mathfrak{M}) \neq 1$. More explicitly, $\pi(\boldsymbol{H}(\mathfrak{M}))$ contains the largest prime in $\pi(\mathfrak{M})$. Note also that $\boldsymbol{H}(\mathfrak{M})$ is a S-subgroup of $\mathfrak{G}$.

If $\mathfrak{M} \in \mathscr{M}_1$, $\boldsymbol{H}_1(\mathfrak{M})$ denotes the unique S_σ-subgroup of $\mathfrak{M}$, where $\sigma = \sigma(\mathfrak{M})$ is the equivalence class of π_0 under $\sim$ associated with $\mathfrak{M}$. That is, $p \in \sigma$ if and only if $p \in \pi_0$ and $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$. Or again, $p \in \sigma$ if and only if $\mathfrak{M}$ contains an elementary subgroup of order p^3. Or again, $p \in \sigma$ if and only if $p \in \pi_0$ and $\mathfrak{M}$ contains an element of $\mathscr{S}\!\!\mathscr{C}_4(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$.

Suppose $\mathfrak{M} \in \mathscr{M}_1$, $q \in \pi(\mathfrak{M}) - \sigma(\mathfrak{M})$ and a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$. Since $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M})$, it follows that $N(\mathfrak{Q}) \subseteq \mathfrak{M}$, so that $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$. Then by Lemma 26.1, $\mathfrak{Q} \subseteq \mathfrak{M}'$. Since the derived group of $\mathfrak{M}/\boldsymbol{H}_1(\mathfrak{M})$ is nilpotent, we have $\mathfrak{Q} \lhd \mathfrak{M}$. Thus, if τ is the largest subset of $\pi(\mathfrak{M}) - \sigma(\mathfrak{M})$ such that some S_τ-subgroup of $\mathfrak{M}$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$, then $\mathfrak{M}$ contains a unique S_τ-subgroup $\boldsymbol{E}_1(\mathfrak{M})$, $\boldsymbol{E}_1(\mathfrak{M})$ is a normal nilpotent S-subgroup of $\mathfrak{M}$, $\boldsymbol{E}_1(\mathfrak{M})$ is a S-subgroup of $\mathfrak{G}$, and the structure of the S_q-subgroups of $\boldsymbol{E}_1(\mathfrak{M})$ is given by Lemma 26.2. We set $\boldsymbol{H}(\mathfrak{M}) = \langle \boldsymbol{E}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M})\rangle = \boldsymbol{E}_1(\mathfrak{M}) \times \boldsymbol{H}_1(\mathfrak{M})$. Since $\boldsymbol{E}_1(\mathfrak{M}) \lhd \mathfrak{M}$ and $\boldsymbol{E}_1(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$, and since $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M})$, it follows that $\boldsymbol{E}_1(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$.

If $p \in \pi_0 \cap \pi^*$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then the definitions of π_0 and π^* imply that $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$ is of type (p, p). In this case, we set $\boldsymbol{T}(\mathfrak{P}) = \boldsymbol{C}_{\mathfrak{P}}(\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})))$, and remark that $\boldsymbol{T}(\mathfrak{P})$ char $\mathfrak{P}$, $|\mathfrak{P} : \boldsymbol{T}(\mathfrak{P})| = p$. Furthermore, if P is an element of order p in $\boldsymbol{T}(\mathfrak{P})$, then $\boldsymbol{C}_{\mathfrak{P}}(P)$ contains an elementary subgroup of order p^3. If $q \in \pi_0 - \pi^*$, set $\boldsymbol{T}(\mathfrak{Q}) = \mathfrak{Q}$, $\mathfrak{Q}$ being any S_q-subgroup of $\mathfrak{G}$. The relevance of $\boldsymbol{T}(\mathfrak{Q})$ lies in the fact that if Q is any element of $\boldsymbol{T}(\mathfrak{Q})$ of order q, then $\boldsymbol{C}(Q)$ is contained in only one maximal subgroup of $\mathfrak{G}$, namely, the one that contains $\mathfrak{Q}$. This statement is an immediate consequence of the theorems proved about $\mathscr{S}\!\!\mathscr{C}_4(\mathfrak{Q})$, explicitly stated in Theorem 25.5.

If $\mathfrak{A} \in \mathscr{X}_1$, then $\mathfrak{A}$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, so we set $\boldsymbol{M}(\mathfrak{A}) = \mathfrak{M}$. The existence of the mapping $\boldsymbol{M}$ from $\mathscr{X}_1$ to $\mathscr{M}_1$ is naturally crucial.

If $\mathfrak{M} \in \mathscr{M}_0$, set $\hat{\boldsymbol{H}}(\mathfrak{M}) = \boldsymbol{H}(\mathfrak{M})^\#$. If $\mathfrak{M} \in \mathscr{M}_1$, let $\hat{\boldsymbol{H}}(\mathfrak{M})$ consist of all elements H in $\boldsymbol{H}(\mathfrak{M})^\#$ with the property that some power of H, say $H_1 = H^n$ is either in $\boldsymbol{E}_1(\mathfrak{M})^\#$ or is in $\boldsymbol{T}(\mathfrak{Q})^\#$ for some S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ with $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$.

Let $q \in \pi_0$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ with $\boldsymbol{T}(\mathfrak{Q}) \subset \mathfrak{Q}$; let $\mathscr{T}(\mathfrak{Q})$ denote the set of subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ of type (q, q) such that

$\mathfrak{Q}_0 = \Omega_1(C_{\mathfrak{Q}}(Q))$ for some element Q in $\mathfrak{Q}_0$. If $\mathfrak{Q}_0 \in \mathscr{T}(\mathfrak{Q})$, then $\mathfrak{Q}_0 \supseteq \Omega_1(\boldsymbol{Z}(\mathfrak{Q}))$. Furthermore, if $q \in \pi_0$ and $\mathfrak{Q}_1$ is a subgroup of $\mathfrak{G}$ of type (q, q), and if $\mathfrak{Q}_1$ is contained in at least two maximal subgroups of $\mathfrak{G}$, then $\mathfrak{Q}_1 \in \mathscr{T}(\mathfrak{Q})$ for every S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ which contains $\mathfrak{Q}_1$.

LEMMA 26.5.
(i) *If* $\mathfrak{M} \in \mathscr{M}_0$, *then* $\boldsymbol{H}(\mathfrak{M})'$ *is a T.I. set in* $\mathfrak{G}$.
(ii) *If* $\mathfrak{M} \in \mathscr{M}_1$, *then* $\hat{\boldsymbol{H}}(\mathfrak{M})$ *is a T.I. set in* $\mathfrak{G}$.

Proof.
(i) $\boldsymbol{H}(\mathfrak{M})'$ is cyclic and normal in $\mathfrak{M}$, by Lemma 26.2. Hence, if $H \in \boldsymbol{H}(\mathfrak{M})'^{\#} \cap \boldsymbol{H}(\mathfrak{M}^G)'^{\#}$ for some G in $\mathfrak{G}$, then $N(\langle H\rangle) \supseteq \langle \mathfrak{M}, \mathfrak{M}^G\rangle$, so $G \in \mathfrak{M}$, as required.

(ii) It is immediate from the definition that $\hat{\boldsymbol{H}}(\mathfrak{M})$ is a normal subset of $\mathfrak{M}$, so $\hat{\boldsymbol{H}}(\mathfrak{M})$ is a T.I. set in $\mathfrak{M}$. Suppose $G \in \mathfrak{G}$ and $H \in \hat{\boldsymbol{H}}(\mathfrak{M}) \cap \hat{\boldsymbol{H}}(\mathfrak{M})^G$. Choose n so that $K = H^n$ is in either $\boldsymbol{E}_1(\mathfrak{M})^{\#}$ or $\boldsymbol{T}(\mathfrak{Q})^{\#}$ for some S_q-subgroup $\mathfrak{Q}$ of $\boldsymbol{H}(\mathfrak{M})$, and such that K is of prime order. If $K \in \boldsymbol{E}_1(\mathfrak{M})^{\#}$, then since $(|\boldsymbol{E}_1(\mathfrak{M})|, |\boldsymbol{H}_1(\mathfrak{M})|) = 1$, it follows that $K \in \boldsymbol{E}_1(\mathfrak{M})^{G\#}$. Hence $C(K) \supseteq \langle \boldsymbol{H}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M})^G\rangle$, and so $G \in \mathfrak{M}$. Suppose $K \in \boldsymbol{H}_1(\mathfrak{M})^{\#}$. Then $C_{\mathfrak{Q}}(K) \in \mathscr{S}\!\mathscr{C}_4(\mathfrak{Q})$ and so $C(K) \subseteq \mathfrak{M}$. This implies that $\boldsymbol{H}_1(\mathfrak{M}) \cap \boldsymbol{H}_1(\mathfrak{M})^G$ contains non cyclic S_q-subgroups. By Theorem 25.6 (ii), we again have $G \in \mathfrak{M}$. The lemma is proved.

With Lemma 26.5 at hand, it is fairly clear that the one remaining obstacle in this chapter is π^*. In dealing with π^*, we will repeatedly use the assumption that $|\mathfrak{G}|$ is odd.

LEMMA 26.6. *Let* $p \in \pi_0$, *let* $\mathfrak{P}$ *be a* S_p-*subgroup of* $\mathfrak{G}$, *and let* $\mathfrak{M} = \boldsymbol{M}(\mathfrak{P})$. *If* $\mathfrak{P}_1$ *is any non identity subgroup of* $\boldsymbol{T}(\mathfrak{P})$ *and* $\mathfrak{P}_1$ *is contained in the* p-*subgroup* $\mathfrak{P}^*$ *of* $\mathfrak{G}$, *then* $N(\mathfrak{P}^*) \subseteq \mathfrak{M}$.

Proof. In any case, $\mathfrak{P}^* \subseteq \mathfrak{M}$, by Theorem 25.6 (iii). If $\mathfrak{P}^*$ is non cyclic, then $N(\Omega_1(\mathfrak{P}^*))$ contains an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P}_0)$ for some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{M}$ and we are done. Otherwise, $\Omega_1(\mathfrak{P}^*) = \Omega_1(\mathfrak{P}_1)$, so $N(\Omega_1(\mathfrak{P}^*))$ contains an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P})$, and we are done.

LEMMA 26.7. *Suppose* $p, q \in \pi_1 \cup \pi_2$, $p \neq q$, $\mathfrak{Q}$ *is a* S_q-*subgroup of* $\mathfrak{G}$ *and* $\mathfrak{P}$ *is a* S_p-*subgroup of* $N(\mathfrak{Q})$. *If* $\mathfrak{P}$ *is cyclic, then* $\mathfrak{P}$ *is prime on* $\mathfrak{Q}$.

Proof. Suppose false. Then $q \equiv \pm 1 \pmod p$, and every p, q-subgroup $\mathfrak{K}$ of $\mathfrak{G}$ is q-closed. Also $\Omega_1(\mathfrak{P}) \subseteq \boldsymbol{Z}(\mathfrak{P}^*)$ for some S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$, by Lemma 26.2 and $|\mathfrak{P}| > p$. If $\mathfrak{P}^*$ is cyclic, or if $\mathfrak{P}^*$ is non abelian, then $\mathfrak{P} \subseteq N(\Omega_1(\mathfrak{P}))'$, by Lemma 26.1. Since every chief

q-factor of $N(\Omega_1(\mathfrak{P}))$ is centralized by $N(\Omega_1(\mathfrak{P}))'$, it follows that $\mathfrak{P}$ centralizes $C_{\mathfrak{Q}}(\Omega_1(\mathfrak{P}))$ and we are done.

If $\mathfrak{P}^*$ is abelian and non cyclic, then $\mathfrak{P}^*$ normalizes some S_q-subgroup $\mathfrak{Q}^*$ of $N(\Omega_1(\mathfrak{P}))$. Since the lemma is assumed false, $C_{\mathfrak{Q}}(\Omega_1(\mathfrak{P})) \neq 1$, so $\mathfrak{Q}^* \neq 1$. If $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}^*\mathfrak{Q}^*$, then $\mathfrak{K}$ is q-closed, so contains a S_q-subgroup of $\mathfrak{G}$. This violates the hypothesis of this lemma.

LEMMA 26.8. *Let $p \in \pi_0$, $q \in \pi(\mathfrak{G})$ and suppose that $q \in \pi_1 \cup \pi_2$ or $p \not\sim q$. If $\mathfrak{K}$ is any p, q-subgroup of $\mathfrak{G}$ and $\mathfrak{K}$ contains an element of $\mathscr{A}_4(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$, then $\mathfrak{K}$ is p-closed.*

Proof. Let $\mathfrak{M} = \boldsymbol{M}(\mathfrak{K})$. The hypotheses imply that $p \,||\, |\boldsymbol{H}_1(\mathfrak{M})|$ and $q \nmid |\boldsymbol{H}_1(\mathfrak{M})|$. The lemma follows.

LEMMA 26.9. *Let $p \in \pi_0$, $q \in \pi(\mathfrak{G})$ and suppose that $q \in \pi_1 \cup \pi_2$ or $p \not\sim q$. If $\mathfrak{Q}$ is a q-subgroup of $\mathfrak{G}$ which is normalized by the cyclic p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$, then $\mathfrak{P}$ is prime on $\mathfrak{Q}$.*

Proof. If $|\mathfrak{P}| = p$, the lemma is trivial. Otherwise, the lemma follows from Lemma 26.8, since $N(\Omega_1(\mathfrak{P}))$ contains an element of $\mathscr{A}_4(\mathfrak{P}_0)$ for some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{G}$.

LEMMA 26.10. *Let $\mathfrak{M} \in \mathscr{M}$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}$ for some prime p. If $\mathfrak{P}$ is non abelian and $\mathfrak{P} \not\subseteq \mathfrak{M}'$, then $\mathfrak{P}$ does not contain a cyclic subgroup of index p.*

Proof. We can suppose that $\mathfrak{P} \in \mathscr{X}_0$, for if $\mathfrak{P} \in \mathscr{X}_1$, then $\mathfrak{M} = \boldsymbol{M}(\mathfrak{P})$ and $\mathfrak{P} \subseteq \mathfrak{M}'$ by Theorem 25.6 (ii). Hence, proceeding by way of contradiction we can suppose that $\mathfrak{P} = gp\langle P_0, P_1 \mid P_0^{p^n} = P_1^p = 1, P_1^{-1}P_0P_1 = P_0^{1+p^{n-1}}\rangle$, where $n \geqq 2$. Note that $\mathfrak{P}' = \langle P_0^{p^{n-1}}\rangle$.

If $\mathfrak{M}'$ is nilpotent, then $\mathfrak{P}' \lhd \mathfrak{M}$, so $\mathfrak{M} = N(\mathfrak{P}')$ by maximality of $\mathfrak{M}$. This implies that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ which is not the case. Hence, $\mathfrak{M}'$ is not nilpotent. In particular, $\mathfrak{M} \in \mathscr{M}_1$. It follows that $p \not\sim q$ for all q in $\pi(\boldsymbol{H}_1(\mathfrak{M}))$.

We first show that $\boldsymbol{E}_1(\mathfrak{M}) = 1$. For $\mathfrak{P}'$ centralizes $\boldsymbol{E}_1(\mathfrak{M})$, so if $\mathfrak{M}_1$ is an element of $\mathscr{M}$ containing $N(\mathfrak{P}')$, then $\boldsymbol{E}_1(\mathfrak{M})$ normalizes some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{M}_1$ with $\mathfrak{P} \subseteq \mathfrak{P}_0$. It follows from Lemma 8.16 that $\boldsymbol{E}_1(\mathfrak{M})$ centralizes $\mathfrak{P}_0$. If $\boldsymbol{E}_1(\mathfrak{M}) \neq 1$, then $\mathfrak{P}_0 \subseteq \mathfrak{M}$, which is not the case, so $\boldsymbol{E}_1(\mathfrak{M}) = 1$.

Choose q in $\pi(\boldsymbol{H}(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$. We can now choose $\mathfrak{A} \subseteq \boldsymbol{T}(\mathfrak{Q})$ such that $\mathfrak{A}$ is normalized by $\Omega_1(\mathfrak{P})$, is centralized by some non identity element P of $\Omega_1(\mathfrak{P})$, but is not centralized by $\Omega_1(\mathfrak{P})$. For otherwise, $\Omega_1(\mathfrak{P})$ centralizes $\boldsymbol{T}(\mathfrak{Q})$, and

$N(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$, which is not the case. For such a choice of $\mathfrak{A}$ and P, let $\mathfrak{K}$ be a $S_{p,q}$-subgroup of $C(P)$ which contains $\mathfrak{A}\Omega_1(\mathfrak{P})$. By Lemma 26.7, there is a S_q-subgroup $\mathfrak{K}_q$ of $\mathfrak{K}$ which contains $\mathfrak{A}$ and is contained in $\mathfrak{M}$. Since $\Omega_1(\mathfrak{P})$ does not centralize $\mathfrak{A}$, and since $p \not\sim q$, a S_p-subgroup $\mathfrak{K}_p$ of $\mathfrak{K}$ is contained in $\mathscr{X}_0$, by Lemma 26.8.

We wish to show that $\mathfrak{K}_q \lhd \mathfrak{K}$. This is clear if $\mathfrak{K}_q$ contains an element of $\mathscr{M}_4(\mathfrak{Q}^*)$ for some S_q-subgroup $\mathfrak{Q}^*$ of $\mathfrak{G}$, by Lemma 26.6. Otherwise, Lemma 8.5 implies that $\mathfrak{K}_q \lhd \mathfrak{K}$, since $q > p$. By Lemma 26.6, $\mathfrak{K} \subseteq \mathfrak{M}$, so $\mathfrak{M}$ contains a S_p-subgroup of $C(P)$. This implies that $\langle P \rangle \neq \langle P_0^{p^{n-1}} \rangle$. Since the p subgroups of $\mathfrak{P}$ of order p different from $\langle P_0^{p^{n-1}} \rangle$ are conjugate in $\mathfrak{P}$, and since $\hat{H}(\mathfrak{M})$ is a normal subset of $\mathfrak{M}$, we can suppose that $P = P_1$.

Let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and let $\mathfrak{W} = \Omega_1(Z_2(\mathfrak{P}^*))$, so that $\mathfrak{W} \cap \mathfrak{P} = \langle P_0^{p^{n-1}} \rangle$, or else $p \in \pi_2$. It follows that P_0W centralizes P_1 for some W in $\mathfrak{W}$. But $\mathfrak{M}$ contains a S_p-subgroup of $C(P_1)$, so $C(P_1) \cap \mathfrak{M}$ contains an element of order equal to that of P_0W. Since P_0W and P_0 have the same order, a S_p-subgroup of $C(P_1) \cap \mathfrak{M}$ has exponent p^n, which is not the case. The proof is complete.

LEMMA 26.11. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}$ for some prime p. If $\mathfrak{P}$ is non abelian, then $\mathfrak{P} \subseteq \mathfrak{M}'$.*

Proof. First, suppose $p \in \pi_0$. If $\mathfrak{P} \in \mathscr{X}_1$, we are done. Otherwise, $\mathfrak{P}$ contains a cyclic subgroup of index p and we are done by the preceding lemma.

We can now suppose that $p \in \pi_2$. If $\mathfrak{M}'$ is nilpotent, the lemma follows readily from Lemmas 26.1 and 26.2. We can suppose that $\mathfrak{M}'$ is not nilpotent and that $\mathfrak{P} \not\subseteq \mathfrak{M}'$. Since $\mathfrak{P}$ is non abelian, Lemma 26.2 implies that $\Omega_1(\mathfrak{P})$ is of order p^3, or else $\mathfrak{P}$ is metacyclic. In the second case, we are done by the preceding lemma.

We first show that $E_1(\mathfrak{M}) = 1$. Since $\Omega_1(Z(\mathfrak{P}))$ centralizes $E_1(\mathfrak{M})$, it follows readily that $N(E_1(\mathfrak{M}))$ dominates $\mathfrak{P}$, by Sylow's theorem. If $E_1(\mathfrak{M}) \neq 1$, then $\mathfrak{M} = N(E_1(\mathfrak{M}))$, and so $\mathfrak{P} \subseteq \mathfrak{M}'$, by Lemma 26.1, and we are done.

Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ which is normalized by $\mathfrak{P}$, with $q \in \pi(H(\mathfrak{M}))$.

We show that $\mathfrak{Q} = T(\mathfrak{Q})$. For otherwise, $\mathfrak{P}'$ centralizes $\mathfrak{Q}$, by Lemma 8.16, so that $N(\mathfrak{P}') \subseteq \mathfrak{M}$. By Lemmas 26.1 and 26.2, $\mathfrak{P} \subseteq N(\mathfrak{P}')'$, contrary to $\mathfrak{P} \not\subseteq \mathfrak{M}'$. Hence, $\mathfrak{Q} = T(\mathfrak{Q})$.

Let $\mathfrak{Z} = Z(\Omega_1(\mathfrak{P}))$. We next show that $\mathfrak{Z}$ has no fixed points on $\mathfrak{Q}^\#$. Let $\mathfrak{Q}_1 = \mathfrak{Q} \cap C(\mathfrak{Z})$, and suppose by way of contradiction that $\mathfrak{Q}_1 \neq 1$. Let $\mathfrak{L} = N(\mathfrak{Z})$, and let $\mathfrak{L}_0$ be the maximal normal subgroup of $\mathfrak{L}$ of order prime to pq. Let $\mathfrak{L}_p$, $\mathfrak{L}_q$ be permutable Sylow subgroups of $\mathfrak{L}$, $\mathfrak{P} \subseteq \mathfrak{L}_p$, $\mathfrak{Q}_1 \subseteq \mathfrak{L}_q$. Since $\mathfrak{L}_p \subseteq \mathfrak{L}'$, it follows that $\mathfrak{L}$ is not contained in any conjugate of $\mathfrak{M}$. This implies that $\mathfrak{L}_q \in \mathscr{X}_0$. This in turn implies that $\mathfrak{L}_p$ centralizes every chief q-factor of $\mathfrak{L}$, by Lemma 8.13. Hence, $\mathfrak{L}_p \lhd \mathfrak{L}_p\mathfrak{L}_q$, and it follows that $N(\mathfrak{L}_q)$ covers $\mathfrak{L}/\mathfrak{L}_0\mathfrak{L}_p$. Since $N(\mathfrak{L}_q) \subseteq \mathfrak{M}$, by Lemma 26.6, we have a contradiction. Hence, $\mathfrak{Q}_1 = 1$.

We next show that if $P \in \Omega_1(\mathfrak{P}) - \mathfrak{Z}$, then $C(P) \subseteq \mathfrak{M}$. This is clear if $C(P) \cap \mathfrak{Q}$ is non cyclic, since $\mathfrak{Q} = T(\mathfrak{Q})$, so suppose $C(P) \cap \mathfrak{Q} = \mathfrak{Q}_1$ is cyclic. We remark that $\mathfrak{Q}_1 \neq 1$, an easy consequence of the preceding paragraph.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $C(P)$, and let $\mathfrak{Q}^*$ be a S_q-subgroup of $\mathfrak{M}_1$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}^*$ is non cyclic, then $\mathfrak{Q}^*$ is contained in a unique maximal subgroup $\mathfrak{M}^G$ of $\mathfrak{G}$, $G \in \mathfrak{G}$, and since $\mathfrak{Q}^* \subseteq \mathfrak{M}_1$, we have $\mathfrak{M}_1 = \mathfrak{M}^G$. Since $\mathfrak{M} \cap \mathfrak{M}^G \supseteq \mathfrak{Q}_1$, and since $\mathfrak{Q}_1 \subseteq T(\mathfrak{Q})$, we have $\mathfrak{M} = \mathfrak{M}^G$. Thus, we can suppose that $\mathfrak{Q}^*$ is cyclic.

Since $\mathfrak{Z}$ acts regularly on $\mathfrak{Q}_1$, we can suppose that a S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{M}_1$ normalizes $\mathfrak{Q}^*$ and that $\langle P, \mathfrak{Z}\rangle \subseteq \mathfrak{P}^*$.

If $\mathfrak{M}_1'$ is nilpotent, then $\Omega_1(\mathfrak{Q}^*) \lhd \mathfrak{M}_1$. Since $\Omega_1(\mathfrak{Q}^*) = \Omega_1(\mathfrak{Q}_1)$, we have $\mathfrak{M} = \mathfrak{M}_1$. Hence, we can suppose that $\mathfrak{M}_1'$ is not nilpotent.

Choose r in $\pi(H_1(\mathfrak{M}_1))$, and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}_1$ normalized by $\mathfrak{P}^*\mathfrak{Q}^*$. Since $\mathfrak{Q}^*$ is cyclic, $q \not\sim r$. Since $q \not\sim r$, $\mathfrak{Q}^*$ does not centralize $\mathfrak{R}$. It follows from $\mathfrak{Q}^* \subseteq (\mathfrak{P}^*\mathfrak{Q}^*)'$ that $\mathfrak{R} = T(\mathfrak{R})$, by Lemma 8.16. Since $\mathfrak{Q}^*\mathfrak{Z}$ is a Frobenius group, it follows that $\mathfrak{R}_1 = \mathfrak{R} \cap C(\mathfrak{Z}) \neq 1$. Let $\mathfrak{C} = N(\mathfrak{Z})$.

Let $\mathfrak{K}$ be a $S_{p,r}$-subgroup of $\mathfrak{C}$ which contains $\mathfrak{R}_1$ and $\mathfrak{P}^*$, and let $\mathfrak{K}_r$ be a S_r-subgroup of $\mathfrak{K}$ containing $\mathfrak{R}_1$. If $\mathfrak{K}_r$ is non cyclic, then $\mathfrak{K}_r \in \mathscr{X}_1$, so $\mathfrak{K} \subseteq \mathfrak{M}_1$. If $\mathfrak{K}_r$ is cyclic, then in any case $\mathfrak{K}_r \subseteq \mathfrak{M}_1$, since $\mathfrak{R} = T(\mathfrak{R})$. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$. If $\mathfrak{P}^*$ does not centralize $\mathfrak{R}_1$, then $r > p$, and so $\mathfrak{K}_r \lhd \mathfrak{K}$, and once again $\mathfrak{K} \subseteq \mathfrak{M}_1$. If $\mathfrak{P}^*$ centralizes $\mathfrak{R}_1$ and $\mathfrak{K}_r \ntriangleleft \mathfrak{K}$, then $\mathfrak{K}_p \lhd \mathfrak{K}$. Since the structure of $\mathfrak{K}_p$ is determined by Lemma 26.2, and since $\mathfrak{R}_1$ centralizes $\mathfrak{P}^*$, it follows that $\mathfrak{R}_1$ centralizes $\mathfrak{K}_p$, so that $\Omega_1(\mathfrak{R}_1) \lhd \mathfrak{K}$, and once again $\mathfrak{K} \subseteq \mathfrak{M}_1$. Thus, in any case, we see that $\mathfrak{K} \subseteq \mathfrak{M}_1$. This implies that $\mathfrak{Z} \subseteq \mathfrak{M}_1'$, so $\mathfrak{Z}$ centralizes every chief q-factor of $\mathfrak{M}_1$. This is absurd, since $\mathfrak{Z}\mathfrak{Q}^*$ is a Frobenius group. We conclude that $C(P) \subseteq \mathfrak{M}$ for every P in $\Omega_1(\mathfrak{P}) - \mathfrak{Z}$.

We will now show directly that $N(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$. Choose $N \in N(\Omega_1(\mathfrak{P}))$. Then $\Omega_1(\mathfrak{P})$ normalizes $\mathfrak{Q}$ and $\mathfrak{Q}^N$. Since $\mathfrak{Z}$ has no fixed points on $\mathfrak{Q}^N$, $\mathfrak{Q}^N$ is generated by its subgroups $\mathfrak{Q}^N \cap C(P)$, $P \in \Omega_1(\mathfrak{P}) - \mathfrak{Z}$. By the preceding paragraph, we conclude that $\mathfrak{Q}^N \subseteq \mathfrak{M}$. Since $\mathfrak{M}^N$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^N$, we have $\mathfrak{M} = \mathfrak{M}^N$, so $N \in \mathfrak{M}$. By Lemma 26.1, $\mathfrak{P} \subseteq N(\Omega_1(\mathfrak{P}))'$, so $\mathfrak{P} \subseteq \mathfrak{M}'$. The proof is complete.

LEMMA 26.12. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{P}$ is an abelian, non cyclic*

S_p-subgroup of $\mathfrak{M}$ for some prime p. Suppose further that a S_p-subgroup of $\mathfrak{G}$ is non abelian. Then $\mathfrak{P} = \mathfrak{P}_1 \times \mathfrak{P}_2$, where $|\mathfrak{P}_1| = p$, $\mathfrak{P}_1$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{P}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group with Frobenius kernel $\boldsymbol{H}(\mathfrak{M})$ and $\mathfrak{P}_2$ contains $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^*))$ for every S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$ which contains $\mathfrak{P}$.

Proof. Let $\mathfrak{P}_0$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. If $p \in \pi_0$, then $\Omega_1(\mathfrak{P}) \in \mathscr{T}(\mathfrak{P}_0)$, and if σ is any automorphism of $\mathfrak{P}_0$ of prime order s, then $s < p$, by Lemma 8.16. The same inequality clearly holds if $p \in \pi_2$.

Choose q in $\pi(\boldsymbol{H}(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$.

Let $\mathfrak{Z} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}_0))$. We will show that $\mathfrak{Q}\mathfrak{Z}$ is a Frobenius group. Let $\mathfrak{C} = \boldsymbol{N}(\mathfrak{Z})$ and suppose by way of contradiction that $\mathfrak{Q}_1 = \mathfrak{Q} \cap \mathfrak{C} \neq 1$. First consider the case that $p \in \pi_0$. Let $\mathfrak{M}_1 = \boldsymbol{M}(\mathfrak{C})$, and let $\mathfrak{P}_{00}$ be a S_p-subgroup of $\mathfrak{M}_1$ normalized by $\mathfrak{Q}_1$ with $\mathfrak{P} \subseteq \mathfrak{P}_{00}$. Then $[\mathfrak{Q}_1, \mathfrak{P}] \subseteq \mathfrak{Q} \cap \mathfrak{P}_{00} = 1$, so $\mathfrak{Q}_1$ centralizes $\mathfrak{P}$. Since $\Omega_1(\mathfrak{P}) \subseteq \mathscr{T}(\mathfrak{P}_{00})$, it follows that $\mathfrak{Q}_1$ centralizes $\mathfrak{P}_{00}$. Thus, if $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$ or $\boldsymbol{T}(\mathfrak{Q}) = \mathfrak{Q}$, we conclude that $\mathfrak{P}_{00} \subseteq \mathfrak{M}$, which is contrary to hypothesis. Otherwise, $\boldsymbol{T}(\mathfrak{Q}) \subset \mathfrak{Q}$, or $q \in \pi_1 \cup \pi_2$, so that $q > p$, or $\mathfrak{P}$ centralizes $\mathfrak{Q}$. But in these cases, we at least have $\boldsymbol{N}(\mathfrak{Q}_1) \subseteq \mathfrak{M}_1$, so $\mathfrak{Q}_1 \neq \mathfrak{Q}$, which yields $q > p$, and so a S_q-subgroup of $\mathfrak{M} \cap \mathfrak{M}_1$ is non cyclic, and centralizes $\mathfrak{P}_{00}$. Again we conclude that $\mathfrak{P}_{00} \subseteq \mathfrak{M}$, which is not the case. Hence, we can suppose that $p \in \pi_2$.

Let $\mathfrak{R}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}\mathfrak{Q}_1$, $\mathfrak{P} \subseteq \mathfrak{R}_p$, $\mathfrak{Q}_1 \subseteq \mathfrak{R}_q$, and let $\mathfrak{R}^*$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{R}$, $\mathfrak{R}_p \subseteq \mathfrak{R}_p^*$, $\mathfrak{R}_q \subseteq \mathfrak{R}_q^*$, where $\mathfrak{R}_p^*$ is a S_p-subgroup of $\mathfrak{R}^*$ and $\mathfrak{R}_q^*$ is a S_q-subgroup of $\mathfrak{R}^*$. Since $\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{R}_p = \mathfrak{R}_p^*$ is a S_p-subgroup of $\mathfrak{G}$. If $\mathfrak{R}_q^*$ contains an elementary subgroup of order q^3, then $\mathfrak{R}_q^* \lhd \mathfrak{R}^*$, and maximality of $\mathfrak{R}^*$ implies that $\mathfrak{R}^*$ is contained in a conjugate of $\mathfrak{M}$, contrary to hypothesis. If $\mathfrak{R}_q^*$ does not contain an elementary subgroup of order q^3, then either $q > p$ or $\mathfrak{P}$ centralizes $\mathfrak{Q}_1$. If $q > p$, then $\mathfrak{R}_q^* \lhd \mathfrak{R}^*$, so once again $\mathfrak{R}^* \subseteq \mathfrak{M}^G$ for some $G \in \mathfrak{G}$. If $q < p$, then $\mathfrak{R}_p^* \lhd \mathfrak{R}^*$, and since $\mathfrak{Q}_1$ centralizes $\mathfrak{P}$, $\mathfrak{Q}_1$ centralizes $\mathfrak{R}_p^*$, by Lemma 26.2. In this case, $\boldsymbol{O}_q(\mathfrak{R}^*) \neq 1$. If $\boldsymbol{O}_q(\mathfrak{R}^*)$ is non cyclic, then $\mathfrak{R}^* \subseteq \mathfrak{M}^G$, either by Lemma 26.6, in case $q \in \pi_0$, or because $\mathfrak{Q} \lhd \mathfrak{M}$ in case $q \in \pi_2$. If $\boldsymbol{O}_q(\mathfrak{R}^*)$ is cyclic, then $\mathfrak{Q}_1 \lhd \mathfrak{R}^*$. In this case $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)\mathfrak{P}$ is conjugate to a subgroup of $\mathfrak{R}^*$, since $\mathfrak{R}^*$ is a S-subgroup of $\boldsymbol{N}(\mathfrak{Q}_1)$. Since $\mathfrak{R}_p^* \lhd \mathfrak{R}^*$, it follows that $\mathfrak{P}$ centralizes $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ so that $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ centralizes some S_p-subgroup of $\boldsymbol{N}(\mathfrak{Q}_1)$. If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, this is not possible. But if $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, then $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ is non cyclic, so $\boldsymbol{N}(\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)) \subseteq \mathfrak{M}$. Thus, in all these cases, $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$. Since this is not possible, $\mathfrak{Z}\mathfrak{Q}$ is a Frobenius group, and so $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Suppose $\mathfrak{M} \in \mathscr{M}_0$. We will show that if $\mathfrak{Z}_0$ is any subgroup of $\mathfrak{P}$ of order p with $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, then $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$. Let $\mathfrak{M}_1 \in \mathscr{M}$ with $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}_1$. First consider the case $\mathfrak{M}_1 = \mathfrak{M}^G$, for some G in $\mathfrak{G}$. Let $\mathfrak{Q}_1$ be a non identity S_q-subgroup of $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M})$ and let $\mathfrak{Q}_2$ be a S_q-subgroup of $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}_1)$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}_1 \subset \mathfrak{Q}_2$, then Lemma 26.2 implies that $\mathfrak{Q}_2$ is a S_q-subgroup of $\mathfrak{G}$. In this case, since $\mathfrak{M}_1$ and $\mathfrak{M}$ are conjugate and since $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}$, $\mathfrak{P}$ contains a subgroup of order p which centralizes the S_q-subgroup of $\mathfrak{M}$. Since $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, this implies that if $\mathfrak{Z}_1$ is any subgroup of $\mathfrak{P}$ of order p, then either $\mathfrak{Z}_1\boldsymbol{H}_q(\mathfrak{M})$ is a Frobenius group, or $\mathfrak{Z}_1$ centralizes $\boldsymbol{H}_q(\mathfrak{M})$, the S_q-subgroup of $\mathfrak{M}$. This violates the choice of $\mathfrak{Q}_1$. Hence, $\mathfrak{Q}_1 = \mathfrak{Q}_2$. If a S_q-subgroup of $\mathfrak{G}$ is abelian, then $\mathfrak{Q}_1 \lhd \langle \mathfrak{M}, \mathfrak{M}_1 \rangle$, so $\mathfrak{M} = \mathfrak{M}_1$. If some S_q-subgroup of $\mathfrak{G}$ contains $\Omega_1(\mathfrak{Q}_1)$ in its center, then by Lemma 8.10, $\mathfrak{M} = \mathfrak{M}_1$. Hence, we can suppose that $\mathfrak{Q}_1$ is of order q and $\mathfrak{Q}_1 \not\subseteq \boldsymbol{Z}(\boldsymbol{H}(\mathfrak{M}))$. In this case, $N(\mathfrak{Q}_1) \cap \mathfrak{M}_1$ is of index q in $\mathfrak{M}_1$ and $N(\mathfrak{Q}_1) \cap \mathfrak{M}$ is of index q in $\mathfrak{M}$, and $N(\mathfrak{Q}_1) \cap \mathfrak{M}_1$ contains $C(\mathfrak{Z}_0)$.

Let $\mathfrak{L} = N(\mathfrak{Q}_1)$. If $\mathfrak{L}$ is contained in a conjugate of $\mathfrak{M}$, then $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{L}$ so $\mathfrak{L} \subseteq \mathfrak{M}_1$, since $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{M}_1$. Similarly, $\mathfrak{L} \subseteq \mathfrak{M}$, and we are done. If $\mathfrak{L}$ is contained in an element of $\mathscr{M}_0$, then since $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, we see that $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}) \lhd \langle \mathfrak{L}, \mathfrak{M} \rangle$, and $\mathfrak{L} \subseteq \mathfrak{M}$.

Hence, in showing that $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$, we can suppose that $C(\mathfrak{Z}_0)$ is contained in an element $\mathfrak{M}_1$ of $\mathscr{M}_1$. Since $\mathfrak{Z} \cdot (C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}))$ is a Frobenius group, this implies that $\mathfrak{Z} \not\subseteq \mathfrak{M}_1'$. Since $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}$, we conclude that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}_1$. By what we have already proved, $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group. This implies that $(C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}))\boldsymbol{H}_1(\mathfrak{M}_1)$ is nilpotent, so $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M}_1)$. Since $\mathfrak{M}_1$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M}_1)$, it follows that $\boldsymbol{H}(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M}_1)$, so that $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is absurd since $\mathfrak{M} \in \mathscr{M}_0$, $\mathfrak{M}_1 \in \mathscr{M}_1$. We conclude that $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$.

We next show that if $\mathfrak{M} \in \mathscr{M}_1$ and $C(\mathfrak{Z}_0)$ contains an element of $\hat{\boldsymbol{H}}(\mathfrak{M})$, then $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$. Here, as above, $\mathfrak{Z}_0$ is a subgroup of $\mathfrak{P}$ of order p. Let $\mathfrak{Q}_1$ be a $\mathfrak{P}$-invariant S_q-subgroup of $C(\mathfrak{Z}_0) \cap \mathfrak{M}$ with $\mathfrak{Q}_1 \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \emptyset$. From Lemma 26.7, we conclude that $C(\mathfrak{Z}_0) \cap \mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}_2$ of $C(\mathfrak{Z}_0)$, and we can assume that $\mathfrak{Q}_1 = \mathfrak{Q}_2$.

Let $\mathfrak{M}_1 \in \mathscr{M}$, $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}_1$. If $\mathfrak{M}_1 = \mathfrak{M}^G$, then $\mathfrak{M} \cap \mathfrak{M}_1 \supseteq \mathfrak{Q}_1$, so $\mathfrak{M} = \mathfrak{M}_1$. If $\mathfrak{M}_1'$ is nilpotent, then by Lemma 26.7, we see that $\mathfrak{M}_1 \cap \mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}_3$ of $\mathfrak{M}_1$ which is $\mathfrak{Z}$-invariant. Since $\mathfrak{Z}\mathfrak{Q}_3$ is a Frobenius group, $\mathfrak{Q}_3 \lhd \mathfrak{M}_1$ and so $\mathfrak{M}_1 = \mathfrak{M}$. We can suppose that $\mathfrak{M}_1'$ is not nilpotent, and that $\mathfrak{M}_1 \neq \mathfrak{M}$. In particular, $\mathfrak{M}_1 \in \mathscr{M}_1$. It follows that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}_1$, so that $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, and so $\mathfrak{Q}_2$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$, and $\mathfrak{M} = \mathfrak{M}_1$ follows.

Thus, $\mathfrak{M} = \mathfrak{M}_1$ in all cases.

Suppose now that $\mathfrak{P}$ contains two distinct subgroups $\mathfrak{Z}_0$, $\mathfrak{Z}_1$ such that $C(\mathfrak{Z}_0) \cap \hat{H}(\mathfrak{M}) \neq \emptyset$ and $C(\mathfrak{Z}_1) \cap \hat{H}(\mathfrak{M}) \neq \emptyset$. We can choose P in $\mathfrak{P}_0$ such that $\mathfrak{Z}_0 = \mathfrak{Z}_1^P$. If $\mathfrak{M} \in \mathscr{M}_1$, we get an easy contradiction. Namely, $C(\mathfrak{Z}_0) \subseteq \mathfrak{M} \cap \mathfrak{M}^P$, and so $\mathfrak{M} = \mathfrak{M}^P$ and $P \in \mathfrak{M} \cap \mathfrak{P}_0 = \mathfrak{P}$, so that $\mathfrak{Z}_0 = \mathfrak{Z}_1$, contrary to assumption.

If $\mathfrak{M} \in \mathscr{M}_0$, then $\mathfrak{M} \cap \mathfrak{M}^P$ contains $C(\mathfrak{Z}_0) \cap H(\mathfrak{M})$. If $H(\mathfrak{M})$ contains an abelian S_q-subgroup $\mathfrak{Q}$ with $C(\mathfrak{Z}_0) \cap \mathfrak{Q} \neq 1$, then $C(\mathfrak{Z}_0) \cap \mathfrak{Q} \lhd \langle \mathfrak{M}, \mathfrak{M}^P \rangle$, and $\mathfrak{M} = \mathfrak{M}^P$, which is the desired contradiction. Otherwise, if $\mathfrak{Q}$ is a S_q-subgroup of $H(\mathfrak{M})$ with $C(\mathfrak{Z}_0) \cap \mathfrak{Q} = \mathfrak{Q}_1 \neq 1$, then $N(\mathfrak{Q}_1) \cap \mathfrak{M}$ is of index q in $\mathfrak{M}$ and $N(\mathfrak{Q}_1) \cap \mathfrak{M}^P$ is of index q in $\mathfrak{M}^P$, while both $N(\mathfrak{Q}_1) \cap H(\mathfrak{M})$ and $N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P)$ are S-subgroups of $N(\mathfrak{Q}_1)$. Furthermore, since a $S_{p,q}$-subgroup $\mathfrak{R}_0$ of $N(\mathfrak{Q}_1)$ is q-closed, it follows that $\mathfrak{P}(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}))$ and $\mathfrak{P}(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))$ are S-subgroups of $N(\mathfrak{Q}_1)$. Furthermore, $\mathfrak{P}$ has a normal complement in $N(\mathfrak{Q}_1)$, since $q \in \pi_2$, and no element of $\mathfrak{P}^\#$ centralizes $N(\mathfrak{Q}_1) \cap \mathfrak{Q}$. By the conjugacy of Sylow systems in $N(\mathfrak{Q}_1)$, we can therefore find $C \in C(\mathfrak{P}) \cap N(\mathfrak{Q}_1)$ such that $(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))^C = N(\mathfrak{Q}_1) \cap H(\mathfrak{M})$. Since $(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))^C = N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^{PC})$, and $N(\mathfrak{Q}_1) \cap H(\mathfrak{M}) \lhd \mathfrak{M}$, we conclude that $\mathfrak{M} = \mathfrak{M}^{PC}$, so $PC \in \mathfrak{M}$, which is not the case, since C is in $\mathfrak{M}$ and P is not.

Hence, there is exactly one subgroup $\mathfrak{Z}_0$ of $\mathfrak{P}$ of order p which has a fixed point on $\hat{H}(\mathfrak{M})$, so $\mathfrak{Z}_0$ centralizes $H(\mathfrak{M})$. Since $\mathfrak{P} = \mathfrak{Z}_0 \times \mathfrak{P}^*$, where $\mathfrak{P}^* \supseteq \mathfrak{Z}$, the lemma follows.

Lemma 26.12 is quite important because, given $\mathfrak{M}$, (and the hypothesis of Lemma 26.12) it produces a unique factorization of $\Omega_1(\mathfrak{P})$. Namely, exactly one subgroup $\mathfrak{Z}$ of $\mathfrak{P}$ of order p is in the center of a S_p-subgroup of $\mathfrak{G}$, and exactly one subgroup $\mathfrak{Z}_0$ of $\mathfrak{P}$ of order p centralizes $H(\mathfrak{M})$, and $\mathfrak{Z} \neq \mathfrak{Z}_0$. This is a critical point in dealing with tamely imbedded subsets. Furthermore, Lemma 26.12 shows that $H(\mathfrak{M})$ is nilpotent, a useful fact.

LEMMA 26.13. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{P}$ is an abelian, non cyclic S_p-subgroup of $\mathfrak{M}$ for some prime p. Suppose further that a S_p-subgroup of $\mathfrak{G}$ is abelian. Then the following statements are true:*

(i) *$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$.*

(ii) $C(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$.

(iii) *If P and P_1 are elements of $\mathfrak{P}$ which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{M}$, either $C(P) \cap H(\mathfrak{M}) = 1$ or $C(P_1) \cap H(\mathfrak{M}) = 1$.*

(iv) *Either $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P})$ or $C(\Omega_1(\mathfrak{P})) \cap H(\mathfrak{M}) = 1$.*

(v) *One of the following conditions holds:*

(a) $\mathfrak{P} \subseteq \mathfrak{M}'$.

(b) *$N(\mathfrak{P}_0) \subseteq \mathfrak{M}$ for every non identity subgroup $\mathfrak{P}_0$ of $\mathfrak{P}$ such* *that* $C(\mathfrak{P}_0) \cap H(\mathfrak{M}) \neq 1$.

Proof. If $p \in \pi_0$, then $\mathfrak{P} \in \mathscr{H}_1$ and all parts of the lemma follow immediately. We can suppose that $p \in \pi_2$.

In proving this lemma, appeal to Lemmas 8.5 and 8.16 will be made repeatedly.

If $\Omega_1(\mathfrak{P})$ centralizes $H(\mathfrak{M})$, then $\mathfrak{M} = N(\Omega_1(\mathfrak{P}))$ and all parts of the lemma follow immediately. We can suppose that $\Omega_1(\mathfrak{P})$ does not centralize $H(\mathfrak{M})$. This implies that $H(\mathfrak{M}) \cap \mathfrak{P} = 1$.

We first prove an auxiliary result: if $\mathfrak{K}$ is any p, q-subgroup of $\mathfrak{G}$ containing $\Omega_1(\mathfrak{P})$ and if $\mathfrak{K} \cap H(\mathfrak{M}) \neq 1$, then $\mathfrak{K}$ is q-closed. To see this, let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{K} \cap H(\mathfrak{M})$, and let $\mathfrak{P}_1$ be a S_p-subgroup of $\mathfrak{K} \cap \mathfrak{M}$ which contains $\Omega_1(\mathfrak{P})$. Let $\mathfrak{K}_q$ be a S_q-subgroup of $\mathfrak{K}$ containing $\mathfrak{Q}$ and let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$ containing $\mathfrak{P}_1$. If $\mathfrak{K}_q \in \mathscr{H}_1$, then $\mathfrak{K} \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$ and so $\mathfrak{K}_q \lhd \mathfrak{K}$. If $\mathfrak{K}_q \in \mathscr{H}_0$, then $\mathfrak{K}$ does not contain elementary subgroups of order p^3 or q^3, so either $\mathfrak{K}_q \lhd \mathfrak{K}$ or $\mathfrak{K}_p \lhd \mathfrak{K}$. If $\mathfrak{K}_p \lhd \mathfrak{K}$, and $\mathfrak{K}_q \not\lhd \mathfrak{K}$, then $p > q$. Suppose $q \in \pi_1 \cup \pi_2$. Then $\mathfrak{P}$ centralizes the S_q-subgroup $\mathfrak{Q}_1$ of $\mathfrak{M}$. There is no loss of generality in supposing that $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$. It follows from this normalization that $O_q(\mathfrak{K})$ is a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{K} = \mathfrak{K}_p \times \mathfrak{K}_q$. Hence, we can suppose $q \in \pi_0$. Since $\mathfrak{K}_q \not\rhd \mathfrak{K}$, $\mathfrak{K}_q \in \mathscr{H}_0$. If $O_q(\mathfrak{K})$ is not of order q, then $\mathfrak{K}$ is contained in a conjugate of $\mathfrak{M}$, by Lemma 26.7, and we are done. Hence, we can suppose that $\mathfrak{Q} = O_q(\mathfrak{K})$ is of order q. But now $N(\mathfrak{Q}) \cap \mathfrak{M}$ contains S_q-subgroups of order exceeding q, so that $S_{p,q}$-subgroups of $N(\mathfrak{Q})$ are q-closed. Since $\mathfrak{K} \subseteq N(\mathfrak{Q})$, $\mathfrak{K}$ is q-closed

(i) is an immediate application of the preceding paragraph, since some element of $\mathfrak{P}^\#$ centralizes an element of $H(\mathfrak{M})^\#$.

We turn next to (iv). Suppose $C(\Omega_1(\mathfrak{P})) \cap H(\mathfrak{M}) \neq 1$, and $\mathfrak{Q}_0$ is a non identity $\mathfrak{P}$-invariant S_q-subgroup of $C(\Omega_1(\mathfrak{P})) \cap H(\mathfrak{M})$. Let $\mathfrak{Q}_1$ be a S_q-subgroup of $N(\Omega_1(\mathfrak{P}))$ permutable with $\mathfrak{P}$. By the first paragraph of the proof, $\mathfrak{P}$ normalizes $\mathfrak{Q}_1$, so by Sylow's theorem $N(\mathfrak{Q}_1)$ dominates $\Omega_1(\mathfrak{P})$. Suppose for some $n \geqq 1$, $\mathfrak{P}$ normalizes $\mathfrak{Q}_n$ and $\mathfrak{Q}_n$ dominates $\Omega_1(\mathfrak{P})$. Let $\mathfrak{Q}_{n+1}$ be a S_q-subgroup of $N(\mathfrak{Q}_n)$ permutable with $\mathfrak{P}$. Then $\mathfrak{P}$ normalizes $\mathfrak{Q}_{n+1}$ and so $\mathfrak{Q}_{n+1}$ dominates $\Omega_1(\mathfrak{P})$. Since $\mathfrak{Q}_0 \subseteq \mathfrak{Q}_1 \subseteq \cdots$, we see that some S_q-subgroup of $\mathfrak{G}$ dominates $\Omega_1(\mathfrak{P})$ and is normalized by $\mathfrak{P}$. It follows that the normalizer of every S_q-subgroup of $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P}^M)$ for some M in $\mathfrak{M}$, and so $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P})$. (iv) is proved.

Notice that if $C(\Omega_1(\mathfrak{P})) \cap H(\mathfrak{M}) \neq 1$, then by (iv), elements of $\mathfrak{P}$ are conjugate in $\mathfrak{C}$ if and only if they are conjugate in $\mathfrak{M}$. Thus, in the case, it only remains to prove (ii). We emphasize that in any case (i) and (iv) are proved.

Since $\mathfrak{P} \subseteq \mathfrak{M}'$, if $\mathfrak{M} \in \mathscr{M}_0$, then $\mathfrak{P} \triangleleft \mathfrak{M}$ and the lemma follows. We can suppose that $\mathfrak{M} \in \mathscr{M}_1$. Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\mathfrak{M}$. If $\Omega_1(\mathfrak{P})$ centralizes $\boldsymbol{T}(\mathfrak{Q})$, then (ii) follows immediately. Thus, we can choose P in $\Omega_1(\mathfrak{P})^{\#}$ such that $\Omega_1(\mathfrak{P})$ does not centralize $\boldsymbol{T}(\mathfrak{Q}) \cap \boldsymbol{C}(\mathfrak{P}) = \mathfrak{Q}_1$. If $\mathfrak{Q}_1 \in \mathscr{X}_1$, then $\boldsymbol{C}(P) \subseteq \mathfrak{M}$, so that (ii) holds. If $\mathfrak{Q}_1 \in \mathscr{X}_0$, then $\mathfrak{Q}_1$ is cyclic, by Lemma 8.16, and the containment $\mathfrak{P} \subseteq \mathfrak{M}'$. Hence $\Omega_1(\mathfrak{P}) = \langle P \rangle \times \mathfrak{P}_0$, where $\mathfrak{P}_0\mathfrak{Q}_1$ is a Frobenius group.

Let $\mathfrak{C} = \boldsymbol{C}(P)$. If $\mathfrak{C}'$ is nilpotent, then $\mathfrak{Q}_1 \subseteq \boldsymbol{O}_q(\mathfrak{C})$, so by Lemma 26.7, $\mathfrak{C} \subseteq \mathfrak{M}$, and (ii) follows. Suppose $\mathfrak{C}'$ is not nilpotent. Hence, $\mathfrak{C}$ contains an elementary subgroup of order r^3 for some prime r. If $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$ then $\mathfrak{C} \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Since $\mathfrak{M} \cap \mathfrak{M}^G \supseteq \mathfrak{Q}_1$, we have $\mathfrak{M} = \mathfrak{M}^G$ and (ii) follows. Suppose $r \notin \pi(\boldsymbol{H}_1(\mathfrak{M}))$. In this case, $\Omega_1(\mathfrak{P})\mathfrak{Q}_1$ normalizes a S_r-subgroup $\mathfrak{R}$ of $\mathfrak{C}$. Since P centralizes $\mathfrak{R}$ and $\mathfrak{P}_0\mathfrak{Q}_1$ is a Frobenius group, and since $q \not\sim r$, it follows that $\mathfrak{R} \cap \boldsymbol{C}(\Omega_1(\mathfrak{P})) \neq 1$. Let $\mathfrak{M}_1 = \boldsymbol{M}(\mathfrak{C})$. By (iv) applied to $\mathfrak{M}_1$, we get $\mathfrak{P} \subseteq \mathfrak{M}_1'$. Since $\mathfrak{Q}_1 \cap \boldsymbol{H}(\mathfrak{M}_1) = 1$, and since the derived group of $\mathfrak{M}_1/\boldsymbol{H}(\mathfrak{M}_1)$ is nilpotent, $\mathfrak{P}$ centralizes $\mathfrak{Q}_1$, which is a contradiction. Hence, $\boldsymbol{C}(P) \subseteq \mathfrak{M}$, and (ii) holds. The lemma is proved in case $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, and (i) is proved in all cases.

Throughout the remainder of the proof, we assume

(26.1) $$\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \quad 1$$

Suppose $\mathfrak{P}_0$ is a non identity subgroup of $\mathfrak{P}$ and

(26.2) $$\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1 .$$

There are three cases:

(a) $\mathfrak{M} \in \mathscr{M}_1$ and $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \varnothing$.

(b) $\mathfrak{M} \in \mathscr{M}_1$ and $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M}) = \varnothing$

(c) $\mathfrak{M} \in \mathscr{M}_0$.

In each of these cases, we will show that

(26.3) $$\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M}$$

Case a_1. $\boldsymbol{N}(\mathfrak{P}_0)'$ is nilpotent.

Choose q so that $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M})$ contains an element of order q, and let $\mathfrak{Q}_0$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$. By (26.1), $\mathfrak{Q}_0 \subseteq \boldsymbol{N}(\mathfrak{P}_0)'$, so $\mathfrak{Q}_0 \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$. If $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, we conclude that $\boldsymbol{N}(\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))) \subseteq \mathfrak{M}$, by Lemma 26.7. If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, then $\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ centralizes $\boldsymbol{H}_1(\mathfrak{M})^G$ for some G in $\mathfrak{G}$, and so $\boldsymbol{N}(\mathfrak{Q}_0) \supseteq \langle \boldsymbol{H}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M})^G \rangle$, and $G \in \mathfrak{M}$ follows.

Case a_2. $\boldsymbol{N}(\mathfrak{P}_0)'$ is not nilpotent.

In this case, $\boldsymbol{N}(\mathfrak{P}_0)$ contains an elementary subgroup of order r^3

for some prime r. If $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, then $\boldsymbol{M}(\boldsymbol{N}(\mathfrak{P}_0)) = \mathfrak{M}^G$, for some G in $\mathfrak{G}$. Since $\mathfrak{M}^G \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \emptyset$, we have $\mathfrak{M} = \mathfrak{M}^G$. If $r \notin \pi(\boldsymbol{H}(\mathfrak{M}))$, let $\mathfrak{R}$ be a S_r-subgroup of $\boldsymbol{N}(\mathfrak{P}_0)$ normalized by $\Omega_1(\mathfrak{P})\mathfrak{Q}_0$, where $\mathfrak{Q}_0$ is a non identity S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$, as in Case a_1. Let $\Omega_1(\mathfrak{P}) = \Omega_1(\mathfrak{P}_0) \times \mathfrak{P}_1$ so that $\mathfrak{Q}_0\mathfrak{P}_1$ is a Frobenius group by (26.1). If $\mathfrak{P}_1\mathfrak{R}$ is a Frobenius group, then $\mathfrak{Q}_0$ centralizes $\mathfrak{R}$, and $\mathfrak{R} \subseteq \mathfrak{M}$. This is not the case, since $r \not\sim r_1$ for all $r_1 \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Hence, $\mathfrak{P}_1$ has a fixed point on $\mathfrak{R}^\#$, so $\Omega_1(\mathfrak{P})$ has a fixed point on $\boldsymbol{H}(\boldsymbol{M}(\mathfrak{R}))$. By (iv) applied to $\boldsymbol{M}(\mathfrak{R})$, it follows that $\Omega_1(\mathfrak{P}) \subseteq \boldsymbol{M}(\mathfrak{R})'$, and so $\Omega_1(\mathfrak{P})$ centralizes $\mathfrak{Q}_0$, which is not the case. Thus (26.3) holds in case (a).

In analysing case (b), we use the fact that $\boldsymbol{E}_1(\mathfrak{M})^\# \subseteq \hat{\boldsymbol{H}}(\mathfrak{M})$, and that if $\mathfrak{B}$ is any subgroup of $\boldsymbol{H}(\mathfrak{M})$ which is disjoint from $\hat{\boldsymbol{H}}(\mathfrak{M})$, then $\mathfrak{B}$ is of square free order and $q \in \pi_0 \cap \pi^*$ for every q in $\pi(\mathfrak{B})$.

Let $\mathfrak{Q}$ be a non identity $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$. so that $|\mathfrak{Q}| = q$. Suppose that (26.3) does not hold.

We will show that $\mathfrak{P}\mathfrak{Q}$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ such that $\mathfrak{M}_1'$ is not nilpotent, and such that $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$.

Case b_1. $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$.

Consider $\boldsymbol{N}(\mathfrak{Q})$. Since $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$ and $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}^G$ have non cyclic S_q-subgroups, and since $\mathfrak{M} \neq \mathfrak{M}^G$, it follows that $\boldsymbol{N}(\mathfrak{Q})$ is contained in no conjugate of $\mathfrak{M}$. Let $\mathfrak{Q}_1$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{N}(\mathfrak{Q}) \cap \boldsymbol{H}(\mathfrak{M})$. If $\boldsymbol{N}(\mathfrak{Q})'$ is nilpotent, then $\mathfrak{Q}_1 \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{Q}))$, and so $\boldsymbol{N}(\mathfrak{Q}) \subseteq \mathfrak{M}$ by Lemma 26.7. This is not the case, since $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}^G$ has non cyclic S_q-subgroups. Hence, $\boldsymbol{N}(\mathfrak{Q})'$ is not nilpotent, so we take $\mathfrak{M}_1 = \boldsymbol{M}(\boldsymbol{N}(\mathfrak{Q}))$.

Case b_2. $\boldsymbol{N}(\mathfrak{P}_0)'$ is nilpotent, but $\boldsymbol{N}(\mathfrak{P}_0)$ is not contained in any conjugate of $\mathfrak{M}$.

Since $\mathfrak{Q} \subseteq \boldsymbol{N}(\mathfrak{P}_0)'$, $\mathfrak{Q} \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$. If $\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ is not of order q, then $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Suppose that $\mathfrak{Q} = \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ is of order q. Let $\mathfrak{N}_1 = \boldsymbol{N}(\mathfrak{Q})$, so that $\mathfrak{N}_1 \cap \mathfrak{M}$ has non cyclic S_q-subgroups and $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{N}_1$. Since $\boldsymbol{N}(\mathfrak{P}_0)$ is contained in no conjugate of $\mathfrak{M}$, neither is $\mathfrak{N}_1$. If $\mathfrak{N}_1'$ is nilpotent, then a S_q-subgroup of $\mathfrak{N}_1 \cap \mathfrak{M}$ is contained in $\boldsymbol{O}_q(\mathfrak{N}_1)$, by (26.1) and so $\mathfrak{N}_1 = \mathfrak{M}$, which is not the case.

We apply (iv) to $\mathfrak{M}_1$. If $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}_1) \neq 1$, then $\mathfrak{P} \subseteq \mathfrak{M}_1'$, so that $\mathfrak{P}$ centralizes $\mathfrak{Q}$, which is not the case. Hence, (26.1) holds with $\mathfrak{M}_1$ replacing $\mathfrak{M}$. Let $\mathfrak{P}_1$ be any subgroup of $\mathfrak{P}$ of order p different from $\Omega_1(\mathfrak{P}_0)$. Then $\mathfrak{P}_1\mathfrak{Q}$ is a Frobenius group. Choose $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}_1))$ and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}_1$ invariant under $\mathfrak{P}\mathfrak{Q}$. If $\mathfrak{Q}$ does not centralize $\boldsymbol{T}(\mathfrak{R})$, then $\boldsymbol{C}(\mathfrak{P}_1) \cap \boldsymbol{T}(\mathfrak{R}) \neq 1$, so that case (a) holds with $\mathfrak{M}_1$ replacing $\mathfrak{M}$, $\mathfrak{P}_1$ replacing $\mathfrak{P}_0$.

Suppose then that $\mathfrak{Q}$ centralizes $\boldsymbol{T}(\mathfrak{R})$. Then $\boldsymbol{N}(\mathfrak{Q}) \subseteq \mathfrak{M}_1$, so a S_q-subgroup $\mathfrak{Q}_1$ of $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$ is contained in $\mathfrak{M}_1$. We suppose without loss of generality that $\mathfrak{Q}_1$ normalizes $\mathfrak{R}$. If now $\mathfrak{P}_2$ is any subgroup of $\mathfrak{P}$ of order p which does not centralize $\mathfrak{Q}_1/\mathfrak{Q}$, then since $\mathfrak{Q}_1$ does not centralize $T(\mathfrak{R})$, we conclude that $C(\mathfrak{P}_2) \cap T(\mathfrak{R}) \neq 1$.

Thus, in all cases, if $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots, \mathfrak{P}_n^*$ are the distinct subgroups of $\mathfrak{P}$ of order p which have fixed points on $\hat{H}(\mathfrak{M}_1)$, then $n \geqq p$, so that $n = p$ or $p + 1$.

Choose $N \in N(\Omega_1(\mathfrak{P}))$. Then there are indices i, j, not necessarily distinct, such that $\mathfrak{P}_i^* = \mathfrak{P}_j^{*N}$. If $i = j$, then $N \in \mathfrak{M}_1$, by (a). If $i \neq j$, then $N(\mathfrak{P}_i^*) \subseteq \mathfrak{M}_1 \cap \mathfrak{M}_1^N$, so that $\hat{H}(\mathfrak{M}_1) \cap \mathfrak{M}_1^N \neq \emptyset$ and $\mathfrak{M}_1 = \mathfrak{M}_1^N$. Hence, $N(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}_1$, so $\Omega_1(\mathfrak{P}) \subseteq \mathfrak{M}_1'$, and $\Omega_1(\mathfrak{P})$ centralizes $\mathfrak{Q}$, which is not the case. Hence, (b) implies (26.3).

We will now complete the proof of this lemma in case $\mathfrak{M} \in \mathscr{M}_1$.

Since some element of $\Omega_1(\mathfrak{P})^\#$ has a fixed point on $\hat{H}(\mathfrak{M})$, (ii) holds by (26.3). Also, by (26.3), alternative (v)b holds. It remains to prove (iii). Suppose P_1, P_2 are elements of $\mathfrak{P}$ which are conjugate in $\mathfrak{G}$, but are not conjugate in $\mathfrak{M}$, and that $C(P_i) \cap H(\mathfrak{M}) \neq 1$, $i = 1, 2$. Theorem 17.1 is violated.

We next verify (26.3) under hypothesis (c).

Suppose by way of contradiction that (26.3) does not hold. Let $\mathfrak{Q}$ be a non identity $\mathfrak{P}$-invariant S_q-subgroup of $C(\mathfrak{P}_0) \cap H(\mathfrak{M})$. We will produce a subgroup $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}'$ is not nilpotent, and such that $\mathfrak{Q}\mathfrak{P} \subseteq \mathfrak{K}$. Once this is done, then it will follow as in case b_2 that p of the $p+1$ subgroups of $\mathfrak{P}$ of order p have fixed points on $H(M(\mathfrak{K}))^\#$, and (26.3) will follow.

Suppose $\mathfrak{M}_1$ is a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{P}_0)$. If $\mathfrak{M}_1'$ is nilpotent, then $\mathfrak{Q} \subseteq O_q(\mathfrak{M}_1)$. If $O_q(\mathfrak{M}_1)$ is non abelian, then $\mathfrak{M}_1 = \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Furthermore, from (26.1) and the fact that $\mathfrak{Q}$ is not a S_q-subgroup of $\mathfrak{G}$, we conclude that $\mathfrak{Q} = O_q(\mathfrak{M}_1) \cap C(\mathfrak{P}_0)$. Hence, $N(\mathfrak{Q})$ contains $C(\mathfrak{P}_0)$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{Q})$. If $\mathfrak{M}_2'$ is nilpotent, then $\mathfrak{M}_2 = \mathfrak{M}$ and (26.3) holds. Hence, $\mathfrak{M}_2'$ is not nilpotent, so we can take $\mathfrak{K} = \mathfrak{M}_2$. If $O_q(\mathfrak{M}_1)$ is abelian, then $\mathfrak{M} = \mathfrak{M}_1$ and (26.3) holds. Thus, (26.3) holds in all cases.

The completion of the proof that (26.3) implies this lemma is a straightforward application of Theorem 17.1.

LEMMA 26.14. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{P}$ is a non abelian S_p-subgroup of $\mathfrak{M}$. Then $N(\Omega_1(Z(\mathfrak{P}))) \subseteq \mathfrak{M}$. Furthermore, one of the following conditions is true:*

(a) *$\Omega_1(Z(\mathfrak{P}))$ centralizes $H(\mathfrak{M})$.*

(b) *$N(\mathfrak{P}_0) \subseteq \mathfrak{M}$ for every non identity subgroup $\mathfrak{P}_0$ of $\mathfrak{P}$.*

(c) *$\mathfrak{P} \subseteq H(\mathfrak{M})$.*

Proof. Suppose $p \in \pi_0$. If $\mathfrak{P} \in \mathscr{X}_1$, then $\mathfrak{M} = M(\mathfrak{P})$, and so

$N(\Omega_1(Z(\mathfrak{P}))) \subseteq \mathfrak{M}$. Since $\mathfrak{P} \subseteq H(\mathfrak{M})$, the lemma is proved. If $\mathfrak{P} \in \mathscr{X}_0$ then $\mathfrak{P}$ contains a cyclic subgroup of index p. Since $\mathfrak{P}$ is assumed to be non abelian, $\mathfrak{P}$ is a non abelian metacyclic group, so $\mathfrak{P} \not\subseteq \mathfrak{M}'$, by 3.8. Lemma 26.10 is violated.

Through the remainder of the proof, we assume $p \in \pi_2$.

Let $\mathfrak{Z} = \Omega_1(Z(\mathfrak{P}))$, so that $\mathfrak{Z}$ is of order p, by Lemma 26.2 and Lemma 26.10.

If $\mathfrak{M}'$ is nilpotent, then $\mathfrak{Z} \triangleleft \mathfrak{M}$, and all parts of the lemma follow. We can suppose that $\mathfrak{M}'$ is not nilpotent. In particular, $\mathfrak{M} \in \mathscr{M}_1$. We can further assume that $p \notin \pi(H(\mathfrak{M}))$.

Since $\mathfrak{P}$ is non abelian, $\mathfrak{Z}$ centralizes $E_1(\mathfrak{M})$.

Choose $q \in \pi(H_1(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\mathfrak{M}$. If $q \in \pi^*$, then $\mathfrak{Z}$ centralizes $\mathfrak{Q}$.

Thus, if $\tilde{\pi} = \pi(E_1(\mathfrak{M})) \cup (\pi^* \cap \pi(H_1(\mathfrak{M})))$, then $\mathfrak{Z}$ centralizes a $S_{\tilde{\pi}}$-subgroup of $\mathfrak{M}$. If $\tilde{\pi} = \pi(H(\mathfrak{M}))$, all parts of the lemma follow.

Let $r \in \pi(H(\mathfrak{M})) - \tilde{\pi}$ and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$, and such that $\mathfrak{Z}$ does not centralize $\mathfrak{R}$. If there are no such primes r, we are done.

Let $\mathfrak{P}_1$ be any subgroup of $\mathfrak{P}$ of order p different from $\mathfrak{Z}$. We will show that $N(\mathfrak{P}_1) \subseteq \mathfrak{M}$.

Since $\mathfrak{Z}$ does not centralize $\mathfrak{R}$, $\mathfrak{R} \cap C(\mathfrak{P}_1) \not\subseteq C(\mathfrak{Z})$. Set $\mathfrak{R}_1 = \mathfrak{R} \cap C(\mathfrak{P}_1)$. If $\mathfrak{R}_1 \in \mathscr{X}_1$, then $N(\mathfrak{P}_1) \subseteq \mathfrak{M}$. Otherwise, $\mathfrak{R}_1$ is a non trivial cyclic subgroup of $\mathfrak{R}$, and $\mathfrak{Z}\mathfrak{R}_1$ is a Frobenius group.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{P}_1)$. If $\mathfrak{M}_1'$ is nilpotent, then $\mathfrak{R}_1 \subseteq O_r(\mathfrak{M}_1)$, so $\mathfrak{M}_1 \subseteq \mathfrak{M}$, by Lemma 26.6. We can suppose that $\mathfrak{M}_1'$ is not nilpotent and that $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. If a S_p-subgroup of $\mathfrak{M}_1$ is non abelian, then $\mathfrak{Z}$ centralizes $\mathfrak{R}_1$, which is not the case. Hence, a S_p-subgroup of $\mathfrak{M}_1$ is abelian and non cyclic. We can apply Lemma 26.12 to $\mathfrak{M}_1$ and a S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{M}_1$ which contains $\mathfrak{P}_1\mathfrak{Z}$. We conclude that $\mathfrak{Z}H(\mathfrak{M}_1)$ is a Frobenius group. Since $\mathfrak{Z}\mathfrak{R}_1$ is a Frobenius group, $\mathfrak{R}_1$ centralizes $H(\mathfrak{M}_1)$, and so $\mathfrak{M} = \mathfrak{M}_1$. We conclude that $\mathfrak{M}$ contains $N(\mathfrak{P}_1)$ in all cases.

Now let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the distinct subgroups of $\mathfrak{P}$ of order p different from $\mathfrak{Z}$. Here $n = p^2 + p$. Let $\mathfrak{L}$ be any proper subgroup of $\mathfrak{G}$ containing $\Omega_1(\mathfrak{P})$. Let $\mathfrak{L}_1 = O_{p'}(\mathfrak{L})$. Since $\mathfrak{L}_1$ is generated by its subgroups $C(\mathfrak{P}_i) \cap \mathfrak{L}_1$, $1 \leqq i \leqq n$, we have $\mathfrak{L}_1 \subseteq \mathfrak{M}$. Let $\mathfrak{L}_2 = \mathfrak{L} \cap N(\Omega_1(\mathfrak{P}))$, and choose L in $\mathfrak{L}_2$. We can then find indices i, j, not necessarily distinct, such that $\mathfrak{P}_i^L = \mathfrak{P}_j$. Hence, $N(\mathfrak{P}_j) \subseteq \mathfrak{M} \cap \mathfrak{M}^L$. Since $N(\mathfrak{P}_j)$ contains an element of $\mathfrak{R}^\# \subseteq \hat{H}(\mathfrak{M})$, we have $\mathfrak{M} = \mathfrak{M}^L$. Hence, $\mathfrak{L} \subseteq \mathfrak{M}$, so in particular, $N(\mathfrak{Z}) \subseteq \mathfrak{M}$.

Let $\mathfrak{P}_0$ be any non identity subgroup of $\mathfrak{P}$. If $\mathfrak{P}_0$ is non cyclic, then $N(\mathfrak{P}_0) \subseteq N(\mathfrak{Z}) \subseteq \mathfrak{M}$. If $\mathfrak{P}_0$ is cyclic, then $N(\Omega_1(\mathfrak{P}_0)) \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 26.15. *Suppose* $\mathfrak{M} \in \mathscr{M}$, $\mathfrak{A}$ *is a cyclic S-subgroup of* $\mathfrak{M}$ *and* $\mathfrak{A} \cap \mathfrak{M}' = 1$. *Then* $\mathfrak{A}$ *is prime on* $\boldsymbol{H}(\mathfrak{M})$, *and* $\boldsymbol{C}(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$ *is a Z-group.*

Proof. Suppose $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, but that $\mathfrak{Q}$ is a non cyclic S_q-subgroup of $\boldsymbol{C}(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$. Choose $p \in \pi(\mathfrak{A})$ and let $\mathfrak{A}_p$ be the S_p-subgroup of $\mathfrak{A}$. Since $\boldsymbol{N}(\mathfrak{A}_p) \not\subseteq \mathfrak{M}$, it follows that $\mathfrak{Q} \in \mathscr{X}_0$. Thus, if $q \in \pi_2$, $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, while if $q \in \pi_0$, $\mathfrak{Q}$ is also a S_q-subgroup of $\mathfrak{G}$, by Lemma 8.12. Since $\mathfrak{Q} \in \mathscr{X}_0$, we have $q \in \pi_2$, so that $\mathfrak{M} = \boldsymbol{N}(\mathfrak{Q})$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{A}_p)$. If a S_p-subgroup of $\mathfrak{G}$ is cyclic, then $\mathfrak{M} = \boldsymbol{N}(\mathfrak{Q})$ dominates $\mathfrak{A}_p$, which is not the case, since $\mathfrak{A}_p \cap \mathfrak{M}' = 1$. Hence, $p \in \pi_0 \cup \pi_2$. Let $\mathfrak{A}_p^*$ be a S_p-subgroup of $\mathfrak{M}_1$ permutable with $\mathfrak{Q}$. If $\mathfrak{A}_p^*$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ normalizes $\mathfrak{A}_p^*$. Otherwise, $\mathfrak{Q}$ normalizes $\mathfrak{A}_p^*$ since $\mathfrak{A}_p \subset \mathfrak{A}_p^*$, and Lemma 8.5 applies to $\mathfrak{Q}\mathfrak{A}_p^*$.

Let $\mathfrak{K}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}\mathfrak{A}_p^*$, and let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$. Then $\mathfrak{K}_p \lhd \mathfrak{K}$, so that $\mathfrak{K}_p$ is a S_p-subgroup of $\mathfrak{G}$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{K}_p)$.

If $\mathfrak{Q}$ were non abelian, then $\mathfrak{M} \subseteq \mathfrak{M}_2$ by Lemma 26.14, which is not the case. Hence, $\mathfrak{Q}$ is abelian. If $p \in \pi_0$, then by Lemma 26.13, we have $\boldsymbol{N}(\Omega_1(\mathfrak{Q})) \subseteq \mathfrak{M}_2$ since $\mathfrak{Q}$ centralizes $\mathfrak{A}_p \neq 1$. Since this is impossible, we see that $p \in \pi_2$.

If $\mathfrak{A}_p \not\subseteq \mathfrak{K}_p'$, then by Lemma 26.1, together with the fact that $\boldsymbol{N}(\mathfrak{Q})$ covers $\boldsymbol{N}(\mathfrak{K}_p)/\mathfrak{K}_p\boldsymbol{C}(\mathfrak{K}_p)$, we see that $\mathfrak{A}_p \cap \mathfrak{M}' \neq 1$, contrary to hypothesis. Hence, $\mathfrak{A}_p \subseteq \mathfrak{K}_p'$. Since $\mathfrak{A}_p = \boldsymbol{C}(\mathfrak{Q}) \cap \mathfrak{K}_p$, this implies that $\mathfrak{K}_p$ is a non abelian group of order p^3 and exponent p.

Since some element of $\mathfrak{Q}^\#$ has a non identity fixed point on $\boldsymbol{H}(\mathfrak{M}_2)^\#$, and since $\mathfrak{M}'$ centralizes $\mathfrak{Q}$, we see that $\mathfrak{M}' \subseteq \mathfrak{M}_2$, by Lemma 26.13. Since $\boldsymbol{N}(\mathfrak{A}_p) \subseteq \mathfrak{M}_2$ and since $\mathfrak{A}_p \cap \mathfrak{M}' = 1$, it follows that $\mathfrak{M} \subseteq \mathfrak{M}_2$, the desired contradiction.

Thus, in proving this lemma, it suffices to show that $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$.

First, suppose that $\mathfrak{A}$ is a p-group for some prime p. We can clearly suppose that $|\mathfrak{A}| \geq p^2$, and that $\boldsymbol{C}(\Omega_1(\mathfrak{A})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$.

Case 1. $p \in \pi_0$. Let $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, so that $q \in \pi_1 \cup \pi_2$. Lemma 26.9 applies. Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Then $p \not\sim q$ since $\mathfrak{A} \cap \mathfrak{M}' = 1$. Lemma 26.10 applies. If $\mathfrak{M} \in \mathscr{M}_0$, Lemma 26.9 applies.

Case 2. $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is abelian.

If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, or $q \in \pi(\boldsymbol{H}(\mathfrak{M}))$ and $\mathfrak{M} \in \mathscr{M}_0$, Lemma 26.7 applies.

Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, and let $\mathfrak{Q}$ be an $\mathfrak{A}$-invariant S_q-subgroup of $\mathfrak{M}$. If $\mathfrak{A}$ centralizes $\mathfrak{Q}$, we have an immediate contradiction. Hence, $\mathfrak{A}$ does not centralize $\mathfrak{Q}$.

We can suppose by way of contradiction that $[\boldsymbol{C}(\Omega_1(\mathfrak{A})) \cap \mathfrak{Q}, \mathfrak{A}] \neq 1$. If $\boldsymbol{C}(\Omega_1(\mathfrak{A})) \cap \mathfrak{Q} \in \mathscr{X}_1$, $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$, which is not the case. Otherwise, $q > p$, so every p, q-subgroup of $\mathfrak{G}$ is q-closed, and $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$, which is not the case.

Case 3. $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is non abelian.

Here, $\mathfrak{A} \subseteq N(\Omega_1(\mathfrak{A}))'$, by Lemma 26.2. Since $C(\Omega_1(\mathfrak{A})) \cap H(\mathfrak{M}) \in \mathscr{X}_0$, the lemma follows.

Case 4. $p \in \pi_1$. In this case, also, we have $\mathfrak{A} \subseteq N(\Omega_1(\mathfrak{A}))'$, and the lemma follows.

Next, suppose that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_i$ is a non identity p_i-group, $i = 1, 2$. Suppose by way of contradiction that $\mathfrak{Q}$ is an $\mathfrak{A}$-invariant S_q-subgroup of $H(\mathfrak{M})$ and that $\mathfrak{A}$ is not prime on $\mathfrak{Q}$. We can suppose that $\mathfrak{A}_2$ does not centralize $\mathfrak{Q} \cap C(\Omega_1(\mathfrak{A}_1)) = \mathfrak{Q} \cap C(\mathfrak{A}_1) = \mathfrak{Q}_1$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\Omega_1(\mathfrak{A}_1))$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, either because $\mathfrak{A}_1$ is not a S-subgroup of $\mathfrak{M}_1$, or because $\mathfrak{A}_1 \subseteq \mathfrak{M}_1'$. Let $\mathfrak{Q}_2$ be a S_q-subgroup of $\mathfrak{M} \cap \mathfrak{M}_1$ which contains $\mathfrak{Q}_1$ and is $\mathfrak{A}$-invariant.

Suppose $\mathfrak{Q}_1 \subset \mathfrak{Q}_2$. Then $\mathfrak{A}_1 \not\subseteq H(\mathfrak{M}_1)$, since $[\mathfrak{Q}_2, \mathfrak{A}_1] \neq 1$, and $q \notin \pi(H(\mathfrak{M}_1))$. Furthermore, $\mathfrak{Q}_2$ is non cyclic. Suppose $q \in \pi_2$. In this case, $q > p_1$, so a S_{p_1}-subgroup $\mathfrak{A}_1^*$ of $\mathfrak{M}_1$ normalizes some S_q-subgroup of $\mathfrak{M}_1$, and it follows that $\mathfrak{A}_1^*$ normalizes some S_q-subgroup of $\mathfrak{G}$. This implies that $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$. But in this case $\mathfrak{A}_1 \subseteq N(\Omega_1(\mathfrak{A}_1))'$ so that $\mathfrak{A}_1$ centralizes $\mathfrak{Q}_2$ and so $\mathfrak{Q}_1 = \mathfrak{Q}_2$. Suppose $q \in \pi_0$. If $\mathfrak{Q}_2 \in \mathscr{X}_1$, then $N(\Omega_1(\mathfrak{A}_1)) \subseteq \mathfrak{M}$, which is not the case. Hence, $\mathfrak{Q}_2 \in \mathscr{X}_0$ so that $q > p_1$. Once again we get that $\mathfrak{Q}_2 = \mathfrak{Q}_1$. Hence, we necessarily have $\mathfrak{Q}_2 = \mathfrak{Q}_1$ in all cases.

Since $\mathfrak{A}_1$ is prime on $H(\mathfrak{M})$, from the first part of the lemma, we conclude that $\mathfrak{Q}_1$ is cyclic.

We next assume that $\mathfrak{M}_1'$ is nilpotent.

Suppose $\Omega_1(O_q(\mathfrak{M}_1)) = \Omega_1(\mathfrak{Q}_1)$. Since $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{M}_1 \cap \mathfrak{M}$, it follows that $q \in \pi_1$ and $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{G}$, so that $\mathfrak{M} = \mathfrak{M}_1$. Since $\mathfrak{Q}_1 = [\mathfrak{Q}_1, \mathfrak{A}_2] \subseteq O_q(\mathfrak{M}_1)$, we can suppose that $O_q(\mathfrak{M}_1)$ is non cyclic. In this case, however, $O_q(\mathfrak{M}_1)$ is a S_q-subgroup of $\mathfrak{G}$ and $\mathfrak{M}_1$ is conjugate to $\mathfrak{M}$, which is not the case.

We can now suppose that $\mathfrak{M}_1'$ is not nilpotent.

Suppose $p_1 \notin \pi(H_1(\mathfrak{M}_1))$. Let $\mathfrak{E}$ be a complement for $H_1(\mathfrak{M}_1)$ in $\mathfrak{M}_1$ which contains $\mathfrak{Q}_1\mathfrak{A}$. Then $\mathfrak{E}'$ is nilpotent and so $[\mathfrak{Q}_1, \mathfrak{A}_2] \subseteq O_q(\mathfrak{E})$.

Case 1. $q \in \pi_1$. In this case, $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$, and $\mathfrak{Q}_1$ dominates $\mathfrak{A}_1$. This violates $\mathfrak{A}_1 \cap \mathfrak{M}' = 1$.

Case 2. $q \in \pi_2$, and a S_q-subgroup of $\mathfrak{G}$ is abelian. In this case, $\Omega_1([\mathfrak{Q}_1, \mathfrak{A}_2]) = \Omega_1(O_q(\mathfrak{E}))$, so once again $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ dominates $\mathfrak{A}_1$.

Case 3. $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is non abelian. Since $\mathfrak{Q}_1$ is cyclic, we have $q > p_1$, so some S_{p_1}-subgroup $\mathfrak{E}_{p_1}$ of $\mathfrak{E}$ normalizes some S_q-subgroup of $\mathfrak{G}$. But now $\mathfrak{M}$ dominates $\mathfrak{A}_1$ since every p_1, q-

subgroup of $\mathfrak{G}$ is q-closed, and $\mathfrak{E}$ dominates $\mathfrak{A}_1$.

Case 4. $q \in \pi_0$. If $q \in \pi^*$, then every p_1, q-subgroup of $\mathfrak{G}$ which contains a $S_{p_1,q}$-subgroup of $\mathfrak{M}_1$ is q-closed, so once again $\mathfrak{M}$ dominates $\mathfrak{A}_1$ and $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$. Hence, $q \notin \pi^*$. Since $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, it follows that if $\mathfrak{Q}_3$ is a S_q-subgroup of $\mathfrak{E}$ containing $\mathfrak{Q}_1$, then $\mathfrak{Q}_3 \in \mathscr{X}_0$, which implies that $\mathfrak{Q}_3$ is cyclic, and $\mathfrak{Q}_3 \subseteq \mathfrak{M}$. Hence, $\mathfrak{Q}_1 = \mathfrak{Q}_3$, since $\mathfrak{A}_1$ centralizes $\mathfrak{Q}_3$. But now $\mathfrak{Q}_1 = [\mathfrak{Q}_1, \mathfrak{A}_2] \lhd \mathfrak{E}$, so $\mathfrak{E} \subseteq \mathfrak{M}$. Thus, once again $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ dominates $\mathfrak{A}_1$.

All these possibilities have led to a contradiction. We now get to the heart of the matter. Suppose $p_1 \in \pi(\boldsymbol{H}_1(\mathfrak{M}_1))$.

We will show that $p_1 \notin \pi^*$.

Let $\mathfrak{P}_1$ be a S_{p_1}-subgroup of $\boldsymbol{H}_1(\mathfrak{M}_1)$ containing $\mathfrak{A}_1$ and invariant under $\mathfrak{A}_2\mathfrak{Q}_1$. Suppose that

$$N([\mathfrak{A}_2, \mathfrak{Q}_1]) \subseteq \mathfrak{M}_1 \tag{26.4}$$

We will derive a contradiction from the assumption that (26.4) holds.

If $q \in \pi_1$, (26.4) is an absurdity, since $N([\mathfrak{A}_2, \mathfrak{Q}_1]) = \mathfrak{M}$. If $q \in \pi_2 \cup \pi_0$, then a S_q-subgroup of $N([\mathfrak{A}_2, \mathfrak{Q}_1]) \cap \mathfrak{M}$ is non cyclic, so $q \in \pi_2$, as already remarked. If $q < p_1$, then $\mathfrak{A}_1$ centralizes a S_q-subgroup of $\mathfrak{M}$, so $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{G}$. In this case, however, $[\mathfrak{A}_2, \mathfrak{Q}_1] \lhd \mathfrak{M}$, an absurdity, by (26.4). Thus, if (26.4) holds, then $q \in \pi_2$ and $q > p_1$.

Since (26.4) is assumed to hold, it follows that $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{M} \cap N([\mathfrak{A}_2, \mathfrak{Q}_1])$. Hence, $\mathfrak{Q}_1$ is non cyclic. We have already shown that $\mathfrak{Q}_1$ is cyclic. We conclude that (26.4) does not hold.

If $p_1 \in \pi^*$, then $[\mathfrak{A}_2, \mathfrak{Q}_1]$ centralizes $\mathfrak{P}_1$, by Lemma 8.16 (ii), so (26.4) holds. Hence, $p_1 \notin \pi^*$.

Since (26.4) does not hold, and since $p_1 \notin \pi^*$, $C([\mathfrak{A}_2, \mathfrak{Q}_1]) \cap \mathfrak{P}_1$ is cyclic. It follows that $C(\mathfrak{A}_2) \cap \mathfrak{P}_1$ is non cyclic. This implies that $N(\mathfrak{A}_2) \subseteq \mathfrak{M}_1$, since $C(\mathfrak{A}_2) \cap \mathfrak{P}_1 \in \mathscr{X}_1$. Since $p_2 \notin \pi(\boldsymbol{H}(\mathfrak{M}_1))$, and since $q > p_2$, it follows that a $S_{p_2,q}$-subgroup of $\mathfrak{M}_1/\boldsymbol{H}(\mathfrak{M}_1)$ is q-closed. This in turn implies that some S_{p_2}-subgroup of $\mathfrak{M}_1$ normalizes some S_q-subgroup of $\mathfrak{G}$. Since $\mathfrak{A}_2$ is a S_{p_2}-subgroup of $\mathfrak{M}$, $\mathfrak{A}_2$ is forced to be a S_{p_2}-subgroup of $\mathfrak{G}$. But $N(\mathfrak{A}_2) \subseteq \mathfrak{M}_1$, and $\mathfrak{A}_2 \subseteq N(\mathfrak{A}_2)'$, so $\mathfrak{A}_2$ centralizes $\mathfrak{Q}_1$. The proof of the lemma is complete in case $\pi(\mathfrak{A}) = \{p_1, p_2\}$.

If $|\pi(\mathfrak{A})| \geqq 3$, the lemma follows immediately by applying the preceding result to all pairs of elements of $\pi(\mathfrak{A})$.

LEMMA 26.16. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\boldsymbol{H}(\mathfrak{M})$ is not nilpotent. Then $|\mathfrak{M} : \mathfrak{M}'|$ is a prime and $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$.*

Proof. Let $p \in \pi(\mathfrak{M}/\mathfrak{M}')$ and let $\mathfrak{A}_p$ be a S_p-subgroup of $\mathfrak{M}$. By Lemma 26.11, $\mathfrak{A}_p$ is abelian. Suppose $\mathfrak{A}_p$ is non cyclic. If a S_p-subgroup of $\mathfrak{G}$ is non abelian, then $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, by Lemma 26.12. Hence, we can suppose that a S_p-subgroup of $\mathfrak{G}$ is abelian. By Lemma 26.13 $\mathfrak{A}_p$ is a S_p-subgroup of $\mathfrak{G}$. By Grün's theorem, the simplicity of $\mathfrak{G}$, and Lemma 26.15, $\mathfrak{A}_p$ contains elements A_1, A_2 which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{M}$. If $\Omega_1(\langle A_1\rangle) = \Omega_1(\langle A_2\rangle)$ and if $\Omega_1(\langle A_1\rangle)$ has a fixed point on $\boldsymbol{H}(\mathfrak{M})^{\sharp}$, then $N(\Omega_1(\langle A_1\rangle)) \subseteq \mathfrak{M}$, so that A_1 and A_2 are conjugate in $\mathfrak{M}$. Since this is not the case, $\Omega_1(\langle A_1\rangle)\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and so $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, contrary to assumption. Hence, $\Omega_1(\langle A_1\rangle) \neq \Omega_1(\langle A_2\rangle)$. By Lemma 26.13, either $\Omega_1(\langle A_1\rangle)\boldsymbol{H}(\mathfrak{M})$ or $\Omega_1(\langle A_2\rangle)\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, which is not the case. Hence, $\mathfrak{A}_p$ is cyclic.

Let $\mathfrak{A}$ be a complement to $\mathfrak{M}'$ in $\mathfrak{M}$, so that $\mathfrak{A}$ is a cyclic S-subgroup of $\mathfrak{M}$.

By Lemma 26.15, $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$ and $C(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$ is a Z-group.

Let $\mathfrak{K} = [\mathfrak{A}, \boldsymbol{H}(\mathfrak{M})]$ and suppose that $|\mathfrak{A}|$ is not a prime. By Lemma 26.3, $\mathfrak{K}$ is nilpotent. By 3.7, $\mathfrak{K} \lhd \boldsymbol{H}(\mathfrak{M})$. Hence $\boldsymbol{F}(\boldsymbol{H}(\mathfrak{M})) \supseteq \mathfrak{K}$, so that $\boldsymbol{H}(\mathfrak{M})/\boldsymbol{F}(\boldsymbol{H}(\mathfrak{M}))$ is a Z-group. It follows that $\boldsymbol{H}(\mathfrak{M}) \not\subseteq \mathfrak{M}'$, the desired contradiction.

LEMMA 26.17. *Suppose* $\mathfrak{M} \in \mathscr{M}$ *and* $\tau_1 = \pi(\boldsymbol{H}(\mathfrak{M})) \cap \pi^*$, $\tau_2 = \pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M})) \cap \pi^*$. *Let* $\tau_1 = \{p_1, \cdots, p_n\}$, $p_1 > p_2 > \cdots > p_n$, *and* $\tau_2 = \{q_1, \cdots, q_m\}$, $q_1 > \cdots > q_m$. *Set* $\tau = \tau_1 \cup \tau_2$. *Then a* S_τ*-subgroup of* $\mathfrak{M}$ *has a Sylow series of complexion* $(p_1, \cdots, p_n, q_1, \cdots, q_m)$. *Furthermore, if* $r \in \tau$, $\mathfrak{M}$ *has* r*-length* 1.

Proof. We first show that $\mathfrak{M}$ has r-length 1 for each r in τ. If $r \notin \pi(\boldsymbol{H}_1(\mathfrak{M}))$, this is clear, so suppose $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}$ and let $\mathfrak{A}$ be a subgroup of $\mathfrak{R}$ of order r such that $C_{\mathfrak{R}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{B}$ is cyclic.

Let $\mathfrak{R}_1 = \mathfrak{R} \cap \boldsymbol{O}_{r',r}(\mathfrak{M})$, and $\mathfrak{M}_1 = N(\mathfrak{R}_1)$. It suffices to show that $\mathfrak{M}_1$ has r-length one, since $\mathfrak{M} = \mathfrak{M}_1\boldsymbol{O}_{r'}(\mathfrak{M})$. Let $\mathfrak{B}$ be a subgroup of $\mathfrak{R}_1$ chosen in accordance with Lemma 8.2, and set $\mathfrak{W} = \Omega_1(\mathfrak{B})$. Then $\ker(\mathfrak{M}_1 \to \operatorname{Aut}\mathfrak{W}) \subseteq \mathfrak{M}_1 \cap \boldsymbol{O}_{r',r}(\mathfrak{M})$. If $\mathfrak{A} \subseteq \mathfrak{R}_1$, then $m(\mathfrak{W}) \leq 2$, and we are done. We can suppose that $\mathfrak{A} \not\subseteq \mathfrak{R}_1$. This implies that $m(\mathfrak{W}) \leq r$, since $C(\mathfrak{A}) \cap \mathfrak{W}$ has order r and $\mathfrak{W}$ is of exponent r. We are assuming by way of contradiction that $\mathfrak{M}$ has r-length $\geqq 2$, so by (B), we have $m(\mathfrak{W}) \geqq r$. Hence, $m(\mathfrak{W}) = r$.

Set $\mathfrak{W}_1 = \mathfrak{W}/\boldsymbol{D}(\mathfrak{W})$ and let $\mathfrak{M}_2 = \mathfrak{M}_1/\ker(\mathfrak{M}_1 \to \operatorname{Aut}\mathfrak{W}_1)$. Then $\mathfrak{A}$ maps onto a S_r-subgroup of $\mathfrak{M}_2$. Hence $\mathfrak{M}_2$ has a normal series $1 \subset \mathfrak{C}_1 \subset \mathfrak{C}_2 \subseteq \mathfrak{M}_2$, where $\mathfrak{C}_1$ and $\mathfrak{M}_2/\mathfrak{C}_2$ are r'-groups and $|\mathfrak{C}_2 : \mathfrak{C}_1| = r$.

Since $m(\mathfrak{W}) = r$, $\mathfrak{C}_1$ is abelian. Also $\mathfrak{M}_2/\mathfrak{C}_2$ is faithfully represented on $\mathfrak{C}_2/\mathfrak{C}_1$ and since $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, $\mathfrak{C}_2 \subset \mathfrak{M}_2$.

By Lemma 26.16, $|\mathfrak{M} : \mathfrak{M}'| = q$ is a prime, and $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$. We let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}_1$, so that $\mathfrak{Q}$ is of order q. Since $|\mathfrak{M} : \mathfrak{M}'| = |\mathfrak{M}_1 : \mathfrak{M}_1'|$, it follows that $\mathfrak{Q}$ maps onto $\mathfrak{M}_2/\mathfrak{C}_2$. Let

$\bar{\mathfrak{A}}$ denote the image of $\mathfrak{A}$ in $\mathfrak{M}_2$ and let $\bar{\mathfrak{Q}}$ denote the image of $\mathfrak{Q}$ in $\mathfrak{M}_2$. Since $\mathfrak{E}_1$ is a r'-group and a q'-group, we assume without loss of generality that $\bar{\mathfrak{Q}}$ normalizes $\bar{\mathfrak{A}}$.

Let α be the linear character of $\bar{\mathfrak{Q}}$ on $\bar{\mathfrak{A}}$, so that $\alpha \neq 1$. Let β be the linear character of $\bar{\mathfrak{Q}}$ on $\mathfrak{W}_1/\gamma\mathfrak{W}_1\mathfrak{A}$. Since q divides $(r-1)/2$, $C_{\mathfrak{W}_1}(\bar{\mathfrak{Q}})$ is non cyclic. Hence, $C(\mathfrak{Q}) \cap \boldsymbol{H}(\mathfrak{M})$ is not a Z-group, contrary to Lemma 26.15.

Thus, $\mathfrak{M}$ has r-length one for each $r \in \tau$. Since a S_{τ_2}-subgroup of $\mathfrak{M}$ has a Sylow series of complexion $(q_1, \cdots, q_m)$ and since a S_τ-subgroup of $\mathfrak{M}$ is τ_1-closed, it suffices to show that a S_{τ_1}-subgroup of $\mathfrak{M}$ has a Sylow series of complexion $(p_1, \cdots, p_n)$.

Let $\mathfrak{K}$ be a S_{p_i,p_j}-subgroup of $\mathfrak{M}$ with Sylow system $\mathfrak{K}_i, \mathfrak{K}_j$ where $p_i > p_j$. By Lemma 8.16, $\mathfrak{K}_i \cap N(\mathfrak{K}_j)$ centralizes $\mathfrak{K}_j$. Hence $\mathfrak{K}$ is p_i-closed, since $\mathfrak{K}$ has p_j-length one. The lemma follows.

LEMMA 26.18. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$. Then there is at most one prime p in $\pi(\mathfrak{E})$ with the following properties*:

(i) *A S_p-subgroup of $\mathfrak{E}$ is a non cyclic abelian group.*

(ii) *A S_p-subgroup of $\mathfrak{G}$ is non abelian.*

Furthermore, if $\pi(\mathfrak{E})$ contains a prime p satisfying (i) *and* (ii), *then a $S_{p'}$-subgroup of $\mathfrak{E}$ is a Z-group.*

Proof. Suppose $p_1, p_2 \in \pi(\mathfrak{E})$, $p_1 \neq p_2$ and both p_1 and p_2 satisfy (i) and (ii). Let $\mathfrak{E}_1$ be a S_{p_1}-subgroup of $\mathfrak{E}$ and let $\mathfrak{E}_2$ be a S_{p_2}-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_1$.

Let $\mathfrak{E}_i = \mathfrak{A}_i \times \mathfrak{B}_i$, where $|\mathfrak{A}_i| = p_i$, $\mathfrak{A}_i$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}_i\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and $\Omega_1(\mathfrak{B}_i) \subseteq \boldsymbol{Z}(\mathfrak{P}_i)$ for some S_{p_i}-subgroup $\mathfrak{P}_i$ of $\mathfrak{G}$, $i = 1, 2$. Assume without loss of generality that $p_1 > p_2$. Then $\mathfrak{E}_2$ normalizes $\mathfrak{E}_1$. It follows that $\Omega_1(\mathfrak{E}_2)$ centralizes $\mathfrak{E}_1/\mathfrak{A}_1$, and this implies that $\Omega_1(\mathfrak{E}_2)$ centralizes $\Omega_1(\mathfrak{B}_1)$. It follows that $\mathfrak{G}$ satisfies E_{p_1,p_2}.

By Lemma 26.17, $N(\mathfrak{P}_1)$ contains a S_{p_2}-subgroup $\mathfrak{P}_2^*$ of $\mathfrak{G}$. By Lemma 8.16, $\mathfrak{P}_2^{*\prime}$ centralizes $\mathfrak{P}_1$, so centralizes $\mathfrak{E}_1$. Since $C(\mathfrak{A}_1) \subseteq \mathfrak{M}$, we see that $p_2 \in \pi_2$. By Lemma 26.2, and Lemma 26.10, $\mathfrak{P}_2$ now centralizes $\mathfrak{P}_1$. This is a contradiction, proving the first assertion.

Now suppose $p \in \pi(\mathfrak{E})$ satisfies (i) and (ii), $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{E}$ and $\mathfrak{E}_q$ is a non cyclic S_q-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_p$, $q \in \pi(\mathfrak{E})$, $q \neq p$.

Case 1. $\mathfrak{E}_q$ is non abelian.

In this case, $\mathfrak{E}_q$ is a S_q-subgroup of $\mathfrak{G}$ and $q \in \pi_2$, by Lemma 26.14. Since $\mathfrak{E}_q \subseteq \mathfrak{M}'$, $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$. Write $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and $\Omega_1(\mathfrak{B}) \subseteq \boldsymbol{Z}(\mathfrak{P})$ for some S_p-subgroup of $\mathfrak{P}$ of $\mathfrak{G}$ with $\mathfrak{E}_p \subseteq \mathfrak{P}$. Then $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$. If $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$, then $\mathfrak{G}$ satisfies $E_{p,q}$ as can be seen by considering

$N(\mathfrak{E}_p)$.

We now show that $\mathfrak{G}$ does not satisfy $\mathfrak{E}_{p,q}$. Otherwise, since $N(\mathfrak{E}_q) \subseteq \mathfrak{M}$, we see that $\mathfrak{E}_q$ normalizes some S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$. Then $\mathfrak{E}_q$ centralizes $\mathfrak{P}^*$ by Lemma 26.2, Lemma 26.14, and Lemma 8.16. This is not possible since $\mathfrak{E}_p$ is abelian.

Hence, $\mathfrak{G}$ does not satisfy $E_{p,q}$, so $\Omega_1(\mathfrak{E}_p)$ does not centralize $\mathfrak{E}_q$ and $q > p$. This implies that $|\mathfrak{E}_q : \mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))| = q$. Hence $\mathfrak{E}_q \cap C(\Omega_1(\mathfrak{E}_p)) = \mathfrak{E}_q^*$ is of order q.

Consider $N(\Omega_1(\mathfrak{E}_p)) = \mathfrak{N}$. Since a S_p-subgroup of $\mathfrak{N}$ has order $p|\mathfrak{E}_p|$, it follows that a $S_{p,q}$-subgroup of $\mathfrak{N}$ is q-closed. Let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{N}$ containing $\mathfrak{E}_q^*$. If $\mathfrak{F}_q$ is not of order q, then $N(\Omega_1(\mathfrak{F}_q))$ contains a S_q-subgroup of $\mathfrak{G}$, a $S_{p,q}$-subgroup of $N(\Omega_1(\mathfrak{F}_q))$ is q-closed, and a S_p-subgroup of $N(\Omega_1(\mathfrak{F}_q))$ has larger order than $\mathfrak{E}_p$. As $N(\mathfrak{E}_q) \subseteq \mathfrak{M}$, this is not possible. Hence $\mathfrak{F}_q = \mathfrak{E}_q^*$ has order q. But now a S_q-subgroup of $N(\mathfrak{F}_q)$ contains $\mathfrak{E}_p$ and $\boldsymbol{Z}(\mathfrak{E}_q)$, so a $S_{p,q}$-subgroup of $N(\mathfrak{F}_q)$ is q-closed. This in turn implies that a S_p-subgroup of $N(\mathfrak{E}_q)$ has order larger than $|\mathfrak{E}_p|$, which is a contradiction.

Case 2. $\mathfrak{E}_q$ is a non cyclic abelian group.

By the first part of the proof, and by Lemma 26.13, $\mathfrak{E}_q$ is a S_q-subgroup of $\mathfrak{G}$. Since $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_q \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{G}$ satisfies $\mathfrak{E}_{p,q}$. This implies that a $S_{p,q}$-subgroup of $\mathfrak{G}$ is p-closed, by Lemma 26.2. Hence, $\mathfrak{E}_q$ centralizes the center $\mathfrak{Z}$ of some S_p-subgroup of $\mathfrak{G}$, since $\Omega_1(\mathfrak{E}_q)$ centralizes $\Omega_1(\mathfrak{B})$, (where $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, as in Case 1). To obtain the relation $[\Omega_1(\mathfrak{E}_q), \Omega_1(\mathfrak{B})] = 1$, we have used Lemma 26.13 to conclude that there are at least 2 subgroups of $\mathfrak{E}_q$ of order q which have no fixed points on $\boldsymbol{H}(\mathfrak{M})$, or else $\mathfrak{E}_q \subseteq \mathfrak{M}'$ in which case $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$ and so $\Omega_1(\mathfrak{B})$ centralizes $\mathfrak{E}_q$.

But now $N(\Omega_1(\mathfrak{B}))$ dominates $\mathfrak{E}_q$, so $\mathfrak{E}_q$ centralizes some S_p-subgroup of $\mathfrak{G}$, contrary to $C(\mathfrak{E}_q) \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 26.19. *Let $\mathfrak{M} \in \mathscr{M}$. Suppose $\mathfrak{M}/\boldsymbol{H}(\mathfrak{M})$ is abelian. Suppose further that either $\boldsymbol{H}(\mathfrak{M})$ is nilpotent or $|\mathfrak{M} : \boldsymbol{H}(\mathfrak{M})|$ is not a prime. Then $\mathfrak{M}$ is of type* I *or* V.

Proof. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$. Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$ by hypothesis (we always have $\boldsymbol{H}(\mathfrak{M}) \subseteq \mathfrak{M}'$), $\mathfrak{E} \cong \mathfrak{M}/\mathfrak{M}'$ is abelian.

Case 1. $\mathfrak{E}$ is cyclic.

We wish to show that $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, so suppose $|\mathfrak{E}|$ is not a prime. Since $|\mathfrak{E}|$ is not a prime, since $\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, since $\mathfrak{E}$ has no fixed points on $\boldsymbol{H}(\mathfrak{M})/\boldsymbol{H}(\mathfrak{M})'$, and since $C(\mathfrak{E}) \cap \boldsymbol{H}(\mathfrak{M})$ is a Z-group, it follows from Lemma 26.3 that $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, so that $C(\mathfrak{E}) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_1$ is cyclic.

Case 1a. $\mathfrak{E}_1 = 1$.

In this case, $\mathfrak{M}$ is a Frobenius group with Frobenius kernel

$\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, so condition (i) in type I holds. If $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, then $\mathfrak{M}$ is of type I, since (ii) (a) holds, so suppose $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$. Let $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{P}_1 \times \cdots \times \mathfrak{P}_n$, where $\mathfrak{P}_i$ is the S_{p_i}-subgroup of $\mathfrak{M}$ and $\{p_1, \cdots, p_n\} = \pi(\boldsymbol{H}(\mathfrak{M}))$. If $p_i \in \pi_1$, then clearly $p_i \in \pi_1^*$. If $p_i \in \pi_0 \cap \pi^*$; then also $p_i \in \pi_1^*$, since $\mathfrak{E}\boldsymbol{Z}(\mathfrak{P}_i)$ is a Frobenius group. Similarly, if $p_i \in \pi_2$ and $\mathfrak{P}_i$ is non abelian, then $p_i \in \pi_1^*$.

Suppose $p_i \notin \pi_1^*$. Then either $p_i \in \pi_2$ and $\mathfrak{P}_i$ is abelian, or $p_i \in \pi_0 - \pi^*$. We will show that the second possibility cannot occur.

Choose G in $\mathfrak{G} - \mathfrak{M}$ such that $\mathfrak{D} = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G \neq 1$, and let H be an element of $\mathfrak{D}$ of prime order p. If $p_i \in \pi_0 - \pi^*$, and $p \neq p_i$, then $C(H) \supseteq \langle \mathfrak{P}_i, \mathfrak{P}_i^G \rangle$, and $\mathfrak{M} = \mathfrak{M}^G$, contrary to assumption. Hence, $p = p_i$. In this case, $C(H) \supseteq \langle C(H) \cap \mathfrak{P}_i, C(H) \cap \mathfrak{P}_i^G \rangle$, and since $p_i \in \pi_0 - \pi^*$, both $C(H) \cap \mathfrak{P}_i$ and $C(H) \cap \mathfrak{P}_i^G$ are in $\mathscr{X}_1$, so $\mathfrak{M} = \mathfrak{M}^G$. Hence, $(\pi_0 - \pi^*) \cap \pi(\boldsymbol{H}(\mathfrak{M})) = \emptyset$.

Thus, if $\pi(\boldsymbol{H}(\mathfrak{M})) \not\subseteq \pi_1^*$, then $\pi(\boldsymbol{H}(\mathfrak{M}))$ contains a prime q such that the S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ is abelian and $q \in \pi_2$. Since $|\mathfrak{E}|$ does not divide $q - 1$ or $q + 1$, but $|\mathfrak{E}|$ does divide $q^2 - 1$, we can find $r_1, r_2 \in \pi(\mathfrak{E})$ such that $r_1 \mid q - 1$ and $r_2 \mid q + 1$. Let $\mathfrak{E}_{r_1}$ be the S_{r_1}-subgroup of $\mathfrak{E}$. Then $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2$, where $\mathfrak{Q}_i$ is normalized by $\mathfrak{E}_{r_1}$ and $\mathfrak{Q}_i$ is cyclic, $i = 1, 2$. Since $r_2 \mid q + 1$, it follows that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are isomorphic $\mathfrak{E}_{r_1}$-modules. Hence, $\mathfrak{E}_{r_1}$ normalizes every subgroup of $\mathfrak{Q}$.

Once again, choose G in $\mathfrak{G} - \mathfrak{M}$ so that $\mathfrak{D} = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G \neq 1$. Then $C(\mathfrak{D}) \supseteq \langle \mathfrak{Q}, \mathfrak{Q}^G \rangle$, so $C(\mathfrak{D})$ is not contained in any conjugate of $\mathfrak{M}$. Let $C(\mathfrak{D}) \subseteq \mathfrak{M}_1 \in \mathscr{M}$. We apply Lemma 26.13 to $\mathfrak{M}_1$ and $\mathfrak{Q}$. Since $C(\Omega_1(\mathfrak{Q})) = \boldsymbol{H}(\mathfrak{M})$, we have $\boldsymbol{H}(\mathfrak{M}) \subseteq \mathfrak{M}_1$.

Suppose $\boldsymbol{H}(\mathfrak{M})$ were not abelian. Let $\mathfrak{R}$ be a non abelian S_r-subgroup of $\boldsymbol{H}(\mathfrak{M})$. Apply Lemma 26.16 to $\mathfrak{M}_1$ and $\mathfrak{R}$, and conclude that $\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{R}))) \subseteq \mathfrak{M}_1$, and so $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is not the case. Thus, alternative (ii) (c) in the definition of type I holds, so $\mathfrak{M}$ is of type I. (Since $\boldsymbol{H}(\mathfrak{M}) \in \mathscr{X}_0$, $\boldsymbol{H}(\mathfrak{M})$ is generated by two elements.)

Case 1b. $\mathfrak{E}_1 \neq 1$.

Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, we have $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})' \subseteq \hat{\boldsymbol{H}}(\mathfrak{M}) \cup \{1\}$. It follows that $\boldsymbol{N}(\mathfrak{E}_0) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{E}_0$ of $\mathfrak{E}_1^\#$. Let $\hat{\mathfrak{E}} = \mathfrak{E}\mathfrak{E}_1 - \mathfrak{E} - \mathfrak{E}_1$. If $\hat{\mathfrak{E}}_0$ is any non empty subset of $\hat{\mathfrak{E}}$, then each element of $\hat{\mathfrak{E}}_0$ is of the form EE_1, $E \in \mathfrak{E}^\#$, $E_1 \in \mathfrak{E}_1^\#$. Thus, if $\hat{\mathfrak{F}}_0 = \{E_0^{|\mathfrak{E}|} \mid E_0 \in \hat{\mathfrak{E}}_0\}$, then $\boldsymbol{N}(\hat{\mathfrak{E}}_0) \subseteq \boldsymbol{N}(\hat{\mathfrak{F}}_0) \subseteq \mathfrak{M}$. Since $\mathfrak{M} \cap \boldsymbol{N}(\hat{\mathfrak{E}}_0) = \mathfrak{E}\mathfrak{E}_1$, $\mathfrak{M}$ is a three step group with $\mathfrak{E}$ in the role of $\mathfrak{Q}^*$, $\boldsymbol{H}(\mathfrak{M})$ in the role of $\mathfrak{H}$, $\mathfrak{E}_1$ in the role of $\mathfrak{H}^*$. Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, we take $\mathfrak{U} = 1$, so that (i) in the definition of type V holds. If (ii) (a) holds, then $\mathfrak{M}$ is of type V, so suppose (ii) (a) does not hold.

Since $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})'$, $\boldsymbol{H}(\mathfrak{M})$ is non abelian. Let $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{P}$ is a non abelian S_p-subgroup of $\boldsymbol{H}(\mathfrak{M})$ (there may be several).

We will show that $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$. Suppose $G \in \mathfrak{G} - \mathfrak{M}$ and $\mathfrak{S}_0 \cap \mathfrak{S}_0^G = \mathfrak{D}$ is a maximal intersection, so that $N(\mathfrak{D})$ is contained in no conjugate of $\mathfrak{M}$. Let $\mathfrak{M}_1 \in \mathscr{M}$ with $N(\mathfrak{D}) \subseteq \mathfrak{M}_1$. Apply Lemma 26.14 to $\mathfrak{M}_1$ and $\mathfrak{P}$ and conclude that $\mathfrak{M} \subseteq \mathfrak{M}_1$, a contradiction. Hence, $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$.

Since $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$, choose $G \in \mathfrak{G} - \mathfrak{M}$ so that $1 \neq \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G$ is a maximal intersection. Since $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$, we see that $\boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G = \mathfrak{D}_1 = \mathfrak{P} \cap \mathfrak{P}^G$, and $N(\mathfrak{D}_1)$ is contained in no conjugate of $\mathfrak{M}$, while $N(\mathfrak{D}_1) \supseteq \mathfrak{S}_0$. Since $\mathfrak{S}_0$ is a T.I. set in $N(\mathfrak{D}_1)$, and since $N(\mathfrak{D}_1) \not\subseteq \mathfrak{M}$, $\mathfrak{S}_0$ is cyclic. By construction, $\mathfrak{P}$ is non abelian, so $p \in \pi^*$. It only remains to show that $p \in \pi_1^*$.

Apply Lemma 8.16 to $\mathfrak{P}$ and $\mathfrak{E}$. If $\mathfrak{E}$ does not centralize $\boldsymbol{Z}(\mathfrak{P})$, then $|\mathfrak{E}|$ divides $p - 1$ and we are done. Suppose that $\mathfrak{E}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$. Then $\mathfrak{E}$ is faithfully represented on $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))/\Omega_1(\boldsymbol{Z}(\mathfrak{P}))$, so if $|\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})) : \Omega_1(\boldsymbol{Z}(\mathfrak{P}))| = p$, we are done. Otherwise, we let P_0 be an element of $\mathfrak{P}$ of order p such that $C_{\mathfrak{P}}(P_0) = \langle P_0 \rangle \times \mathfrak{A}$, where $\mathfrak{A}$ is cyclic. Since $|\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})) : \Omega_1(\boldsymbol{Z}(\mathfrak{P}))| \geq p^2$, we have $P_0 \in \Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$, so $\langle P_0, \Omega_1(\boldsymbol{Z}(\mathfrak{P})) \rangle \lhd \mathfrak{P}$. By Lemma 8.9, $\mathscr{SCN}_3(\mathfrak{P})$ is empty. By Lemma 26.2, $\mathfrak{P}$ is a central product of a cyclic group and $\Omega_1(\mathfrak{P})$, with $|\Omega_1(\mathfrak{P})| = p^3$. Since $\mathfrak{P} \subseteq \mathfrak{M}'$ and since $\mathfrak{E}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$, we have $|\mathfrak{P}| = p^3$. $\mathfrak{E}$ is faithfully represented on $\mathfrak{P}/\mathfrak{P}'$, and since $\mathfrak{E}$ centralizes $\mathfrak{P}'$, each element of $\mathfrak{E}$ induces a linear transformation of $\mathfrak{P}/\mathfrak{P}'$ of determinant 1. Thus, $|\mathfrak{E}|$ divides either $p - 1$ or $p + 1$, since $\mathfrak{E}$ is isomorphic to a cyclic p'-subgroup of $SL(2, \text{p})$. Hence, $p \in \pi_1^*$, and $\mathfrak{M}$ is of type V.

Case 2. $\mathfrak{E}$ is non cyclic.

Case 2a. There is an element $p \in \pi(\mathfrak{E})$ such that the S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{E}$ is non cyclic and a S_p-subgroup of $\mathfrak{G}$ is non abelian. In this case, Lemma 26.18 implies that $\mathfrak{E} = \mathfrak{E}_p \times \mathfrak{F}$ where $\mathfrak{F}$ is cyclic.

Let $\mathfrak{E}_p = \mathfrak{E}_{p0} \times \mathfrak{E}_{p1}$, with $|\mathfrak{E}_{p0}| = p$, $\mathfrak{E}_{p0} \subseteq \boldsymbol{Z}(\mathfrak{M})$, and with $\mathfrak{E}_{p1}\boldsymbol{H}(\mathfrak{M})$ a Frobenius group. Also $\mathfrak{F}$ is a cyclic S-subgroup of $\mathfrak{M}$.

We will show that $\mathfrak{E}_{p1}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. If $\mathfrak{F} = 1$, this is the case, so suppose $\mathfrak{F} \neq 1$. By Lemma 26.16, $\mathfrak{F}$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Let $\mathfrak{H}^* = \boldsymbol{C}(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M})$, and suppose $\mathfrak{H}^* \neq 1$. Then $\mathfrak{E}_{p1}\mathfrak{H}^*$ is a Frobenius group. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_1)$, $\mathfrak{F}_1$ being a fixed subgroup of $\mathfrak{F}$ of prime order. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. Hence, $\mathfrak{M} \cap \mathfrak{M}_1 \in \mathscr{X}_0$. Since $\mathfrak{E}_{p1}\mathfrak{H}^*$ is a Frobenius group, $\mathfrak{E}_{p1} \cap \mathfrak{M}_1' = 1$, so a S_p-subgroup of $\mathfrak{M}_1$ is abelian. By Lemma 26.12, $\mathfrak{E}_{p1}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, so $\boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$. Since $1 \subset \mathfrak{H}^* \subseteq \boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$, we see that $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is not the case. Hence, $\mathfrak{H}^* = 1$, so $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, as is $\mathfrak{E}_{p1}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$. $\mathfrak{M}$ itself is a group of Frobenius type.

Suppose $\mathfrak{M}'$ is not a T.I set in $\mathfrak{G}$ and $\pi(\mathfrak{M}') \not\subseteq \pi_1^*$. It follows readily that $\mathfrak{M}'$ is abelian and is generated by two elements. $\mathfrak{M}$ is of type I.

Case 2b. Whenever a S_p-subgroup of $\mathfrak{E}$ is non cyclic, a S_p-subgroup of $\mathfrak{G}$ is abelian.

Let $\tilde{\pi}$ be the set of primes p in $\pi(\mathfrak{E})$ such that a S_p-subgroup of $\mathfrak{E}$ is non cyclic. Let $\mathfrak{E} = \mathfrak{E}_1 \times \mathfrak{E}_2$, where $\mathfrak{E}_1$ is the $S_{\tilde{\pi}}$-subgroup of $\mathfrak{E}$. Thus $\mathfrak{E}_2$ is a cyclic S-subgroup of $\mathfrak{M}$, and $\tilde{\pi} \neq \emptyset$. By Lemma 26.13, $\mathfrak{E}_1$ is a S-subgroup of $\mathfrak{G}$.

We first show that if $p \in \tilde{\pi}$ and $\mathfrak{E}_p$ is the S_p-subgroup of $\mathfrak{E}_1$, then

$$C(\Omega_1(\mathfrak{E}_p)) \cap \boldsymbol{H}(\mathfrak{M}) = 1 \tag{26.6}$$

This is an immediate consequence of Lemma 26.13 (iv) and Grün's theorem, since $\mathfrak{M}' \cap \mathfrak{E}_p = 1$.

We next show that either $\mathfrak{E}_2 = 1$ or $\mathfrak{E}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Suppose $\mathfrak{E}_2 \neq 1$. By Lemma 26.15 $\mathfrak{E}_2$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Suppose $\mathfrak{H}^* = C(\mathfrak{E}_2) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$. Let $\mathfrak{E}_q$ be the S_q-subgroup of $\mathfrak{E}_2$ for some $q \in \pi(\mathfrak{E}_2)$, and let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{E}_q)$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$.

By Lemma 26.13 (ii), together with $\mathfrak{M}' \cap \mathfrak{E}_p = 1$, there is some element of $\Omega_1(\mathfrak{E}_p)^\sharp$ which has no fixed points on $\boldsymbol{H}(\mathfrak{M})^\sharp$, so $\mathfrak{H}^*$ is cyclic. By construction $\langle \mathfrak{E}, \mathfrak{H}^* \rangle \subseteq \mathfrak{M}_1$. Suppose $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$ is non cyclic for some G in $\mathfrak{G}$. Let $\mathfrak{R}$ be a non cyclic S_r-subgroup of $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$. If a S_r-subgroup of $\mathfrak{G}$ is abelian, then $\boldsymbol{H}(\mathfrak{M}^G) \subseteq \mathfrak{M}_1$ by Lemma 26.13 (i) and (ii). Since $\mathfrak{E} \subseteq \mathfrak{M}_1$, we have $\mathfrak{M}^G = \mathfrak{M}_1$, which is not the case. Hence, a S_r-subgroup of $\mathfrak{G}$ is non abelian. If $\mathfrak{R}$ were non abelian, then $\mathfrak{M}_1 = \mathfrak{M}^{G_1}$ for some G_1 in $\mathfrak{G}$, by Lemma 26.14 with $\mathfrak{R}$ in the role of $\mathfrak{P}$. Hence, $\mathfrak{R}$ is abelian. By Lemma 26.13, $\mathfrak{R} = \mathfrak{R}_0 \times \mathfrak{R}_1$, $|\mathfrak{R}_0| = r$, $\mathfrak{R}_0$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$ and $\mathfrak{R}_1\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group. By (26.6), $\mathfrak{R} \subseteq \mathfrak{M}_1'$, so $\mathfrak{R}_0 \lhd \mathfrak{M}_1$. Since $\mathfrak{R}\boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{M}_1$, we can find a S_r-subgroup $\mathfrak{R}^*$ of $\mathfrak{M}_1$ which is normalized by $\mathfrak{E}_p$. Since $\mathfrak{M}$ and $\mathfrak{M}_1$ are not conjugate, $\pi(\boldsymbol{H}(\mathfrak{M})) \cap \pi(\boldsymbol{H}(\mathfrak{M}_1)) = \emptyset$, so $\mathfrak{R}^*$ does not lie in $\boldsymbol{H}(\mathfrak{M}_1)$, and $\mathfrak{R}^*$ does not centralize $\boldsymbol{H}(\mathfrak{M}_1)$. There are at least p subgroups $\mathfrak{P}_0$ of $\Omega_1(\mathfrak{E}_p)$ with the property that $\mathfrak{P}_0\mathfrak{R}^*/\mathfrak{R}_0$ is a Frobenius group, by (26.6). Each of these has a fixed point on $\boldsymbol{H}(\mathfrak{M}_1)^\sharp$. It follows from Lemma 26.13 (iii) that $\mathfrak{M}_1$ dominates $\mathfrak{E}_p$. This is absurd, by (26.6) and Lemma 8.13. Hence, $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$ is cyclic for all G in $\mathfrak{G}$. In particular, $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M})$ is cyclic. This implies that $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M})$ is faithfully represented on $\boldsymbol{H}(\mathfrak{M}_1)$, so $\mathfrak{H}^*$ is faithfully represented on $\boldsymbol{H}(\mathfrak{M}_1)$. By (26.6), at least p subgroups of $\mathfrak{E}_p$ or order p have fixed points on $\boldsymbol{H}(\mathfrak{M}_1)$, so $\mathfrak{M}_1$ dominates $\mathfrak{E}_p$, which violates (26.6), by Lemma 8.13. $\mathfrak{E}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Thus, in the definition of a group of Frobenius type, the primes in $\pi(\mathfrak{E}_2)$ are taken care of. Let $\mathfrak{E}_p = \mathfrak{E}_{p1} \times \mathfrak{E}_{p2}$, with $|\mathfrak{E}_{p1}| \leqq |\mathfrak{E}_{p2}|$, $p \in \tilde{\pi}$, and where $\mathfrak{E}_{pi}$ is cyclic, $i = 1, 2$. If $|\mathfrak{E}_{p1}| < |\mathfrak{E}_{p2}|$, then $\Omega_1(\mathfrak{E}_{p2})$ char $\mathfrak{E}_p$. By Lemma 26.14

(v), it follows that $\mathfrak{E}_{p_2}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. If $|\mathfrak{E}_{p_1}| = |\mathfrak{E}_{p_2}|$, then by Lemma 26.14 (iii), there is some element P of order p in $\mathfrak{E}_p$ such that $\langle P\rangle\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Thus, $\mathfrak{E}$ contains a subgroup $\mathfrak{E}^*$ of the same exponent as $\mathfrak{E}$ with the property that $\mathfrak{E}^*\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. $\mathfrak{M}$ is of Frobenius type.

If $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$, and $\pi(\boldsymbol{H}(\mathfrak{M})) \not\subseteq \pi_1^*$, it follows readily that $\boldsymbol{H}(\mathfrak{M})$ is abelian and is generated by two elements. The proof is complete.

LEMMA 26.20. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\tilde{\pi}$ be the subset of primes p in $\pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M}))$ such that a S_p-subgroup of $\mathfrak{M}$ is a non cyclic abelian group and a S_p-subgroup of $\mathfrak{G}$ is abelian. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$. Then a $S_{\tilde{\pi}}$-subgroup $\mathfrak{P}$ of $\mathfrak{E}$ is a normal abelian subgroup of $\mathfrak{E}$ and $\mathfrak{P} \cap \mathfrak{E}' = 1$ or $\mathfrak{P}$.*

Proof. We can suppose $\mathfrak{P} \neq 1$. Let $p \in \tilde{\pi}$ and let $\mathfrak{E}_p$ be a S_p-subgroup of $\mathfrak{E}$. We first show that $\mathfrak{E}_p \lhd \mathfrak{E}$. Let $q \in \pi(\mathfrak{E})$ and let $\mathfrak{E}_q$ be a S_q-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_p$. If $\mathfrak{E}_q$ is non abelian, then $N(\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))) \subseteq \mathfrak{M}$, by Lemma 26.14. If $\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))$ centralizes $\Omega_1(\mathfrak{E}_p)$, then $\mathfrak{E}_p \subseteq \mathfrak{M}'$ so that $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$. We can suppose that $\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))$ does not centralize $\Omega_1(\mathfrak{E}_p)$. Since $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_q \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}_p \subseteq \mathfrak{M}'$ so that $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$.

If $\mathfrak{E}_q$ is a non cyclic abelian group, then $q \in \tilde{\pi}$ by Lemma 26.18. If $\mathfrak{E}_p \ntriangleleft \mathfrak{E}_p\mathfrak{E}_q$, then $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$ and $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$. If $\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M})) = 1$, then $N(\mathfrak{E}_q)$ dominates $\Omega_1(\mathfrak{E}_p)$, so $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$. If $\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M})) \neq 1$, then $\mathfrak{E}_q \cap C(\Omega_1(\mathfrak{E}_p))$ dominates $\mathfrak{E}_p$, so that $\mathfrak{E}_q$ dominates $\mathfrak{E}_p$ and once again $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$.

Suppose $\mathfrak{E}_q$ is cyclic. We can suppose that $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$. Then $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q$. If $q \in \pi_1 \cup \pi_2$, then $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$, since $\mathfrak{E}_p \subseteq N(\Omega_1(\mathfrak{E}_q))'$. We can suppose $q \in \pi_0$ and that a S_q-subgroup $\mathfrak{Q}$ of $C(\Omega_1(\mathfrak{E}_p))$ is in $\mathscr{X}_1$. In this case, however, $C(P) \subseteq \boldsymbol{M}(\mathfrak{Q})$ for all $P \in \mathfrak{E}_p^\sharp$, so $\mathfrak{M} = \boldsymbol{M}(\mathfrak{Q})$ which is absurd. Hence, $\mathfrak{E}_p \lhd \mathfrak{E}$, so that $\mathfrak{P}$ is a normal abelian subgroup of $\mathfrak{E}$.

Suppose $\mathfrak{E}$ contains a non abelian S_q-subgroup $\mathfrak{E}_q$ for some prime q. Then $N(\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))) \subseteq \mathfrak{M}$, which implies that $\mathfrak{P} \subseteq \mathfrak{M}'$, since $N(\mathfrak{E}_q)$ dominates each Sylow subgroup of $\mathfrak{P}$.

Thus, in showing that $\mathfrak{P} \cap \mathfrak{M}' = 1$ or $\mathfrak{P}$, we can suppose that every Sylow subgroup of $\mathfrak{E}$ is abelian. By Lemma 26.18 and the definition of $\tilde{\pi}$, this implies that a $S_{\tilde{\pi}'}$-subgroup $\mathfrak{F}$ of $\mathfrak{E}$ is a Z-group. This in turn implies that $\mathfrak{F} \cap \mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$. Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F} \cap \mathfrak{M}'$ in $\mathfrak{F}$. Then $\mathfrak{F}_0$ is cyclic. If $\mathfrak{F}_0 = 1$, then $\mathfrak{E}$ is abelian and we are done. We can suppose $\mathfrak{F}_0 \neq 1$.

Suppose $\mathfrak{F}_0$ is not of prime order. Let $\mathfrak{X} = [\mathfrak{F}_0, \mathfrak{P}\boldsymbol{H}(\mathfrak{M})]$. By

Lemma 26.3, and Lemma 26.16 $\mathfrak{T}$ is nilpotent. If $[\mathfrak{F}_0, \mathfrak{P}] \neq 1$, then $[\mathfrak{F}_0, \mathfrak{E}_p] \neq 1$, for some S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{P}$. Hence, $N([\mathfrak{F}_0, \mathfrak{E}_p])$ dominates every Sylow subgroup of $\mathfrak{P}$. Since $[\mathfrak{F}_0, \boldsymbol{H}(\mathfrak{M})]$ can be assumed non cyclic, $\mathfrak{P} \subseteq \mathfrak{M}'$, and we are done. If $[\mathfrak{F}_0, \mathfrak{P}] = 1$, then $\mathfrak{P} \cap \mathfrak{M}' = 1$, and we are done.

We can now suppose that $\mathfrak{F}_0$ is of prime order r. We can now write $\mathfrak{P} = \mathfrak{P}_0 \times \mathfrak{P}_1$, where $\mathfrak{P}_0 = \mathfrak{P} \cap C(\mathfrak{F}_0)$ and $\mathfrak{P}_1 = [\mathfrak{P}, \mathfrak{F}_0]$, and we suppose by way of contradiction that $\mathfrak{P}_i \neq 1, i = 0, 1$.

Choose p so that $\mathfrak{E}_p \cap \mathfrak{P}_0 \neq 1$, where $\mathfrak{E}_p$ is the S_p-subgroup of $\mathfrak{P}$.

If $\mathfrak{P}_0 \cap \mathfrak{E}_p$ centralizes $\boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0)$, then $N(\mathfrak{P}_0) \subseteq \mathfrak{M}$, by Lemma 26.13, since $\boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0) \neq 1$. Since $\mathfrak{P}_0 \cap \mathfrak{E}_p \lhd N(\mathfrak{E}_p)$, $\mathfrak{P}_0 \cap \mathfrak{E}_p \subseteq \mathfrak{M}'$, contrary to construction. Hence we can assume that $(\mathfrak{P}_0 \cap \mathfrak{E}_p)\mathfrak{H}^*$ is a Frobenius group, where $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0)$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_0)$. Since $\mathfrak{P}_0 \cap \mathfrak{E}_p \lhd N(\mathfrak{E}_p)$, it follows that $\mathfrak{P}_0 \cap \mathfrak{E}_p \subseteq \mathfrak{M}_1'$, since $N(\mathfrak{F}_0)$ dominates $\mathfrak{E}_p$. Since $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, it follows that $\pi(\boldsymbol{H}(\mathfrak{M}_1)) \cap \pi(\boldsymbol{H}(\mathfrak{M})) = \emptyset$, so that $\mathfrak{H}^* \cap \boldsymbol{H}(\mathfrak{M}_1) = 1$. Since $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p] \neq 1$, both $\mathfrak{P}_0 \cap \mathfrak{E}_p$ and $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p]$ are in $\mathfrak{M}_1'$, so commute elementwise. Thus $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p] = 1$, contrary to the above argument. The lemma is proved.

LEMMA 26.21. *Let $\mathfrak{M} \in \mathscr{M}$ and suppose $\pi(\mathfrak{M}/\mathfrak{M}')$ contains a prime p such that a S_p-subgroup of $\mathfrak{M}$ is non cyclic. Then $\mathfrak{M}$ is of type I.*

Proof.

Case 1. A S_p-subgroup of $\mathfrak{G}$ is abelian.

Case 2. A S_p-subgroup of $\mathfrak{G}$ is non abelian.

In Case 1, let $\tilde{\pi}$ be the subset of those q in $\pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M}))$ such that a S_q-subgroup of $\mathfrak{M}$ is an abelian non cyclic S_q-subgroup of $\mathfrak{G}$. Then $p \in \tilde{\pi}$, and if $\mathfrak{E}$ is a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, then a $S_{\tilde{\pi}}$-subgroup $\mathfrak{P}$ of $\mathfrak{E}$ is an abelian direct factor of $\mathfrak{E}$ by Lemma 26.20. Let $\mathfrak{E} = \mathfrak{P} \times \mathfrak{F}$. If $\mathfrak{F}$ were not a Z-group, then some Sylow subgroup $\mathfrak{F}_r$ of $\mathfrak{F}$ would be non abelian, by Lemma 26.18 and the definition of $\tilde{\pi}$. But then $N(\mathfrak{F}_r) \subseteq \mathfrak{M}$, by Lemma 26.14. Since $N(\mathfrak{F}_r)$ dominates every Sylow subgroup of $\mathfrak{P}$, we would find $\mathfrak{P} \subseteq \mathfrak{M}'$, which is not the case. Hence, $\mathfrak{F}$ is a Z-group.

Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F}'$ in $\mathfrak{F}$, and let $\mathfrak{F}_r$ be the S_r-subgroup of $\mathfrak{F}_0$. Let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\Omega_1(\mathfrak{F}_r))$. Since $\mathfrak{H}^*$ is a Z-group, and since $N(\Omega_1(\mathfrak{F}_r))$ dominates every Sylow subgroup of $\mathfrak{P}$, $\mathfrak{P}$ centralizes $\mathfrak{H}^*$. By Lemma 26.13, $\mathfrak{H}^* = 1$. Hence $\mathfrak{F}_0\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Let $\mathfrak{F}_s$ be the S_s-subgroup of $\mathfrak{F}'$, and let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\Omega_1(\mathfrak{F}_s))$. If $\mathfrak{H}^*$ is a Z-group, then $\mathfrak{H}^* = 1$ as in the preceding paragraph. If

$\mathfrak{H}^*$ is not a Z-group, then since $N(\Omega_1(\mathfrak{F}_s))$ dominates every Sylow subgroup of $\mathfrak{P}$, we find $\mathfrak{P} \subseteq \mathfrak{M}'$, which is not the case. Hence, $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

If $\mathfrak{F}$ is non abelian, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, so that $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$. By Lemma 26.13, $\mathfrak{M}$ is of Frobenius type, so $\mathfrak{M}$ is of type I. If $\mathfrak{F}$ is abelian, $\mathfrak{E}$ is abelian, so $\mathfrak{M}$ is of type I by Lemma 26.19.

In Case 2, let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, let $\mathfrak{E}_p$be a S_p-subgroup of $\mathfrak{E}$, and let $\mathfrak{F}$ be a $S_{p'}$-subgroup of $\mathfrak{E}$. Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F} \cap \mathfrak{M}'$ in $\mathfrak{F}$. Then $\mathfrak{F}_0$ is a S-subgroup of $\mathfrak{M}$, and $\mathfrak{F}_0 = 1$ is a possibility. We can suppose $\mathfrak{F}_0$ is permutable with $\mathfrak{E}_p$, so that $\mathfrak{F}_0$ normalizes $\mathfrak{E}_p$, since by Lemma 26.18, $\mathfrak{F}$ is a Z-group, and $\mathfrak{F}_0 \cap \mathfrak{M}' = 1$.

Let $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, $\mathfrak{F}_0$ normalizes both $\mathfrak{A}$ and $\mathfrak{B}$, and $\Omega_1(\mathfrak{B}) \subseteq \boldsymbol{Z}(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. By hypothesis, $[\mathfrak{F}_0, \mathfrak{E}_p] \subset \mathfrak{E}_p$.

Suppose $\mathfrak{F}_0 \neq 1$. Let $\mathfrak{F}^* = \mathfrak{F}_0 \cap \boldsymbol{C}(\mathfrak{B})$, and suppose that $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$. Let $\mathfrak{F}_0^*$ be a fixed subgroup of $\mathfrak{F}^*$ of prime order. Then $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0^*) = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0)$ is a Z-group normalized by $\mathfrak{F}_0\mathfrak{B}$. Since $\mathfrak{F}_0\mathfrak{B}$ is non abelian, $\mathfrak{H}^* = 1$. Hence $\mathfrak{F}^*\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Since $\mathfrak{F}_0$ is prime on $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{F}_0\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. In particular, every subgroup of $\mathfrak{F}_0$ of prime order centralizes $\mathfrak{B}$.

Let $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{M}'$, and suppose that $\mathfrak{F}_1 \neq 1$, so that our running assumptions are: $\mathfrak{F}_0 \neq 1$, $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$, $\mathfrak{F}_1 \neq 1$. Suppose $\mathfrak{F}_1\boldsymbol{H}(\mathfrak{M})$ is not a Frobenius group, and let $\mathfrak{F}_1^*$ be a subgroup of prime order such that $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_1^*) \neq 1$. It follows that $N(\mathfrak{F}_1^*) \subseteq \mathfrak{M}$. But $\mathfrak{F}_1^*$ centralizes $\Omega_1(\mathfrak{E}_p)$, so $\mathfrak{E}_p$ is not a S_p-subgroup of $N(\mathfrak{F}_1^*)$. Hence $\mathfrak{B}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, in case $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$. Hence, $\mathfrak{M}$ is of Frobenius type in this case. If $\mathfrak{B}\mathfrak{F}$ is non abelian, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, so $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$ and $\mathfrak{M}$ is of type I. If $\mathfrak{B}\mathfrak{F}$ is abelian, and $\boldsymbol{H}(\mathfrak{M})$ is not a T.I set in $\mathfrak{G}$, and $\pi(\boldsymbol{H}(\mathfrak{M})) \nsubseteq \pi_1^*$, then $m(\boldsymbol{H}(\mathfrak{M})) = 2$ and $\boldsymbol{H}(\mathfrak{M})$ is abelian. $\mathfrak{M}$ is of type I in this case.

Suppose now that $\mathfrak{F}_0 = \mathfrak{F}^* \neq 1$. In this case $\mathfrak{A}\mathfrak{F} \lhd \mathfrak{E}$. Since $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, it follows readily that $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and that $\mathfrak{M}$ is of type I.

Next suppose $\mathfrak{F}^* = 1, \mathfrak{F}_0 \neq 1$. Since $\mathfrak{F}_0$ is prime on $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{F}_0$ is of prime order. Since $\mathfrak{F}_0$ does not centralize $\mathfrak{B}$, $\mathfrak{F}_0$ does centralize $\mathfrak{A}$. Let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0)$, so that $\mathfrak{H}^* \neq 1$. Since $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{A}$ centralizes $\mathfrak{H}^*$. Since $\mathfrak{B}\mathfrak{F}_0$ is non abelian and $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, it follows that $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$ and that $\mathfrak{H}^*$ is cyclic.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_0)$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. Let $\mathfrak{E}_1$ be a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ which contains $\mathfrak{H}^*$. If $\mathfrak{A} \subseteq \boldsymbol{H}(\mathfrak{M}_1)$, then since $C(\mathfrak{A}) \subseteq \mathfrak{M}$, $\mathfrak{H}^*$ centralizes a non cyclic p-group, which is not the case. Hence, $\mathfrak{A} \not\subseteq \boldsymbol{H}(\mathfrak{M})$, and we can suppose that $\mathfrak{A} \subseteq \mathfrak{E}_1$.

Since $N(\mathfrak{D}) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{D}$ of $(\mathfrak{A}\mathfrak{H}^*)^{\sharp}$, it follows that $\mathfrak{A}\mathfrak{H}^*$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$. Let $\mathfrak{H}_0^* = \boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$, so that $\mathfrak{H} \subseteq \mathfrak{H}_0^*$, and $\mathfrak{H}_0^*$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$. Since $N(\mathfrak{F}_0) \subseteq \mathfrak{M}_1$, it follows that $\mathfrak{H}_0^* = \mathfrak{H}^*$.

If $\mathfrak{A}$ is not a S_p-subgroup of $\mathfrak{M}_1$, then $\Omega_1(\mathfrak{B})^M \subseteq \mathfrak{M}_1$ for some M in $\mathfrak{M}$. But then $\Omega_1(\mathfrak{B})^M\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, as is $\Omega_1(\mathfrak{B})^M\mathfrak{H}^*$, so that $\mathfrak{H}^*$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$, which is absurd. Hence $\mathfrak{A}$ is a S_p-subgroup of $\mathfrak{M}_1$.

If $\mathfrak{F}_0 \not\subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$, then either $|\mathfrak{F}_0| \in \pi_1$ or a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{M}_1$ is abelian. But in the first case, $\mathfrak{H}^*$ dominates $\mathfrak{F}_0$, contrary to $\mathfrak{F}_0 \cap \mathfrak{M}' = 1$, while in the second case $\mathfrak{H}^*\mathfrak{A}$ normalizes some $S_{|\mathfrak{F}_0|}$-subgroup $\mathfrak{K}$ of $\mathfrak{M}_1$ with $\mathfrak{F}_0 \subseteq \mathfrak{K}$, and $[\mathfrak{K}, \mathfrak{H}^*\mathfrak{A}]\mathfrak{H}^*\mathfrak{A}$ is a Frobenius group. As $\mathfrak{H}^*\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$ and $|\mathfrak{H}^*\mathfrak{A}|$ is not a prime, it follows that $[\mathfrak{H}^*\mathfrak{A}, \mathfrak{K}]$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$. If $\mathfrak{K}$ is a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$, then $\mathfrak{H}^*\mathfrak{A}$ dominates $\mathfrak{K}$, so $\mathfrak{F}_0 \subseteq \mathfrak{M}'$, which is not the case. Otherwise, a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$ is non abelian, and $\Omega_1([\mathfrak{K}, \mathfrak{H}^*\mathfrak{A}])$ is contained in the center of some $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$. But $N([\mathfrak{H}^*\mathfrak{A}, \mathfrak{K}]) \subseteq \mathfrak{M}_1$, and a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{M}_1$ is abelian. Hence, $\mathfrak{F}_0 \subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$.

We next show that $\mathfrak{H}^*\mathfrak{A}$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Namely, turning back to the definition of $\mathfrak{F}_0$, we have $\mathfrak{F} = \mathfrak{F}_0(\mathfrak{F} \cap \mathfrak{M}')$. But $\mathfrak{B} \subseteq \mathfrak{M}'$, and $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$. Hence, $\mathfrak{F} = \mathfrak{F}_0$ or $\mathfrak{F}$ is a Frobenius group with Frobenius kernel $\mathfrak{F} \cap \mathfrak{M}'$. Now, since $\mathfrak{F}_0 \subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$, it follows that $\mathfrak{M}_1 \cap \mathfrak{M} \subseteq \mathfrak{H}^*\mathfrak{A}\boldsymbol{H}(\mathfrak{M}_1)$. This implies that $\mathfrak{H}^*\mathfrak{A}$ has a normal complement in $\mathfrak{E}_1$. If $\mathfrak{H}^*\mathfrak{A} \neq \mathfrak{E}_1$, then $\mathfrak{E}_1$ is a Frobenius group with Frobenius kernel $\mathfrak{E}_1'$ and $\mathfrak{E}_1 = \mathfrak{E}_1'\mathfrak{H}^*\mathfrak{A}$. This is absurd since $\mathfrak{H}^*\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$, and $|\mathfrak{H}^*\mathfrak{A}|$ is not a prime. Thus $\mathfrak{H}^*\mathfrak{A}$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Now, however, $\boldsymbol{H}(\mathfrak{M}_1)$ is nilpotent. Since $\mathfrak{F}_0$ has no fixed points on $(\mathfrak{E} \cap \mathfrak{M}')^{\sharp}$, it follows that $\mathfrak{M} \cap \mathfrak{M}_1 = \mathfrak{F}_0\mathfrak{H}^*\mathfrak{A}$.

Since $\mathfrak{H}^*\mathfrak{A}$ centralizes $\mathfrak{F}_0$, it follows that $\mathfrak{F}_0 \subseteq \boldsymbol{H}(\mathfrak{M}_1)'$. We next show that $\boldsymbol{H}(\mathfrak{M}_1)$ is a T.I. set in $\mathfrak{G}$. Namely, $|\mathfrak{F}_0|$ divides $p-1$, since $[\mathfrak{B}, \mathfrak{F}_0] = \mathfrak{B}$. Hence $p > |\mathfrak{F}_0|$; since $|\mathfrak{F}_0|$ is a prime, $|\mathfrak{F}_0| \in \pi_0 - \pi^*$, so $\boldsymbol{H}(\mathfrak{M}_1)$ is a T.I. set in $\mathfrak{G}$.

We now turn to $N(\mathfrak{E}_p)$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ which contains $N(\Omega_1(\mathfrak{B}))$. Then $\mathfrak{M}_2$ is not conjugate to either $\mathfrak{M}$ or $\mathfrak{M}_1$, since the S_p-subgroups of these three maximal subgroups are pairwise non isomorphic. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}_2$ containing $\mathfrak{E}_p$ and normalized by $\mathfrak{F}_0$. If $p \in \pi_2$, then $\mathfrak{F}_0$ does not map onto $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$, since $\mathfrak{F}_0$ centralizes $\mathfrak{A}$. But then $N(\mathfrak{F}_0)$ covers $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$. This is not the case since $N(\mathfrak{F}_0) \subseteq \mathfrak{M}_1$, and $\mathfrak{A} \not\subseteq \mathfrak{M}_1'$. Hence, $p \notin \pi_2$, so $p \in \pi_0$, and $\mathfrak{P} \subseteq \boldsymbol{H}_1(\mathfrak{M}_2)$.

Since $C(\mathfrak{F}_0) \cap H_1(\mathfrak{M}_2) \subseteq \mathfrak{M}_1$, and since

$$1 = (|H_1(\mathfrak{M}_2)|, |H(\mathfrak{M}_1)| \cdot |H(\mathfrak{M})|),$$

it follows that $C(\mathfrak{F}_0) \cap H_1(\mathfrak{M}_2) = \mathfrak{A}$. Hence, $N(\mathfrak{F}_0) \cap \mathfrak{M}_2$ normalizes $\mathfrak{A}$. But $N(\mathfrak{F}_0) \cap N(\mathfrak{A}) = \mathfrak{F}_0\mathfrak{A}\mathfrak{H}^*$. (This turns the tide.) Suppose $N(\mathfrak{F}_0) \cap \mathfrak{M}_2 \supset \mathfrak{A}\mathfrak{F}_0$. Then $\mathfrak{M}_2$ contains a non identity subgroup $\mathfrak{H}^{**}$ of $\mathfrak{H}^*$. But $H(\mathfrak{M}_2)$ contains $\mathfrak{B}$, and we find that $[\mathfrak{H}^{**}, \mathfrak{B}] = \mathfrak{H}^{**} \subseteq H(\mathfrak{M}_2)$, which is not the case. Hence $N(\mathfrak{F}_0) \cap \mathfrak{M}_2 = \mathfrak{A}\mathfrak{F}_0$.

By Lemma 26.17, $\mathfrak{M}_2$ has p-length one. Let $\mathfrak{K}_2 = O_{p'}(\mathfrak{M}_2)$, so that $\mathfrak{P}\mathfrak{K}_2/\mathfrak{K}_2 = \bar{\mathfrak{P}} \lhd \bar{\mathfrak{M}}_2 = \mathfrak{M}_2/\mathfrak{K}_2$. Then $\mathfrak{M}_2/\mathfrak{P}\mathfrak{K}_2$ is a Frobenius group whose Frobenius kernel is of index $|\mathfrak{F}_0|$, or else $\mathfrak{M}_2 = \mathfrak{P}\mathfrak{K}_2\mathfrak{F}_0$. In any case, by Lemma 8.16, $\mathfrak{M}_2'$ centralizes $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}'$. But now $\mathfrak{A} \not\subseteq \mathfrak{M}_2'$, which is a contradiction to $H(\mathfrak{M}_2) \subseteq \mathfrak{M}_2'$.

We have now exhausted all possibilities under the assumption that $\mathfrak{F}_0 \neq 1$.

Suppose $\mathfrak{F}_0 = 1$. In this case, $\mathfrak{F} \subseteq \mathfrak{M}'$, $\mathfrak{F}$ is cyclic and $\mathfrak{F}$ is normalized by $\mathfrak{E}_p$. Since $\mathfrak{B}H(\mathfrak{M})$ is a Frobenius group, $\Omega_1(\mathfrak{B})$ centralizes $\mathfrak{F}$, so $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{F}$. This implies that $\mathfrak{F}H(\mathfrak{M})$ is a Frobenius group, or $\mathfrak{F} = 1$. In both cases, $\mathfrak{M}$ is of Frobenius type. If $\mathfrak{F} \neq 1$, then $\mathfrak{B}\mathfrak{F}$ is non abelian, so $m(Z(\mathfrak{H}_r)) \geq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $H(\mathfrak{M})$, $r \in \pi(H(\mathfrak{M}))$, and $H(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{F} = 1$, then $\mathfrak{E} = \mathfrak{E}_p$ is abelian, and the lemma follows from Lemma 26.19.

LEMMA 26.22. *Let $\mathscr{X}$ be the set of Z-subgroups $\mathfrak{Z}$ of $\mathfrak{G}$ with the following properties:*

(i) *If p, q are primes, every subgroup of $\mathfrak{Z}$ of order pq is cyclic.*

(ii) $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$, $|\mathfrak{Z}_i| = z_i \neq 1$, $i = 1, 2$ *and for any non empty subset $\mathfrak{Z}_0$ of $\mathfrak{Z} - \mathfrak{Z}_1 - \mathfrak{Z}_2$, $N(\mathfrak{Z}_0) \subseteq \mathfrak{Z}$.*

Then $\mathscr{X}$ is empty or consists of a unique conjugate class of subgroups.

Proof. If $\mathfrak{Z} \in \mathscr{X}$, and $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ satisfies (i) and (ii), then $\hat{\mathfrak{Z}} = \mathfrak{Z} - \mathfrak{Z}_1 - \mathfrak{Z}_2$ contains $(z_1 - 1)(z_2 - 1)$ elements. Since $\mathfrak{Z}$ is a Z-group, $(z_1, z_2) = 1$. $\hat{\mathfrak{Z}}$ is clearly a normal subset of $\mathfrak{Z}$, so $N(\hat{\mathfrak{Z}}) = \mathfrak{Z}$. Suppose $G \in \mathfrak{G}$ and $Z \in \hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^G$. Then there is a power of Z, say $Z_1 = Z^k$ such that $Z_1 \in \hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^G$ and such that Z_1 has order p_1p_2 where p_i is a prime divisor of $|\mathfrak{Z}_i| = z_i$. Then $\langle \mathfrak{Z}_1 \rangle \lhd \langle \mathfrak{Z}, \mathfrak{Z}^G \rangle$ and so $\mathfrak{Z} = \mathfrak{Z}^G$, $G \in \mathfrak{Z}$. Thus, the number of elements of $\mathfrak{G}$ which are conjugate to an element of $\hat{\mathfrak{Z}}$ is

$$\frac{|\mathfrak{G}|}{|\mathfrak{Z}|}(z_1 - 1)(z_2 - 1) > \frac{|\mathfrak{G}|}{2}. \tag{26.7}$$

Suppose $\mathfrak{Z}^*$ is another subgroup of $\mathscr{X}$ and $\mathfrak{Z}^* = \mathfrak{Z}_1^* \times \mathfrak{Z}_2^*$ satisfies (i) and (ii). Set $\hat{\mathfrak{Z}}^* = \mathfrak{Z}^* - \mathfrak{Z}_1^* - \mathfrak{Z}_2^*$. We can assume that $\hat{\mathfrak{Z}}^* \cap \hat{\mathfrak{Z}} \neq \varnothing$, by (26.7), and it follows that $\mathfrak{Z}^* = \mathfrak{Z}$. The proof is complete.

LEMMA 26.23. *Let $\mathfrak{M} \in \mathscr{M}$, and suppose $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$, $|\mathfrak{M} : \mathfrak{M}'|$ is not a prime, and $\mathfrak{M}/\mathfrak{M}'$ is cyclic. Then $\mathfrak{M}$ is of type I or V, or $\mathfrak{M}$ has the following properties:*

(i) *$\boldsymbol{H}(\mathfrak{M})$ is a nilpotent T.I. set in $\mathfrak{G}$.*

(ii) *If $\mathfrak{E}$ is a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$ then*

 (a) *$\mathfrak{E}$ is a non abelian Z-group and every subgroup of $\mathfrak{E}$ of order pq is cyclic, p, q primes.*

 (b) *$\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, and $\mathfrak{E}_1 = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{E})$ is a non identity cyclic group.*

(iii) *$\mathfrak{E}\mathfrak{E}_1 = \mathfrak{Z}$ satisfies the hypotheses of Lemma 26.22.*

Proof. If $\mathfrak{M}' = \boldsymbol{H}(\mathfrak{M})$, the lemma follows from Lemma 26.19. We can therefore suppose that $\boldsymbol{H}(\mathfrak{M}) \subset \mathfrak{M}'$. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, let $\mathfrak{F}$ be a complement to $\mathfrak{E}_0 = \mathfrak{E} \cap \mathfrak{M}'$ in $\mathfrak{E}$. Then $\mathfrak{F}$ is a cyclic S-subgroup of $\mathfrak{M}$, and $|\mathfrak{F}|$ is not a prime.

If $\mathfrak{M}$ is a Frobenius group, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every non identity S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, so $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, and we are done. We can suppose that $\mathfrak{M}$ is not a Frobenius group.

Suppose $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group with Frobenius kernel $\boldsymbol{H}(\mathfrak{M})$. With this hypothesis, we will show that $\mathfrak{M}$ is of type I.

Let $\mathfrak{E}_p$ be a cyclic S_p-subgroup of $\mathfrak{E}_0$. Suppose $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\Omega_1(\mathfrak{E}_p)) \neq 1$. Then $\mathfrak{E}_p\mathfrak{F}_0$ normalizes $\mathfrak{H}^*$. Consider $N(\Omega_1(\mathfrak{E}_p)) \supseteq \langle \mathfrak{H}^*, \mathfrak{E}_p, \mathfrak{F} \rangle$. Since $|\mathfrak{F}|$ is not a prime and $\mathfrak{F}\mathfrak{H}^*$ is a Frobenius group, it follows that $N(\Omega_1(\mathfrak{E}_0)) \subseteq \mathfrak{M}$. Hence, $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$. Since $\mathfrak{E}_p$ does not centralize $\boldsymbol{H}(\mathfrak{M})$, it follows that every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p$. Since $\mathfrak{E}_p \subseteq \mathfrak{M}'$, $|\mathfrak{F}|$ is not square free, and $\mathfrak{F}$ contains a S_q-subgroup $\mathfrak{F}_q$ such that $[\mathfrak{E}_p, \mathfrak{F}_q] \neq 1$. Consider $N(\Omega_1(\mathfrak{F}_q))$. If $q \in \pi_0$, then $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$. If $q \in \pi_1$ or $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is non abelian, then $\mathfrak{F}_q \subseteq N(\Omega_1(\mathfrak{F}_q))'$, so once again $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$. If $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is abelian, then $N(\Omega_1(\mathfrak{E}_p))$ contains a S_q-subgroup of $\mathfrak{G}$, contrary to $N(\Omega_1(\mathfrak{E}_p)) \subseteq \mathfrak{M}$. Hence $\mathfrak{H}^* = 1$ and $\mathfrak{E}_p\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Since $\mathfrak{M}$ is not a Frobenius group, $\mathfrak{E}_0$ contains a non cyclic S_p-subgroup $\mathfrak{E}_p$ for some prime p. If $\mathfrak{E}_p$ is abelian, and a S_p-subgroup of $\mathfrak{G}$ is non abelian, then $\mathfrak{E} = \mathfrak{E}_p \cdot \mathfrak{E}_{p'}$, and $\mathfrak{E}_{p'}$ is a Z-group. In this case, $\mathfrak{E}_{p'}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and so $\mathfrak{M}$ is of type I. If $\mathfrak{E}_p$ is abelian, and a S_p-subgroup of $\mathfrak{G}$ is abelian, then $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$. In this case, every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap \boldsymbol{C}(\boldsymbol{H}(\mathfrak{M}))$, so centralizes $\mathfrak{E}_p^*$ for some non identity subgroup of $\mathfrak{E}_p$. Since $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is abelian, it follows that if $\mathfrak{F}_q$ is a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p^*$, then $q \in \pi_2$, a S_q-subgroup of $\mathfrak{G}$ is abelian, and $\mathfrak{E}_p$ is normalized by a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ with $\mathfrak{F}_q \subset \mathfrak{Q}$. Since $C(\Omega_1(\mathfrak{E}_p)) \subset \mathfrak{M}$, $C(\mathfrak{E}_p) \cap \mathfrak{Q} \subseteq \mathfrak{F}_q$. Since $\mathfrak{Q}$ is of type (q^a, q^b), $ab > 0$, there is a direct factor of $\mathfrak{Q}$ which normalizes every subgroup of $\mathfrak{E}_p$. Hence, $\mathfrak{F}_q$ is this direct factor. Hence, q divides $p - 1$, so we have $\mathfrak{E}_p = \mathfrak{E}_{p1} \times \mathfrak{E}_{p2}$, where $\mathfrak{E}_{pi}$ is normalized by $\mathfrak{Q}$. It follows that $\mathfrak{E}_{pi}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group for $i = 1, 2$.

Suppose every Sylow subgroup of $\mathfrak{E}$ is abelian. Let $\tilde{\pi}$ be the subset of p in $\pi(\mathfrak{E})$ such that a S_p-subgroup of $\mathfrak{E}$ is non cyclic, and let $\mathfrak{P}$ be a $S_{\tilde{\pi}}$-subgroup of $\mathfrak{E}$. By Lemma 26.18 and the preceding paragraph, $\mathfrak{P}$ is a normal abelian subgroup of $\mathfrak{E}$. Hence, $\mathfrak{M}$ is of Frobenius type. Since $\mathfrak{E}$ is non abelian, $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, so $\mathfrak{M}$ is of type I.

Thus, if $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and every Sylow subgroup of $\mathfrak{E}$ is abelian, then $\mathfrak{M}$ is of type I.

Suppose $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and $\mathfrak{E}_p$ is a non abelian S_p-subgroup of $\mathfrak{E}$. Then $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $p \in \pi_2$. Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_p \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, Lemma 26.9 implies that $\mathfrak{F}$ centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$. This violates the containment $\mathfrak{E}_p \subseteq \mathfrak{M}'$. Hence, if $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, $\mathfrak{M}$ is of type I.

Suppose now that $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is not a Frobenius group. Let $\mathfrak{E}_1 = C(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M})$. By Lemma 26.15, $\mathfrak{E}_1$ is a Z-group. By Lemma 26.3, $\boldsymbol{H}(\mathfrak{M})$ is nilpotent so $\mathfrak{E}_1$ is cyclic. Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}'/\mathfrak{E}' \cap C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}$ normalizes $\mathfrak{E}_1$, so centralizes $\mathfrak{E}_1$ since Aut $\mathfrak{E}_1$ is abelian. Hence, $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})'$.

Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}'/\mathfrak{E}' \cap C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}'$ is abelian. Suppose $\mathfrak{E}'$ were non cyclic. Let $\mathfrak{E}_p$ be a non cyclic S_p-subgroup of $\mathfrak{E}'$. By Lemma 26.12, together with $\mathfrak{E}_1 \neq 1$, $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$.

Let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$, and let $\mathfrak{E}_p^* = \mathfrak{E}_p \cap C(\Omega_1(\mathfrak{F}_q)) \neq 1$. Then $\mathfrak{N} = N(\Omega_1(\mathfrak{F}_q)) \supseteq \langle \mathfrak{F}, \mathfrak{E}_p^*, \mathfrak{E}_1 \rangle$. It follows now from $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})' \subseteq \hat{\boldsymbol{H}}(\mathfrak{M}) \cup \{1\}$ that either $\mathfrak{F}_q$ is not a S_q-subgroup of $\mathfrak{G}$ or $\mathfrak{F}_q \subseteq \mathfrak{M}'$, both of which are false. Hence, $\mathfrak{E}'$ is cyclic. This yields that every subgroup of $\mathfrak{E}$ of order pq is cyclic, p, q being primes.

We next show that $\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Since $C(E) \supseteq C(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_1$, for all $E \in \mathfrak{E}$, it suffices to show that $\mathfrak{E}_1 = C(E) \cap \boldsymbol{H}(\mathfrak{M})$ for all $E \in \mathfrak{E}^{\sharp}$. Suppose false and $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{E}$ such that $C(\Omega_1(\mathfrak{E}_p)) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_2 \supset \mathfrak{E}_1$. Since $\mathfrak{F}\mathfrak{E}_2/\mathfrak{E}_1$ is a Frobenius group, it follows that $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $N(\mathfrak{E}_p) \subseteq \mathfrak{M}$. In this case, let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p$ and consider $N(\Omega_1(\mathfrak{F}_q)) \supseteq \langle \mathfrak{E}_p, \mathfrak{F} \rangle$. If $q \in \pi_0$, Lemma 26.9 is violated; if $q \in \pi_1$, then $\mathfrak{F}_q \subseteq N(\Omega_1(\mathfrak{F}_q))'$ so $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$; if $q \in \pi_2$, $\mathfrak{F}_q$ is not a S_q-subgroup of $N(\mathfrak{E}_p)$, contrary to $N(\mathfrak{E}_p) \subseteq \mathfrak{M}$. Hence, $\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, and so $\mathfrak{E}_1 = \boldsymbol{C}(E) \cap \boldsymbol{H}(\mathfrak{M})$ for all $E \in \mathfrak{E}^\#$. Since $\mathfrak{E}$ is non abelian, $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$.

Let $\mathfrak{Z} = \mathfrak{E}\mathfrak{E}_1$, and let $\hat{\mathfrak{Z}} = \mathfrak{E}\mathfrak{E}_1 - \mathfrak{E} - \mathfrak{E}_1$. By construction, $\mathfrak{E} \neq 1$, $\mathfrak{E}_1 \neq 1$, and $N(\hat{\mathfrak{Z}}) \cap \mathfrak{M} = \mathfrak{Z}$. Since $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})' \subseteq \hat{\boldsymbol{H}}(\mathfrak{M}) \cup \{1\}$, $N(\mathfrak{Z}_0) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{Z}_0$ of $\mathfrak{E}_1^\#$. Since $(|\mathfrak{E}|, |\mathfrak{E}_1|) = 1$, this implies that $N(\hat{\mathfrak{Z}}) = \mathfrak{Z}$ and $N(\hat{\mathfrak{Z}}_0) \subseteq \mathfrak{Z}$ for every non empty subset $\hat{\mathfrak{Z}}_0$ of $\hat{\mathfrak{Z}}$. Thus, $\mathfrak{Z}$ satisfies the hypotheses of Lemma 26.22. The proof is complete.

LEMMA 26.24. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{M}$ is of type* V. *Then $\mathfrak{M}'$ is tamely imbedded in $\mathfrak{G}$.*

Proof. We can suppose that $\mathfrak{M}'$ is not a T.I. set in $\mathfrak{G}$. Let $\mathfrak{E}_1 = \mathfrak{M}' \cap \boldsymbol{C}(\mathfrak{E})$, where $\mathfrak{E}$ is a complement to $\mathfrak{M}'$ in $\mathfrak{M}$. Then $\mathfrak{E}_1 \neq 1$, and $\mathfrak{E}_1 \subseteq \mathfrak{M}''$. Hence, $\mathfrak{M}'$ is non abelian. Let $\mathfrak{M}' = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{P}$ is a non abelian S_p-subgroup of $\mathfrak{M}'$, and $\mathfrak{S}_0$ is the $S_{p'}$-subgroup of $\mathfrak{M}'$ for some prime p (there may be several).

We show that $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{S}_0 = 1$, this is the case. Suppose $\mathfrak{S}_0 \neq 1$, and $S \in \mathfrak{S}_0 \cap \mathfrak{S}_0^G$, $S \neq 1$. Then $\boldsymbol{C}(S) \supseteq \langle \mathfrak{P}, \mathfrak{P}^G \rangle$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{C}(S)$. By Lemma 26.14, $N(\Omega_1(\boldsymbol{Z}(\mathfrak{P}))) \subseteq \mathfrak{M}_1$. Hence $\mathfrak{M} \subseteq \mathfrak{M}_1$, so $\mathfrak{M} = \mathfrak{M}_1 \supseteq \mathfrak{P}^G$ and so $\mathfrak{P} = \mathfrak{P}^G$ and $G \in \mathfrak{M}$.

Since $\mathfrak{M}'$ is not a T.I. set in $\mathfrak{G}$, it follows that $\mathfrak{S}_0$ is cyclic.

Suppose $M \in \mathfrak{M}'$, $M \neq 1$, and $\boldsymbol{C}(M) \not\subseteq \mathfrak{M}$. Since every subgroup of $\mathfrak{S}_0$ is normal in $\mathfrak{M}$, it follows that $M \in \mathfrak{P}$. Furthermore, $\langle M \rangle \cap \boldsymbol{T}(\mathfrak{P}) = \langle 1 \rangle$, so M is of order p, and $\boldsymbol{C}_{\mathfrak{M}}(M) = \langle M \rangle \times \mathfrak{B} \times \mathfrak{S}_0$, where $\mathfrak{B}$ is a non identity cyclic subgroup of $\mathfrak{P}$, and $\mathfrak{B} \supseteq \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$. (Notice that since $M \notin \mathfrak{M}''$, $\boldsymbol{C}_{\mathfrak{M}}(M) \subseteq \mathfrak{M}'$.)

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{C}(M)$. Then a S_p-subgroup of $\mathfrak{M}_1$ is abelian, by Lemma 26.14, so $\langle M \rangle \times \mathfrak{B}$ is a S_p-subgroup of $\mathfrak{M}_1$, by Lemma 26.6. By Lemma 26.12 $\mathfrak{B}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group.

Let $\mathfrak{L}$ be a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$ which contains $\boldsymbol{C}_{\mathfrak{M}}(M)$. Since $\mathfrak{B}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, it follows that $\langle M \rangle \times \Omega_1(\mathfrak{B}) \lhd \mathfrak{L}$. This implies that $\mathfrak{L} \subseteq \mathfrak{M}$, so that $\mathfrak{L} = \mathfrak{M} \cap \mathfrak{M}_1$.

We next show that $(|\mathfrak{M}|, |\boldsymbol{H}(\mathfrak{M}_1)|) = 1$. This is equivalent to showing that $(|\mathfrak{E}|, |\boldsymbol{H}(\mathfrak{M}_1)|) = 1$. Suppose false and q is a prime divisor of $(|\mathfrak{E}|, |\boldsymbol{H}(\mathfrak{M}_1)|)$. Since $p \in \pi^*$, q divides $p+1$ or $p-1$. Since p divides $|\mathfrak{M}_1 : \boldsymbol{H}(\mathfrak{M}_1)|$, and $\mathfrak{B}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, $q \in \pi_0 - \pi^*$. Thus, if Q is any element of $\mathfrak{G}$ of order q, then $\boldsymbol{C}(Q)$

is contained in a unique maximal subgroup of $\mathfrak{G}$. Let Q be an element of $\mathfrak{F}$ of order q, and let $\mathfrak{M}_2 = \boldsymbol{M}(C(Q))$. Then $\mathfrak{F}\mathfrak{E}_1 \subseteq \mathfrak{M}_2$. Since $q \in \pi_0 - \pi^*$, $\mathfrak{M}_2$ is conjugate to $\mathfrak{M}_1$ in $\mathfrak{G}$. Since $\mathfrak{F}\mathfrak{S}_0$ is a Frobenius group or $\mathfrak{S}_0 = 1$, $\mathfrak{E}_1$ is a p-group. We can thus find G in $\mathfrak{G}$ such that $\mathfrak{M}_2^G = \mathfrak{M}_1$, and we can suppose that $\langle \mathfrak{E}_1^G, M, \mathfrak{B}\rangle$ is a p-group. This implies that $\mathfrak{E}_1^G \subseteq \mathfrak{M}$, so that $G \in \mathfrak{M}$. Since $\langle M, \mathfrak{B}\rangle$ is a S_p-subgroup of $\mathfrak{M}_1$, we have $\mathfrak{E}_1^G \subseteq \langle M, \mathfrak{B}\rangle$. Since $\mathfrak{E}_1 \subseteq \mathfrak{M}''$ and $G \in \mathfrak{M}$, $\mathfrak{E}_1^G \subseteq \mathfrak{M}'' \cap \langle M, \mathfrak{B}\rangle$, and so $\Omega_1(\mathfrak{E}_1^G) = \Omega_1(\mathfrak{B})$. But now $[\Omega_1(\mathfrak{E}_1^G), \mathfrak{E}^G] = 1$, contrary to $Q^G \in \boldsymbol{H}(\mathfrak{M}_1)$ and $\Omega_1(\mathfrak{B})\boldsymbol{H}(\mathfrak{M}_1)$ a Frobenius group. Hence, $(|\mathfrak{M}|, |\boldsymbol{H}(\mathfrak{M}_1)|) = 1$.

By construction, $C(M) \subseteq \mathfrak{M}_1$. We next show that $N_{\mathfrak{M}}(\langle M\rangle)$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Since $\mathfrak{L} = \mathfrak{M} \cap \mathfrak{M}_1$, it follows that $\langle M\rangle \lhd \mathfrak{L}$, since $\langle M, \mathfrak{B}\rangle \lhd \mathfrak{L}$ and $\langle M\rangle \subseteq C(\boldsymbol{H}(\mathfrak{M}_1))$. Thus, $\mathfrak{L} = N_{\mathfrak{M}}(\langle M\rangle)$.

We next show that two elements of $\mathfrak{M}'$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $\mathfrak{M}$. Let M, $M_1 \in \mathfrak{M}'^{\sharp}$, and $M = M_1^G$, $G \in \mathfrak{G}$. Since $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$, we can suppose $M, M_1 \in \mathfrak{P}$. If $M \in \hat{\boldsymbol{H}}(\mathfrak{M})$, then $C(M) \subseteq \mathfrak{M}$, so $\mathfrak{P}^G \cap \mathfrak{M}$ is non cyclic, and so $G \in \mathfrak{M}$. We can suppose $M \notin \hat{\boldsymbol{H}}(\mathfrak{M})$. In this case $C_{\mathfrak{P}}(M)$ is a S_p-subgroup of $C(M)$. Now $C(M) \supseteq \langle \Omega_1(\boldsymbol{Z}(\mathfrak{P})), \Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))\rangle$, so we can find $C \in C(M)$ so that $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C \subseteq C_{\mathfrak{P}}(M)$. As observed earlier, this implies that $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$. Since $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))^{GC}$, and $\mathfrak{M} = N(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$, we have $GC \in \mathfrak{M}$. Then $M_1^{GC} = M^C$, so M and M_1 are conjugate in $\mathfrak{M}$, namely, by GC, since $C \in C(M)$.

Let $M_1, \cdots, M_m$ be a set of representatives for the conjugate classes $\mathfrak{C}_1, \cdots, \mathfrak{C}_m$ of elements in $\mathfrak{M}$ which are in $\mathfrak{M}'^{\sharp}$ and satisfy $C(M_i) \not\subseteq \mathfrak{M}$, $1 \leqq i \leqq m$. As we saw in the preceding paragraph, $C(M_i)$ is contained in a unique maximal subgroup of $\mathfrak{G}$, for each i, in fact, $N(\langle M_i\rangle)$ is the unique maximal subgroup of $\mathfrak{G}$ which contains $C(\langle M_i\rangle)$. Let $\mathfrak{N}_i = N(\langle M_i\rangle)$, $1 \leqq i \leqq m$, and suppose notation is chosen so that $\mathfrak{N}_1, \cdots, \mathfrak{N}_n$ are non conjugate in $\mathfrak{G}$, while $\mathfrak{N}_j$ is conjugate to some $\mathfrak{N}_i$ with $1 \leqq i \leqq n$, if $n + 1 \leqq j \leqq m$. Set $\mathfrak{H}_i = \boldsymbol{H}(\mathfrak{N}_i)$, $1 \leqq i \leqq n$, so that $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ if $1 \leqq i, j \leqq n, i \neq j$.

Let

$$\hat{\mathfrak{N}}_i = \bigcup_{H \in \mathfrak{H}_i^{\sharp}} C_{\mathfrak{N}_i}(H) - \mathfrak{H}_i^{\sharp} .$$

Since $M_i\mathfrak{H}_i \subseteq \hat{\mathfrak{N}}_i$, it follows that $N(\hat{\mathfrak{N}}_i) = \mathfrak{N}_i$. Also, $\mathfrak{N}_i = \mathfrak{H}_i(\mathfrak{N}_i \cap \mathfrak{M})$ and $\mathfrak{H}_i \cap \mathfrak{M} = 1$. If $\mathfrak{N}_i \cap \mathfrak{M} \subseteq \mathfrak{M}'$, then $\mathfrak{N}_i \cap \mathfrak{M}$ is abelian, and in fact $\mathfrak{N}_i \cap \mathfrak{M} = \langle M_i\rangle \times \mathfrak{B}_i \times \mathfrak{S}_0$, where $\mathfrak{B}_i$ is a cyclic subgroup of $\mathfrak{P}$. Since $(\mathfrak{B}_i \times \mathfrak{S}_0)\mathfrak{H}_i$ is a Frobenius group,

$$\hat{\mathfrak{N}}_i = \bigcup_{M \in \langle M_i\rangle^{\sharp}} M\mathfrak{H}_i \cup \{1\} , \tag{26.8}$$

so is a T.I. set in $\mathfrak{G}$.

Suppose $\mathfrak{N}_i \cap \mathfrak{M} \not\subseteq \mathfrak{M}'$. Then $\mathfrak{N}_i \cap \mathfrak{M}' \lhd \mathfrak{N}_i \cap \mathfrak{M}$, and $\mathfrak{N}_i \cap \mathfrak{M} = (\mathfrak{N}_i \cap \mathfrak{M}')\cdot\mathfrak{F}$, where $\mathfrak{F} \cap \mathfrak{M}' = 1$, and $\mathfrak{F}\langle M_i \rangle$ is a Frobenius group so that $|\mathfrak{F}|$ divides $p-1$. Now $\mathfrak{F}$ normalizes $\mathfrak{B}_i \times \mathfrak{S}_0$. ($\mathfrak{B}_i$ can be so chosen.) If $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is abelian, then $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0\mathfrak{H}_i$ is a Frobenius group by Lemma 26.21, (together with $\mathfrak{F}\langle M_i \rangle$ a Frobenius group), and $\hat{\mathfrak{N}}_i$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is non abelian, then since $\mathfrak{F}$ is prime on $\mathfrak{M}'$, and $\mathfrak{F}$ is prime on $\mathfrak{H}_i$, $\mathfrak{F}$ is prime on $\mathfrak{B}_i\mathfrak{S}_0\mathfrak{H}_i$. If $|\mathfrak{F}|$ is not a prime, then $[\mathfrak{F}, \mathfrak{B}_i\mathfrak{S}_0]$ centralizes $\mathfrak{H}_i$. Since $\mathfrak{S}_0$ is cyclic and every subgroup of $\mathfrak{S}_0$ is normal in $\mathfrak{M}$, we have $\mathfrak{S}_0 = 1$. But $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$ since $\Omega_1(\mathfrak{B}_i) \subseteq \boldsymbol{Z}(\mathfrak{M}')$. Thus, we can suppose $|\mathfrak{F}|$ is a prime. If $\mathfrak{F}$ centralizes $\mathfrak{B}_i$, Lemma 26.21 implies that $\mathfrak{N}_i$ is of type I. Thus, we can suppose that $\mathfrak{F}\mathfrak{B}_i$ is a Frobenius group. Hence $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is a Frobenius group, as is $\mathfrak{F}\langle M_i \rangle\mathfrak{B}_i\mathfrak{S}_0$. Since $\mathfrak{B}_i\mathfrak{H}_i$ is a Frobenius group and $\mathfrak{F}\mathfrak{B}_i$ is also a Frobenius group, $\mathfrak{H}_i$ is a nilpotent T.I. set in $\mathfrak{G}$. Hence $\mathfrak{F}^* = \boldsymbol{C}_{\mathfrak{H}_i}(\mathfrak{F})$ is a non identity cyclic subgroup and $\mathfrak{F}\mathfrak{F}^*$ satisfies the hypotheses of Lemma 26.22 with the obvious factorization $\mathfrak{F}\mathfrak{F}^* = \mathfrak{F} \times \mathfrak{F}^*$. But $\mathfrak{E}\mathfrak{E}_1$ also satisfies the hypotheses of Lemma 26.22, so $\mathfrak{F}\mathfrak{F}^*$ and $\mathfrak{E}\mathfrak{E}_1$ are isomorphic. In particular, p divides $|\mathfrak{F}\mathfrak{F}^*|$, so divides $|\mathfrak{F}^*|$. This is absurd, since p divides $|\mathfrak{B}_i|$ and $\mathfrak{B}_i\mathfrak{H}_i$ is a Frobenius group with Frobenius kernel $\mathfrak{H}_i \supseteq \mathfrak{F}^*$. Hence, this case cannot arise. Hence, $\hat{\mathfrak{N}}_i$ is a T.I. set in $\mathfrak{G}$, and in fact (26.8) holds. Since $\mathfrak{H}_i$ is a S-subgroup of $\mathfrak{N}_i$, we have $\mathfrak{N}_i = \boldsymbol{N}(\hat{\mathfrak{N}}_i)$.

Since $\mathfrak{N}_i$ and $\mathfrak{N}_j$ are not conjugate in $\mathfrak{G}$, $1 \leq i, j \leq n, i \neq j$, by construction, we have $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ if $i \neq j$. The factorization of $\boldsymbol{C}(M_k)$ is now immediate, $1 \leq k \leq m$. We have already shown that $(|\mathfrak{M}|, |\mathfrak{H}_i|) = 1$. Thus, $\mathfrak{M}'$ is tamely imbedded in $\mathfrak{G}$.

Hypothesis 26.1.

(i) *$\mathfrak{S} \in \mathscr{M}$ and $\mathfrak{S}'$ is a S-subgroup of $\mathfrak{S}$.*

(ii) *$|\mathfrak{S} : \mathfrak{S}'| = q$ is a prime and $\mathfrak{Q}^*$ is a complement to $\mathfrak{S}'$ in $\mathfrak{S}$.*

(iii) *$\mathfrak{S}'$ is not nilpotent.*

(iv) *$\mathfrak{H}^* = \boldsymbol{C}_{\mathfrak{S}'}(\mathfrak{Q}^*)$.*

LEMMA 26.25. *Under Hypothesis* 26.1, *$\mathfrak{H}^*$ is cyclic and $\mathfrak{Q}^*\mathfrak{H}^*$ satisfies the hypotheses of Lemma* 26.22 *with the factorization $\mathfrak{Q}^*\mathfrak{H}^* = \mathfrak{Q}^* \times \mathfrak{H}^*$; $N(\mathfrak{Q}^*)$ is contained in a unique maximal subgroup $\mathfrak{T}$ of $\mathfrak{G}$; $\mathfrak{S} \cap \mathfrak{T} = \mathfrak{Q}^*\mathfrak{H}^*$; $\mathfrak{Q}^* \subseteq \mathfrak{T}'$; every element of $\mathscr{M}$ is of type I or is conjugate to $\mathfrak{S}$ or $\mathfrak{T}$.*

Proof. Since $\mathfrak{S}'$ is not nilpotent, $\mathfrak{H}^* \neq 1$. Let $\mathfrak{T}$ be any maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{Q}^*)$.

Let $\tilde{\pi}$ consist of those p in $\pi(\mathfrak{S}')$ such that either $p \in \pi^*$ or $p \notin \pi(\mathfrak{H}^*)$ or $p \notin \pi(\boldsymbol{H}(\mathfrak{S}))$, and let $\mathfrak{U}$ be a $\mathfrak{Q}^*$-invariant $S_{\tilde{\pi}}$-subgroup of

$\mathfrak{S}$, and let $\mathfrak{H}$ be a $S_{\tilde{\pi}'}$-subgroup of $\mathfrak{S}'$. We will show that $\mathfrak{U}$ is nilpotent and that $\mathfrak{H} \lhd \mathfrak{S}$.

Choose $p \in \tilde{\pi}$ and let $\mathfrak{P}$ be a $\mathfrak{Q}^*$-invariant S_p-subgroup of $\mathfrak{S}$. If $p \in \pi^*$ or $p \notin \pi(\boldsymbol{H}(\mathfrak{S}))$, then $\mathfrak{S}$ has p-length one, by Lemma 26.17. Hence, $\mathfrak{S}'$ centralizes $\boldsymbol{O}_{p',p}(\mathfrak{S})/\boldsymbol{O}_{p'}(\mathfrak{S})$, so $\mathfrak{S}'$ has a normal p-complement. If $p \notin \pi(\mathfrak{H}^*)$, then by 3.16 (i) or Lemma 13.4, $\mathfrak{S}'$ centralizes $\boldsymbol{O}_{p',p}(\mathfrak{S})/\boldsymbol{O}_{p'}(\mathfrak{S})$, so in this case, too, $\mathfrak{S}'$ has a normal p-complement. Hence, $\mathfrak{U}$ is nilpotent and $\mathfrak{H} \lhd \mathfrak{S}$. Since $\mathfrak{S}'$ is not nilpotent, $\mathfrak{H} \neq 1$. Furthermore, $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$. By construction, $\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, so $\boldsymbol{N}(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^\sharp$. Thus, $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Since $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$, Lemma 26.14 implies that $\boldsymbol{N}(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$. Thus $\mathfrak{H}^*\mathfrak{Q}^* = \mathfrak{H}^* \times \mathfrak{Q}^*$ satisfies Hypothesis (ii) in Lemma 26.22.

Let $\mathfrak{H}^{**} = \mathfrak{S}' \cap \mathfrak{T} \supseteq \mathfrak{H}^*$. $\mathfrak{T}$ is not conjugate to $\mathfrak{S}$, either because $\mathfrak{Q}^*$ is not a S_q-subgroup of $\mathfrak{G}$ or because $\mathfrak{Q}^* \subseteq \mathfrak{T}'$. Thus, $\mathfrak{H}^{**} \cap \boldsymbol{H}(\mathfrak{T}) = 1$. If $\mathfrak{H}^* \subset \mathfrak{H}^{**}$, then $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$ since $[\mathfrak{Q}^*, \mathfrak{H}^{**}] \neq 1$. But in that case, some S_q-subgroup of $\mathfrak{T}$ normalizes $\mathfrak{H}^{**}$, so $\mathfrak{Q}^*$ is a S_q-subgroup of $\mathfrak{G}$. But in that case, $\mathfrak{Q}^* \subseteq \boldsymbol{N}(\mathfrak{Q}^*)' \subseteq \mathfrak{T}'$. Hence, $\mathfrak{H}^* = \mathfrak{S}' \cap \mathfrak{T}$, so $\mathfrak{H}^*\mathfrak{Q}^* = \mathfrak{S} \cap \mathfrak{T}$. Since $\boldsymbol{N}(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$, it follows that $\mathfrak{H}^*$ has a normal complement in $\mathfrak{T}$, say $\mathfrak{T}_1$, and $\mathfrak{T}_1$ is a S-subgroup of $\mathfrak{T}$. Suppose $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$. Then $\mathfrak{T}_1 \cap \mathfrak{T}'$ is disjoint from $\mathfrak{Q}^*$, $\mathfrak{H}^*(\mathfrak{T}_1 \cap \mathfrak{T}')$ is a Frobenius group, and $\mathfrak{T}_1 = (\mathfrak{T}_1 \cap \mathfrak{T}')\mathfrak{Q}^*$. Furthermore, a $\mathfrak{H}^*$-invariant S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{T}_1$ has a normal complement in $\mathfrak{T}_1$, and $\mathfrak{Q}$ is abelian, by Lemmas 26.10 and 26.11. Thus $\mathfrak{Q}^*$ is a direct factor of $\mathfrak{Q}$, and $\mathfrak{Q}^* \subset \mathfrak{Q}$, since $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$ and $\boldsymbol{N}(\mathfrak{Q}^*) \subseteq \mathfrak{T}$. If a S_q-subgroup of $\mathfrak{G}$ is abelian, then $\boldsymbol{N}(\mathfrak{H}^*)$ dominates $\mathfrak{Q}$, so $\mathfrak{Q}^* \subseteq \mathfrak{S}'$, which is not the case. If a S_q-subgroup of $\mathfrak{G}$ is non abelian, then since $\mathfrak{T}_1 \cap \mathfrak{T}'$ is nilpotent, $\mathfrak{Q}^*$ is contained in the center of some S_q-subgroup of $\mathfrak{G}$. This is absurd, since $\boldsymbol{N}(\mathfrak{Q}^*) \subseteq \mathfrak{T}$ and $\mathfrak{Q}$ is an abelian S_q-subgroup of $\mathfrak{T}$. Hence, $\mathfrak{Q}^* \subseteq \mathfrak{T}'$.

Again, let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{T}$ normalized by $\mathfrak{H}^*$, and let $\mathfrak{B}$ be a $S_{q'}$-subgroup of $\mathfrak{T}_1$ normalized by $\mathfrak{H}^*$. Then either $\mathfrak{B} = 1$ or $\mathfrak{H}^*\mathfrak{B}$ is a Frobenius group. In both these case, we conclude that $\mathfrak{Q} \lhd \mathfrak{T}$. If $\mathfrak{B}$ does not centralize $\mathfrak{Q}$, then by Lemma 26.16, $q \in \pi_0 - \pi^*$, so $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{Q}^*)$. If $\mathfrak{B}$ centralizes $\mathfrak{Q}$, then $\mathfrak{Q}^* \subseteq \mathfrak{Q}'$, so if $q \in \pi_0$, $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$. But if $q \notin \pi_0$, then $\mathfrak{Q}^* \lhd \mathfrak{T}$, so of course $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{Q}^*)$. Thus, in all cases, $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$.

We next see that if p_1, p_2 are primes then every subgroup of $\mathfrak{H}^*$ of order p_1p_2 is cyclic. We next show that $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Suppose false and $\mathfrak{H}^*_r = \mathfrak{H}^* \cap \mathfrak{U}_r \not\subseteq \boldsymbol{Z}(\mathfrak{H}^*)$ where $\mathfrak{U}_r$ is the S_r-subgroup of $\mathfrak{U}$. If $r \in \pi_1 \cup \pi_2$, then since $\mathfrak{U}_r \subseteq \mathfrak{S}'$, it follows that $r \in \pi_2$ and $\mathfrak{U}_r$ is the non abelian group of order r^3 and exponent r, so that $|\mathfrak{H}_r^*| = r$. Since $\mathfrak{H}^* \cap \mathfrak{U}$ has a normal complement in $\mathfrak{H}^*$ and every subgroup of $\mathfrak{H}^*$ of order p_1p_2 is cyclic, $\mathfrak{H}_r^* \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Thus, we can suppose that $r \in \pi_0$. By definition of $\tilde{\pi}$, we also have $r \in \pi^*$. Apply Lemma 8.17 and conclude that q divides $r-1$. Since $\mathfrak{H}^*$ is a Z-group, Lemma 13.4 applied to $\mathfrak{Q}^*\mathfrak{U}_r$ acting on the $S_{r'}$-subgroup of $\mathfrak{S}'$ implies that $\mathfrak{U}_r'$ centralizes the $S_{r'}$-subgroup of $\mathfrak{S}'$; since $\mathfrak{H}_r^* \subseteq \mathfrak{U}_r'$, it follows once again that $\mathfrak{H}_r^* \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Hence, $\mathfrak{H}^* = (\mathfrak{H}^* \cap \mathfrak{U}) \times (\mathfrak{H}^* \cap \mathfrak{H})$ with cyclic $\mathfrak{H}^* \cap \mathfrak{U}$.

If $\mathfrak{H}^* \cap \mathfrak{H} \subseteq \boldsymbol{F}(\mathfrak{S})$, then $\mathfrak{H}^*$ is cyclic. Suppose $\mathfrak{H}^*$ is non cyclic. Since $\mathfrak{U}$ is nilpotent and since $\mathfrak{S}'/\boldsymbol{F}(\mathfrak{S})$ is nilpotent by Lemma 26.4, it follows that $\pi(\mathfrak{H}^* \cap \mathfrak{H})$ contains a prime s such that a S_s-subgroup of $\mathfrak{S}'/\boldsymbol{F}(\mathfrak{S}) \cap \mathfrak{H}$ is non abelian. Hence, $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{U})$ contains a non abelian S_s-subgroup. By construction, $s \in \pi_0 - \pi^*$, so $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{U}) \in \mathscr{H}_1$. This implies that $\mathfrak{S}'$ is a T.I. set in $\mathfrak{G}$.

Since $\mathfrak{H}^*$ is assumed non cyclic, hence non abelian, and since every subgroup of $\mathfrak{H}^*$ of order p_1p_2 is cyclic, it follows that $|\mathfrak{H}^* : \mathfrak{H}^{*\prime}|$ is not a prime. By Lemma 26.23 (i), $\mathfrak{T}_1$ is a nilpotent T.I. set in $\mathfrak{G}$. Set $g = |\mathfrak{G}|$, $|\mathfrak{S}'| = m_1$, $|\mathfrak{T}_1| = m_2$, $|\mathfrak{H}^*| = h$, $|\mathfrak{Q}^*| = q$. If $G_1, G_2, G_3 \in \mathfrak{G}$, the sets $G_1^{-1}\mathfrak{S}'^{\sharp}G_1$, $G_2^{-1}\mathfrak{T}_1^{\sharp}G_2$, $G_3^{-1}(\mathfrak{H}^*\mathfrak{Q}^* - \mathfrak{H}^* - \mathfrak{Q}^*)G_3$ have pairwise empty intersections. Hence,

$$g \geqq \frac{g}{m_1q}(m_1 - 1) + \frac{g}{m_2h}(m_2 - 1) + \frac{g}{hq}(h-1)(q-1) ,$$

so that

$$\frac{1}{m_1q} + \frac{1}{m_2h} \geqq \frac{1}{hq} .$$

Since $m_1 \geqq 3h$, $m_2 \geqq 3q$, the last inequality is not possible. Hence, $\mathfrak{H}^*$ is cyclic.

Let $\mathfrak{L}$ be a maximal subgroup of $\mathfrak{G}$ which is not conjugate to either $\mathfrak{S}$ or $\mathfrak{T}$. If $\mathfrak{L}'$ is not a S-subgroup of $\mathfrak{L}$, then Lemmas 26.10, 26.11 and 26.21 imply that $\mathfrak{L}$ is of type I. If $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$ but $\mathfrak{L}/\mathfrak{L}'$ is non cyclic, Lemma 26.21 implies that $\mathfrak{L}$ is of type I. If $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$, $\mathfrak{L}/\mathfrak{L}'$ is cyclic, and $|\mathfrak{L} : \mathfrak{L}'|$ is not a prime, then by Lemma 26.23, $\mathfrak{L}$ is of type I or $\mathfrak{L}$ contains a subgroup $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ which satisfies the hypotheses of Lemma 26.22. But $\mathfrak{Q}^*\mathfrak{H}^*$ also satisfies the hypotheses of Lemma 26.22, so $\mathfrak{Z}$ is conjugate to $\mathfrak{Q}^*\mathfrak{H}^*$. Since $\mathfrak{Z}_1 \subseteq \boldsymbol{H}(\mathfrak{L})$ can be assumed, either $(|\mathfrak{Z}_1|, |\mathfrak{Q}^*|) \neq 1$, or $(|\mathfrak{Z}_1|, |\mathfrak{H}^*|) \neq 1$. The first case yields $\mathfrak{L} = \mathfrak{T}^G$, $G \in \mathfrak{G}$, the second case yields $\mathfrak{L} = \mathfrak{S}^{G_1}$, $G_1 \in \mathfrak{G}$ and we are done in this case. Lemmas 26.22 and 26.23 complete the proof.

LEMMA 26.26. *Under Hypothesis* 26.1 $\mathfrak{T}$ *is either of type* V, *or*
(i) $|\mathfrak{H}^*| = p$ *is a prime.*
(ii) $\mathfrak{T}$ *satisfies*
 (a) $|\mathfrak{T} : \mathfrak{T}'| = p$, *and* $\mathfrak{T}'$ *is a S-subgroup of* $\mathfrak{T}$.
 (b) $\mathfrak{T}'$ *is not nilpotent.*

Proof. By Lemma 26.25, $\mathfrak{Q}^* \subseteq \mathfrak{T}'$ and $\mathfrak{H}^*$ is cyclic. As $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$ and $\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, it follows that $N(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$. Since $\mathfrak{S} \cap \mathfrak{T} = \mathfrak{Q}^*\mathfrak{H}^*$, this implies that $\mathfrak{H}^*$ has $\mathfrak{T}'$ as a complement. If $|\mathfrak{H}^*|$ is not a prime, $\mathfrak{T}'$ is nilpotent, by Lemma 26.3. This implies directly that $\mathfrak{T}$ is of type V, condition (ii) in the definition of type V following easily, since $\mathfrak{T}'$ is non abelian.

We can suppose that $\mathfrak{T}$ is not of type V. Hence, (i) is satisfied. Since $\mathfrak{T}'$ is not nilpotent, (ii) (a) and (ii) (b) also hold.

Lemma 26.26 is important, since if $\mathfrak{T}$ is not of type V, then $\mathfrak{T}$ satisfies Hypothesis 26.1, as does $\mathfrak{S}$.

LEMMA 26.27. *Under Hypothesis* 26.1, *one of the following holds*: (i) $N(\mathfrak{U}) \not\subseteq \mathfrak{S}$; (ii) $\mathfrak{S}'$ *is a tamely imbedded subset of* $\mathfrak{G}$, *and* $\mathfrak{U}$ *is a S-subgroup of* $\mathfrak{G}$.

Proof. Suppose $N(\mathfrak{U}) \subseteq \mathfrak{S}$. If $\mathfrak{S}'$ is a T.I. set in $\mathfrak{G}$ we are done. Hence, we can suppose that $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$.

Since $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$ and since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$ ($\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, so Lemma 26.5 (ii) applies), $\mathfrak{U} \neq 1$. We first treat the case in which $\mathfrak{U}$ is non abelian. Let $\mathfrak{U} = \mathfrak{R} \times \mathfrak{R}_0$, where $\mathfrak{R}$ is a non abelian S_r-subgroup of $\mathfrak{R}$, and $\mathfrak{R}_0$ is the $S_{r'}$-subgroup of $\mathfrak{U}$. We show that $\mathfrak{S}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{R}$.

Suppose $\mathfrak{R} \subseteq \mathfrak{L}, \mathfrak{L} \in \mathscr{M}$. By Lemma 26.1, $N(\Omega_1(\boldsymbol{Z}(\mathfrak{R}))) \subseteq \mathfrak{L} \cap \mathfrak{S}$. In particular, $N(\mathfrak{R}) \subseteq \mathfrak{L} \cap \mathfrak{S}$, so $\mathfrak{R}$ is a S_r-subgroup of $\mathfrak{G}$. If $\mathfrak{L} = \mathfrak{S}^G$, $G \in \mathfrak{G}$, then by Sylow's theorem, $\mathfrak{R}$ is conjugate to $G\mathfrak{R}G^{-1}$ in $\mathfrak{S}$, $\mathfrak{R} = S^{-1}G\mathfrak{R}G^{-1}S$, so that $S^{-1}G \in N(\mathfrak{R}) \subseteq \mathfrak{S}$, and $G \in S$. Hence, we can suppose $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$. Clearly, $\mathfrak{L}$ is not conjugate to $\mathfrak{T}$, since $q \nmid |\mathfrak{T} : \mathfrak{T}'|$. Hence, $\mathfrak{L}$ is of type I. But then $\mathfrak{R} \subseteq \boldsymbol{H}(\mathfrak{L})$, so that $\mathfrak{L} = N(\mathfrak{R}) \subseteq \mathfrak{S}$, contrary to assumption. Hence, $\mathfrak{R}$ is contained in $\mathfrak{S}$ and no other maximal subgroup of $\mathfrak{G}$. This implies that $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$.

Choose $S \in \mathfrak{S}'^{\sharp} \cap \mathfrak{S}'^{G}$, $G \in \mathfrak{G} - \mathfrak{S}$. There are such elements S and G since $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$. If S is not a $\tilde{\pi}$-element, then $S_1 = S^n \in \mathfrak{H}^{\sharp} \cap \mathfrak{H}^{\sharp G}$ for some integer n, contrary to the fact that $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Hence S is a $\tilde{\pi}$-element and we can suppose that $S \in \mathfrak{U}$. If $S \notin \mathfrak{R}$, then $S_2 = S^m \in \mathfrak{R}_0^{\sharp} \cap S'^{\sharp G}$ for some m, and $C(S_2)$ contains a S_r-subgroup of both $\mathfrak{S}$ and $\mathfrak{S}^G$, which is not the case. Hence,

$S \in \mathfrak{R}$. Since $\mathfrak{R}$ was any non abelian Sylow subgroup of $\mathfrak{U}$, it follows that $\mathfrak{R}_0$ is abelian.

Let $\mathfrak{L} \in \mathscr{M}$, $C(S) \subseteq \mathfrak{L}$. A S_r-subgroup of $\mathfrak{L}$ is non cyclic. Let $\tilde{\mathfrak{R}}$ be a S_r-subgroup of $\mathfrak{L}$ containing $C_{\mathfrak{R}}(S)$. If $r \in \pi_0$, then by Lemma 26.7, $N(C_{\mathfrak{R}}(S)) \subseteq \mathfrak{S}$, so $\tilde{\mathfrak{R}} = C_{\mathfrak{R}}(S)$. If $r \in \pi_2$, the same equality holds by Lemma 26.14 and the containment $N(C_{\mathfrak{R}}(S)) \subseteq N(\Omega_1(Z(\mathfrak{R})))$. Thus, $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$. Since $\tilde{\mathfrak{R}}$ is non cyclic, $\mathfrak{L}$ is not conjugate to $\mathfrak{T}$. Hence, $\mathfrak{L}$ is of type I, and this implies directly that $\mathfrak{L} = H(\mathfrak{L})(\mathfrak{L} \cap \mathfrak{S})$, $\mathfrak{S} \cap H(\mathfrak{L}) = 1$. Since a S_r-subgroup of $\mathfrak{G}$ is non abelian, Lemmas 26.12 and 26.18 imply that

$$\left\{\bigcup_{H \in H(\mathfrak{L})^{\#}} C_{\mathfrak{L}}(H)\right\} - H(\mathfrak{L})^{\#} = H(\mathfrak{L})\langle S\rangle^{\#},$$

and it is obvious that $H(\mathfrak{L})\langle S\rangle^{\#}$ is a T.I. set in $\mathfrak{G}$ with $\mathfrak{L}$ as its normalizer. We have verified all the properties in the definition of a tamely imbedded subset except the conjugacy condition for $\mathfrak{S}'$ and the coprime conditions. By definition of $H(\mathfrak{L})$, together with the fact that $\mathfrak{S}'$ is a S-subgroup of $\mathfrak{G}$, it follows that $(|H(\mathfrak{L})|, |\mathfrak{S}'|) = 1$. If $(|H(\mathfrak{L})|, |\mathfrak{Q}^*|) \neq 1$, then $\mathfrak{L}$ is conjugate to $\mathfrak{T}$. This is not the case, as $\tilde{\mathfrak{R}}$ is non cyclic. Thus, if $\mathfrak{L}_1, \cdots, \mathfrak{L}_m$ is a set of representatives for the conjugate classes of maximal subgroups of $\mathfrak{G}$ which contain $C(S)$ for some S in $\mathfrak{S}'$ and are different from $\mathfrak{S}$, it follows that $(|H(\mathfrak{L}_i)|, |H(\mathfrak{L}_j)|) = 1$ for $i \neq j$. It remains only to verify the conjugacy condition for elements of $\mathfrak{S}'^{\#}$. Let S, S_1 be elements of $\mathfrak{S}'^{\#}$ which are conjugate in $\mathfrak{G}$. We can suppose that S and S_1 have order r and are in $\mathfrak{R}$; otherwise it is immediate that S and S_1 are conjugate in $\mathfrak{S}$. Let $S = G^{-1}S_1G$, then $C(S) \supseteq \langle \Omega_1(Z(\mathfrak{R})), \Omega_1(Z(\mathfrak{R}^G))\rangle$. Since $N(\Omega_1(Z(\mathfrak{R}))) \subseteq \mathfrak{S}$, it follows that S and S_1 are conjugate in $\mathfrak{S}$. (It is at this point that we once again have made use of the fact that the subgroups in $\mathscr{T}(\mathfrak{R})$ have two conjugate classes of subgroups of order r.) Thus, $\mathfrak{S}'$ is a tamely imbedded subset of $\mathfrak{G}$ in this case.

We now assume that $\mathfrak{U}$ is abelian. We first show that $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$. Otherwise, $\mathfrak{U}$ is not a S-subgroup of $N(\mathfrak{U}_r)$ for some non identity S_r-subgroup $\mathfrak{U}_r$ of $\mathfrak{U}$. Let $N(\mathfrak{U}_r) \subseteq \mathfrak{L} \in \mathscr{M}$. Then $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$, since $|\mathfrak{L}|_{\pi} \neq |\mathfrak{S}|_{\pi}$. Suppose $\mathfrak{L}$ is conjugate to $\mathfrak{T}$. Since $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group, we have $\mathfrak{U} \subseteq \mathfrak{L}'$. Thus $\mathfrak{L}'$ is not nilpotent, since by hypothesis $N(\mathfrak{U}) \subseteq \mathfrak{S}$. Hence, $\mathfrak{T}$ is not of type V. By Lemma 26.26, $|\mathfrak{P}^*| = p$ is a prime. Since $|\mathfrak{Q}^*| = q$ is also a prime, it follows that if $\mathfrak{V}$ is a S_q-subgroup of $\mathfrak{T}'$ normalized by $\mathfrak{P}^*$, then $\mathfrak{P}^*\mathfrak{V}$ is a Frobenius group, ($\mathfrak{V} \neq 1$, since $\mathfrak{T}'$ is not nilpotent). If $\pi(\mathfrak{U}) \subseteq \pi(\mathfrak{V})$, then since $N(\mathfrak{U}) \subseteq \mathfrak{S}$, it follows that $\mathfrak{U}$ is conjugate to $\mathfrak{V}$. But p divides $|N(\mathfrak{V}) : C(\mathfrak{V})|$, and so $p = q$, which is not the case. Hence $\pi(\mathfrak{U}) \not\subseteq \pi(\mathfrak{V})$. But $\pi(\mathfrak{U}) \subseteq \pi(\mathfrak{S}) \cap \pi(\mathfrak{T}') \subseteq$

$\pi(\mathfrak{V}) \cup \{q\}$, so $q \in \pi(\mathfrak{U})$, which is absurd since $\mathfrak{S}'$ is a q'-group. Hence, $\mathfrak{L}$ is not conjugate to either $\mathfrak{S}$ or $\mathfrak{T}$, so $\mathfrak{L}$ is of type I. Since $\mathfrak{Q}^*$ is of prime order and $\mathfrak{Q}^*\mathfrak{U}$ is a Frobenius group, $\mathfrak{U} \subseteq \boldsymbol{H}(\mathfrak{L})$. Since $N(\mathfrak{U}) \subseteq \mathfrak{S}$, we have $\mathfrak{U} = \boldsymbol{H}(\mathfrak{L})$. Hence $\mathfrak{L} \subseteq N(\mathfrak{U}) \subseteq \mathfrak{S}$, which is absurd. Hence, $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$. This implies directly that $N(\mathfrak{U}_r) \subseteq \mathfrak{S}$ for all non identity Sylow subgroups $\mathfrak{U}_r$ of $\mathfrak{S}$.

Since $\mathfrak{U}$ is an abelian S-subgroup of $\mathfrak{G}$, and $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$, the condition $N(\mathfrak{U}) \subseteq \mathfrak{S}$ implies that two elements of $\mathfrak{S}'$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $\mathfrak{S}$.

Suppose $S \in \mathfrak{S}'^{\sharp}$, and $C(S) \not\subseteq \mathfrak{S}$. Then S is a $\tilde{\pi}$-element, and we can suppose $S \in \mathfrak{U}$. Let $\mathfrak{L} \in \mathscr{M}$, $\boldsymbol{C}(S) \subseteq \mathfrak{L}$. Since $\mathfrak{U}$ is an abelian S-subgroup of $\mathfrak{G}$ and since $\mathfrak{U} \subseteq \boldsymbol{C}(S) \subseteq \mathfrak{L}$, it follows that $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$ or $\mathfrak{T}$. It is now straightforward to verify that $\mathfrak{S}'$ is tamely imbedded in $\mathfrak{G}$.

LEMMA 26.28. *Under Hypothesis* 26.1, *either* $\mathfrak{S}$ *or* $\mathfrak{T}$ *is of type* II. *If* $\mathfrak{S}$ *is of type* II, *then*

$$\bigcup_{H \in \mathfrak{H}^{\sharp}} C'_{\mathfrak{S}}(H)$$

is a T.I. *set in* $\mathfrak{G}$. *Both* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are of type* II, III, IV *or* V.

Proof. First, suppose $\mathfrak{T}$ is of type V, but that $\mathfrak{S}$ is not of type II. Suppose $N(\mathfrak{U}) \subseteq \mathfrak{S}$. By Lemma 26.27, $\mathfrak{S}'$ is a tamely imbedded subset of $\mathfrak{G}$. As $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$ in this case, we have $(|\mathfrak{S}'|, |\mathfrak{T}'|) = 1$. By Lemma 26.24, $\mathfrak{T}'$ is a tamely imbedded subset of $\mathfrak{G}$. We now use the notation of section 9. Suppose $S \in \mathfrak{S}'^{\sharp}$, $T \in \mathfrak{T}'^{\sharp}$ and some element of $\mathfrak{A}_S$ is conjugate to some element of $\mathfrak{A}_T$. This implies the existence of $\mathfrak{L} \in \mathscr{M}$ such that $|\mathfrak{L} : \boldsymbol{H}(\mathfrak{L})|$ divides $(|\mathfrak{S}'|, |\mathfrak{T}'|) = 1$, which is not the case. Setting $\hat{\mathfrak{W}} = \mathfrak{H}^*\mathfrak{Q}^* - \mathfrak{H}^* - \mathfrak{Q}^*$, it follows that no element of $\hat{\mathfrak{W}}$ is conjugate to an element of $\mathfrak{A}_S$ or $\mathfrak{A}_T$. We find, with $h = |\mathfrak{H}^*|$, $s = |\mathfrak{S}'|$, $t = |\mathfrak{T}'|$, that by Lemma 9.5,

$$g \geqq \frac{(h-1)(q-1)}{hq} g + \frac{s-1}{sq} g + \frac{t-1}{th} g , \tag{26.9}$$

which is not the case. Hence $N(\mathfrak{U}) \not\subseteq \mathfrak{S}$. If $\mathfrak{U}_r$ were a non abelian S_r-subgroup of $\mathfrak{G}$, then $N(\Omega_1(\boldsymbol{Z}(\mathfrak{U}_r))) \subseteq \mathfrak{S}$, by Lemma 26.14. Since $N(\mathfrak{U}) \subseteq N(\Omega_1(\boldsymbol{Z}(\mathfrak{U}_r)))$, this is impossible. Hence $\mathfrak{U}$ is abelian, and $m(\mathfrak{U}) \leqq 2$. Thus, $\mathfrak{S}$ is of type II in this case, since the above information implies directly that $\mathfrak{H}$ is nilpotent.

Suppose now that $\mathfrak{T}$ is not of type V. Then from Lemma 26.26 we have $\mathfrak{T} = \mathfrak{H}^*\mathfrak{V}\mathfrak{Q}$, where $\mathfrak{Q}$ is a normal S_q-subgroup of $\mathfrak{T}$, $\mathfrak{H}^*\mathfrak{V}$ is a Frobenius group with Frobenius kernel $\mathfrak{V}$, and $\mathfrak{V}$ is a non identity q'-group. Since $\mathfrak{Q}^*$ is of prime order q, it follows from 3.16 that $\mathfrak{Q}$ contains a subgroup $\mathfrak{Q}_0$ such that $\mathfrak{Q}_0 \lhd \mathfrak{T}$, $\mathfrak{Q}/\mathfrak{Q}_0$ is elementary of order q^p ($p = |\mathfrak{H}^*|$), and $\mathfrak{V}$ centralizes $\mathfrak{Q}_0$.

We next show that $\mathfrak{V}'$ centralizes $\mathfrak{Q}$. This is an immediate application of 3.16. If $N(\mathfrak{V}) \subseteq \mathfrak{T}$, then $\mathfrak{T}$ is of type III or IV according as $\mathfrak{V}$ is abelian or non abelian. If neither $\mathfrak{S}$ nor $\mathfrak{T}$ is of type II, then both $\mathfrak{S}'$ and $\mathfrak{T}'$ are tamely imbedded subsets of $\mathfrak{G}$, by Lemma 26.27, since both $\mathfrak{S}$ and $\mathfrak{T}$ satisfy Hypothesis 26.1. Once again, (26.9) yields a contradiction.

If $\mathfrak{S}$ is of type II, then $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Suppose

$$X, Y \in \bigcup_{H \in \mathfrak{H}^\sharp} C_{\mathfrak{S}'}(H)$$

and $X = G^{-1}YG$. Choose $H_1 \in C_{\mathfrak{H}}(X)^\sharp$, $H_2 \in C_{\mathfrak{H}}(Y)^\sharp$. Then $C(X) \supseteq \langle H_1, G^{-1}H_2G\rangle$. If $C(X) \subseteq \mathfrak{S}$, then $G \in \mathfrak{S}$, since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. We can suppose $C(X) \not\subseteq \mathfrak{S}$, and without loss of generality, we assume that X has prime order r, $X \in \mathfrak{U}$. If a S_r-subgroup of $\mathfrak{U}$ is non cyclic, then by Lemmas 26.12 and 26.13, $C(X) \subseteq \mathfrak{S}$. We can suppose that the S_r-subgroup $\mathfrak{U}_r$ of $\mathfrak{U}$ is cyclic, so that $\langle X\rangle = \Omega_1(\mathfrak{U}_r)$. Since $N(\mathfrak{U}) \not\subseteq \mathfrak{S}$, it follows that $N(\langle X\rangle) \not\subseteq \mathfrak{S}$. Choose $\mathfrak{L} \in \mathscr{M}$ with $N(\langle X\rangle) \subseteq \mathfrak{L}$. If $C(X) \cap \mathfrak{H}^* \neq 1$, it follows readily that $C(X) \subseteq \mathfrak{S}$, so we can suppose $C(X) \cap \mathfrak{H}^* = 1$. In this case, $C_{\mathfrak{H}}(X)\mathfrak{Q}^*$ is a Frobenius group, and this implies that $C_{\mathfrak{H}}(X) \subseteq \boldsymbol{H}(\mathfrak{L})$, which is not the case. The proof is complete.

LEMMA 26.29. *If $\mathfrak{L} \in \mathscr{M}$ and $\mathfrak{L}$ is of type* I, *then*

$$\bigcup_{H \in \boldsymbol{H}(\mathfrak{L})^\sharp} C_{\mathfrak{L}}(H) = \hat{\mathfrak{L}}$$

is a tamely imbedded subset of $\mathfrak{G}$.

Proof. We first show that $\boldsymbol{H}(\mathfrak{L})$ is tamely imbedded in $\mathfrak{G}$.

If $\boldsymbol{H}(\mathfrak{L})$ is a T.I. set in $\mathfrak{G}$ we are done. If $\boldsymbol{H}(\mathfrak{L})$ is abelian, the conjugacy property for elements of $\boldsymbol{H}(\mathfrak{L})$ holds. Suppose $\boldsymbol{H}(\mathfrak{L})$ is abelian, $L \in \boldsymbol{H}(\mathfrak{L})$, and $C(L) \not\subseteq \mathfrak{L}$. Let $\mathfrak{N} \in \mathscr{M}$ with $C(L) \subseteq \mathfrak{N}$.

Suppose $\mathfrak{N}$ is of type I. Then $\mathfrak{N} \cap \mathfrak{L}$ is disjoint from $\boldsymbol{H}(\mathfrak{N})$, since $\boldsymbol{H}(\mathfrak{L}) \subseteq \mathfrak{N} \cap \mathfrak{L}$. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{N})$ in $\mathfrak{N}$ which contains $\mathfrak{N} \cap \mathfrak{L}$. Lemmas 26.12 and 26.13 imply that $\mathfrak{E} = \mathfrak{N} \cap \mathfrak{L}$.

If $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$ is a set of representatives for the conjugate classes of maximal subgroups of $\mathfrak{G}$ constructed in this fashion, then $(|\boldsymbol{H}(\mathfrak{L}_i)|, |\boldsymbol{H}(\mathfrak{L}_j)|) = 1$ for $i \neq j$. Also, $(|\boldsymbol{H}(\mathfrak{L}_i)|, |\boldsymbol{H}(\mathfrak{L})|) = 1$. Suppose $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|) \neq 1$ for some $L \in \boldsymbol{H}(\mathfrak{L})^\sharp$, and some i. We can suppose that

L has prime order r. Let s be a prime divisor of $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|)$, so that $s \in \pi(\mathfrak{L}) - \pi(\boldsymbol{H}(\mathfrak{L}))$, Since $\mathfrak{L}$ is of type I, this implies that a S_r-subgroup $\mathfrak{S}$ of $\mathfrak{L}$ is non cyclic so that $s \in \pi^*$. Since $\mathfrak{S}$ does not centralize a S_r-subgroup of $\mathfrak{L}$, $s < r$. But now Lemma 8.16 implies that the S_r-subgroup of $\mathfrak{L}$ centralizes a S_s-subgroup of $\boldsymbol{H}(\mathfrak{L}_i)$, which is not the case. Hence, $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|) = 1$ for every $L \in \boldsymbol{H}(\mathfrak{L})^\sharp$.

By construction

$$\hat{\mathfrak{L}}_i = \bigcup_{H \in \boldsymbol{H}(\mathfrak{L}_i)^\sharp} C_{\mathfrak{L}_i}(H) - \boldsymbol{H}(\mathfrak{L}_i)^\sharp$$

contains a non identity element. From Lemma 26.13 we have $N(\hat{\mathfrak{L}}_i) = \mathfrak{L}_i$ and $\hat{\mathfrak{L}}_i$ is a T.I. set in $\mathfrak{G}$. Thus, if $\boldsymbol{H}(\mathfrak{L})$ is abelian and every $\mathfrak{N}$ with the property that $\mathfrak{N} \in \mathscr{M}$ and $C(L) \subseteqq \mathfrak{N}$ for some $L \in \boldsymbol{H}(\mathfrak{L})^\sharp$ is of type I, then $\boldsymbol{H}(\mathfrak{L})$ is tamely imbedded in $\mathfrak{G}$.

Suppose $\mathfrak{N}$ is not of type I. Since $\boldsymbol{H}(\mathfrak{L}) \subseteqq \mathfrak{N}$, it is obvious that $\mathfrak{N}$ is not of type V. It is equally obvious that $\mathfrak{N}$ is not of type III or IV. Hence, $\mathfrak{N}$ is of type II. Since $\boldsymbol{H}(\mathfrak{L})$ is a S-subgroup of $\mathfrak{G}$, it is a S-subgroup of $\mathfrak{N}$, and it follows that $\mathfrak{N} \cap \mathfrak{L}$ is a complement to $\boldsymbol{H}(\mathfrak{N})$. Since $|\boldsymbol{H}(\mathfrak{N})|$ is relatively prime to $|\boldsymbol{H}(\mathfrak{L})|$ and to each $|\boldsymbol{H}(\mathfrak{L}_i)|$, we only need to show that $|\boldsymbol{H}(\mathfrak{N})|$ is relatively prime to $|C_{\mathfrak{L}}(L)|$, $L \in \boldsymbol{H}(\mathfrak{L})$. Let $q = |\mathfrak{N} : \mathfrak{N}'|$, so that q is a prime and $\mathfrak{N} \cap \mathfrak{L}$ contains a S_q-subgroup $\mathfrak{Q}^*$ of $\mathfrak{N}$. Since $\pi(\boldsymbol{H}(\mathfrak{N})) \subseteqq \pi_0 - \pi^*$, it follows that if $\mathfrak{R}$ is a S_ϖ-subgroup of $\mathfrak{L}$, $\varpi = \pi(\boldsymbol{H}(\mathfrak{N})) \cap \pi(\mathfrak{L})$, either $\mathfrak{R} = 1$, or $\mathfrak{R}\boldsymbol{H}(\mathfrak{L})$ is a Frobenius group. Thus $(|\boldsymbol{H}(\mathfrak{N})|, |C_{\mathfrak{L}}(\mathfrak{L})|) = 1$ for $L \in \boldsymbol{H}(\mathfrak{L})^\sharp$, and $\boldsymbol{H}(\mathfrak{L})$ is a tamely imbedded subset of $\mathfrak{G}$. Since $C(L) \subseteqq \mathfrak{L}$ for every element of

$$\{\bigcup_{H \in \boldsymbol{H}(\mathfrak{L})^\sharp} C_{\mathfrak{L}}(H)\} - \boldsymbol{H}(\mathfrak{L}) ,$$

by Lemmas 26.12 and 26.13, the lemma is proved if $\boldsymbol{H}(\mathfrak{L})$ is abelian.

We can now suppose that $\boldsymbol{H}(\mathfrak{L})$ is non abelian, and is not a T.I. set in $\mathfrak{G}$. Let $\mathfrak{R}$ be a non abelian $\mathfrak{S}$-subgroup of $\boldsymbol{H}(\mathfrak{L})$, and let $\boldsymbol{H}(\mathfrak{L}) = \mathfrak{R} \times \mathfrak{R}_0$. Since $\boldsymbol{H}(\mathfrak{L})$ is not a T.I. set in $\mathfrak{G}$ Lemmas 26.14 and 26.13 imply that $\mathfrak{R}_0$ is a cyclic T.I. set in $\mathfrak{G}$. It follows directly from Lemma 26.12 that $\boldsymbol{H}(\mathfrak{L})$ is a tamely imbedded subset of $\mathfrak{G}$.

It remains to show that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{G}$. This is an immediate consequence of Lemmas 26.12 and 26.13.

LEMMA 26.30. *If $\mathfrak{H}$ is a nilpotent S-subgroup of $\mathfrak{G}$, then two elements of $\mathfrak{H}$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $N(\mathfrak{H})$.*

Proof. Let $\mathfrak{L} \in \mathscr{M}$, $N(\mathfrak{H}) \subseteqq \mathfrak{L}$. If $\mathfrak{H} \subseteqq \boldsymbol{H}(\mathfrak{L})$ and $\mathfrak{L}$ is of type I, we are done. If $\mathfrak{H} \subseteq \boldsymbol{H}(\mathfrak{L})$ and $\mathfrak{L}$ is not of type I, we are done. If $\mathfrak{H} \not\subseteq \boldsymbol{H}(\mathfrak{L})$, then $\mathfrak{H} \cap \boldsymbol{H}(\mathfrak{L}) = 1$. If $\mathfrak{L}$ is of type I, $\mathfrak{H}$ is abelian, and we are done. If $\mathfrak{L}$ is not of type I, then $\mathfrak{L}$ is of type III or IV, and we are done.

We now summarize to show that the proofs of Theorems 14.1 and 14.2 are complete. By Lemma 26.30, the conjugacy property for nilpotent S-subgroups holds. If every element of $\mathscr{M}$ is of type I, we are done by Lemma 26.29. We can therefore suppose that $\mathscr{M}$ contains an element not of type I. Choose $\mathfrak{L} \in \mathscr{M}$, $\mathfrak{L}$ not of type I. By Lemma 26.21, if $p \in \pi(\mathfrak{L}/\mathfrak{L}')$, a S_p-subgroup of $\mathfrak{L}$ is cyclic. This implies that $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$. First, suppose $|\mathfrak{L} : \mathfrak{L}'|$ is not a prime. Then by Lemma 26.23, $\mathfrak{L}$ is of type V or satisfies the conditions listed in Lemma 26.23. Suppose that $\mathfrak{L}$ is not of type V, and $\mathfrak{E}$ is a complement to $\boldsymbol{H}(\mathfrak{L})$ in $\mathfrak{L}$. Let p be the smallest prime such that a S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{E}$ is not contained in $\boldsymbol{Z}(\mathfrak{E})$ and choose $\mathfrak{L}_1 \in \mathscr{M}$, $N(\Omega_1(\mathfrak{E}_p)) \subseteq \mathfrak{L}_1$. By Lemmas 26.12 and 26.13, $\mathfrak{L}_1$ is not of type I. Lemma 26.21 implies that $\mathfrak{L}_1'$ is a S-subgroup of $\mathfrak{L}_1$ and $\mathfrak{L}_1/\mathfrak{L}_1'$ is cyclic. By construction, $\mathfrak{L}_1'$ is not nilpotent, and also by construction $\mathfrak{L}_1$ is not conjugate to $\mathfrak{L}$. We will now show that $|\mathfrak{L}_1 : \mathfrak{L}_1'|$ is a prime. Otherwise, since $\mathfrak{L}_1$ is not of type I or V, $\mathfrak{L}_1$ satisfies the conditions of Lemma 26.23. In this case, both $\boldsymbol{H}(\mathfrak{L})$ and $\boldsymbol{H}(\mathfrak{L}_1)$ are nilpotent T.I. sets in $\mathfrak{G}$ and $\mathfrak{L} \cap \mathfrak{L}_1$ satisfies the hypotheses of Lemma 26.22. Let $\ell = |\mathfrak{L}|$, $\ell_1 = |\mathfrak{L}_1|$, $|\mathfrak{L} : \boldsymbol{H}(\mathfrak{L})| = e$, $|\mathfrak{L}_1 : \boldsymbol{H}(\mathfrak{L}_1)| = e_1$, $g = |\mathfrak{G}|$, so that

$$(26.10) \qquad g \geqq \frac{(e-1)(e_1-1)}{ee_1} g + \frac{\ell - 1}{\ell e} g + \frac{\ell_1 - 1}{\ell_1 e_1} g \,,$$

which is not the case. Hence $|\mathfrak{L}_1 : \mathfrak{L}_1'|$ is a prime, so that $\mathfrak{L}_1$ satisfies Hypothesis 26.1. But then Lemma 26.25 implies that $\mathfrak{L}$ is of type V. Thus, whenever $\mathfrak{L} \in \mathscr{M}$ satisfies the hypotheses of Lemma 26.23, $\mathfrak{L}$ is of type I or V.

Suppose every element of $\mathscr{M}$ is of type I or V, and there is an element $\mathfrak{L}$ of type V. Let $p \in \pi(\mathfrak{L}/\mathfrak{L}')$, and let $\mathfrak{E}_p$ be a S_p-subgroup of $\mathfrak{L}$. Choose $\mathfrak{L}_1$ so that $N(\mathfrak{E}_p) \subseteq \mathfrak{L}_1 \in \mathscr{M}$. Then $\mathfrak{L}_1$ is not of type I. Suppose $\mathfrak{L}_1$ is of type V. By Lemma 26.20, $\mathfrak{L}'$ and $\mathfrak{L}_1'$ are tamely imbedded subsets of $\mathfrak{G}$. Since $(|\mathfrak{L}'|, |\mathfrak{L}_1'|) = 1$, it follows that $\mathfrak{A}_L$ and $\mathfrak{A}_{L_1}$ do not contain elements in the same conjugate class of $\mathfrak{G}$, $L \in \mathfrak{L}'$, $L_1 \in \mathfrak{L}_1'$. Setting $g = |\mathfrak{G}|$, $|\mathfrak{L}'| = \ell$, $|\mathfrak{L}_1'| = \ell_1$, $|\mathfrak{L} : \mathfrak{L}'| = e$, $|\mathfrak{L}_1 : \mathfrak{L}_1'| = e_1$, then (26.10) holds, by Lemma 9.5, which is not the case.

We can now suppose that $\mathscr{M}$ contains an element $\mathfrak{L}$ not of type I or V. Lemmas 26.21, 26.23 and the previous reduction imply that $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$, $\mathfrak{L}'$ is not nilpotent, and $|\mathfrak{L} : \mathfrak{L}'|$ is a prime. Lemmas 26.25 and 26.28 complete the proof of Theorem 14.1.

As for Theorem 14.2, Lemmas 26.28 and 26.29, together with Theorem 14.1, imply all parts of the theorem, since if $\mathfrak{L}$ is of type II, III, IV, or V, $\hat{\mathfrak{L}}$ is any tamely imbedded subset of $\mathfrak{G}$ which satisfies $N(\hat{\mathfrak{L}}) = \mathfrak{L}$, and $\mathfrak{W} = \mathfrak{W}_1\mathfrak{W}_2$ is a cyclic subgroup of $\mathfrak{L}$ which satisfies the hypothesis of Lemma 26.22, then adjoining all $L^{-1}(\mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2)L$, $L \in \mathfrak{L}$, to $\hat{\mathfrak{L}}$ does not alter the set of supporting subgroups for $\hat{\mathfrak{L}}$, as $C(W) \subseteq \mathfrak{L}$ for all $W \in \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$. The proofs are complete.

CHAPTER V

27. Statement of the Result Proved in Chapter V

The following result is proved in this chapter.

THEOREM 27.1. *Let $\mathfrak{G}$ be a minimal simple group of odd order. Then $\mathfrak{G}$ satisfies the following conditions:*

(i) *p and q are odd primes with $p > q$. $\mathfrak{G}$ contains elementary abelian subgroups $\mathfrak{P}$ and $\mathfrak{Q}$ with $|\mathfrak{P}| = p^q$, $|\mathfrak{Q}| = q^p$. $\mathfrak{P}$ and $\mathfrak{Q}$ are* T.I. *sets in $\mathfrak{G}$.*

(ii) *$N(\mathfrak{P}) = \mathfrak{P}\mathfrak{U}\mathfrak{Q}^*$, where $\mathfrak{P}\mathfrak{U}$ and $\mathfrak{U}\mathfrak{Q}^*$ are Frobenius groups with Frobenius kernels $\mathfrak{P}$, $\mathfrak{U}$ respectively. $|\mathfrak{Q}^*| = q$, $|\mathfrak{U}| = (p^q - 1)/(p - 1)$, $\mathfrak{Q}^* \subseteq \mathfrak{Q}$ and $((p^q - 1)/(p - 1), p - 1) = 1$.*

(iii) *If $\mathfrak{P}^* = C_{\mathfrak{P}}(\mathfrak{Q}^*)$, then $|\mathfrak{P}^*| = p$ and $\mathfrak{P}^*\mathfrak{Q}^*$ is a self-normalizing cyclic subgroup of $\mathfrak{G}$. Furthermore, $C(\mathfrak{P}^*) = \mathfrak{P}\mathfrak{Q}^*$, $C(\mathfrak{Q}^*) = \mathfrak{Q}\mathfrak{P}^*$, and $\mathfrak{P}^* \subseteq N(\mathfrak{Q})$.*

(iv) *$C(\mathfrak{U})$ is a cyclic group which is a* T.I. *set in $\mathfrak{G}$. Furthermore, $\mathfrak{Q}^* \subseteq N(\mathfrak{U}) = N(C(\mathfrak{U}))$, $N(\mathfrak{U})/C(\mathfrak{U})$ is a cyclic group of order pq and $N(\mathfrak{U})$ is a Frobenius group with Frobenius kernel $C(\mathfrak{U})$.*

In this chapter we take the results stated in Section 14 as our starting point. The notation introduced in that section is also used. There is no reference to any result in Chapter IV which is not contained in Section 14. The theory of group characters plays an essential role in the proof of Theorem 27.1. In particular we use the material contained in Chapter III.

Sections 28–31 consist of technical results concerning the characters of various subgroups of $\mathfrak{G}$. In Section 32 the troublesome groups of type V are eliminated. In Section 33 it is shown that groups of type I are Frobenius groups. By making use of the main theorem of [10] it is then easy to show that the first possibility in Theorem 14.1 cannot occur. The rest of the chapter consists of a detailed study of the groups $\mathfrak{S}$ and $\mathfrak{T}$ until in Section 36 we are able to supply a proof of Theorem 27.1.

28. Characters of Subgroups of Type I

Hypothesis 28.1.

(i) *$\mathfrak{X}$ is of Frobenius type with Frobenius kernel $\mathfrak{H}$ and complement $\mathfrak{E}$.*

(ii) *$\mathfrak{E} = \mathfrak{A}\mathfrak{B}$, where $\mathfrak{A}$ is abelian, $\mathfrak{B}$ is cyclic, and $(|\mathfrak{A}|, |\mathfrak{B}|) = 1$.*

(iii) $\mathfrak{E}_0$ *is a subgroup of* $\mathfrak{E}$ *with the same exponent as* $\mathfrak{E}$ *such that* $\mathfrak{E}_0\mathfrak{H}$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}$.

LEMMA 28.1. *Under Hypothesis* 28.1, $\mathfrak{X}$ *has an irreducible character of degree* $|\mathfrak{E}_0|$ *which does not have* $\mathfrak{H}$ *in its kernel.*

Proof. If $\mathfrak{A}$ is cyclic, then $\mathfrak{X}$ is a Frobenius group and the lemma is immediate. We may assume that $\mathfrak{A}$ is non cyclic.

Let $\mathfrak{H}_1/D(\mathfrak{H})$ be a chief factor of $\mathfrak{A}\mathfrak{H}$ with $\mathfrak{H}_1 \subseteq \mathfrak{H}$. Let $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{H}_1/D(\mathfrak{H}))$. Then $\mathfrak{A}/\mathfrak{A}_1$ is cyclic. Since $\mathfrak{X}$ is of Frobenius type, the exponent of $\mathfrak{A}/\mathfrak{A}_1$ is the exponent of $\mathfrak{A}$. Hence, $|\mathfrak{E}:\mathfrak{A}_1| = |\mathfrak{E}_0|$. Let $\mathfrak{A}_2$ be the normal closure of $\mathfrak{A}_1$ in $\mathfrak{E}$. Then $\mathfrak{A}_2$ is abelian. Let μ be a non principal linear character of $\mathfrak{H}_1/D(\mathfrak{H})$. Then $\mathfrak{J}(\mu) = \mathfrak{H}\mathfrak{A}_1$, so Lemma 4.5 completes the proof.

LEMMA 28.2. *Suppose* $\mathfrak{L}$ *is of type* I, *and* $\mathfrak{L} = \mathfrak{X}$ *satisfies Hypothesis* 28.1. *Suppose further that* $Z(\mathfrak{E})$ *contains an element* E *such that* $C_{\mathfrak{H}}(E) \not\subseteq \mathfrak{H}'$ *and* $C_{\mathfrak{H}}(E) \neq \mathfrak{H}$. *Then the set* $\mathscr{L}$ *of irreducible characters of* $\mathfrak{L}$ *which do not have* $\mathfrak{H}$ *in their kernel is coherent.*

Proof. By Lemmas 28.1 and 4.5, it follows that Hypothesis 11.1 and (11.4) are satisfied if we take $\mathfrak{H}_0 = 1$, $\mathfrak{K} = \mathfrak{L}$, $d = |\mathfrak{E}_0|$ and let $\mathscr{L}$ play the role of $\mathscr{S}$.

Since E is in the center of $\mathfrak{E}$, it follows that $\mathfrak{H}'C_{\mathfrak{H}}(E) \lhd \mathfrak{L}$. Thus, by assumption, $\mathfrak{H}/\mathfrak{H}'$ is not a chief factor of $\mathfrak{L}$. Therefore,

$$(28.1) \qquad \mathfrak{H}:\mathfrak{H}'| > 4|\mathfrak{E}_0|^2 + 1\,.$$

Let $\mathscr{S}(\mathfrak{H}') = \{\lambda_{is} \mid s = 1, \cdots, n_i;\ i = 1, \cdots, k\}$, where the notation is chosen so that $\lambda_{is}(1) = \lambda_{jt}(1)$ if and only if $i = j$, and where $\lambda_{11}(1) < \cdots < \lambda_{k1}(1)$. By (28.1) we get that (11.5) holds with $\mathfrak{H}_1 = \mathfrak{H}'$ and by Theorem 11.1 the lemma will follow as soon as it is shown that $\mathscr{S}(\mathfrak{H}')$ is coherent.

Set $\ell_i = \lambda_{i1}(1)/d$ for $1 \leqq i \leqq k$. Then each ℓ_i is an integer and $1 = \ell_1 < \cdots < \ell_k$. By Theorem 10.1, the coherence of $\mathscr{S}(\mathfrak{H}')$ will follow once inequality (10.2) is established. Suppose (10.2) does not hold. Then for some m with $1 < m \leqq k$,

$$(28.2) \qquad \sum_{i=1}^{m-1} \ell_i^2 n_i \leqq 2\ell_m\,.$$

Every character in $\mathscr{S}(\mathfrak{H}')$ is a constituent of a character induced by a linear character of $\mathfrak{H}$. Therefore,

$$(28.3) \qquad \ell_k \leqq |\mathfrak{E}:\mathfrak{E}_0|\,.$$

Let $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H}'$ and let $\bar{\mathfrak{H}}_1 = C_{\bar{\mathfrak{H}}}(E)$, $\bar{\mathfrak{H}}_2 = [\bar{\mathfrak{H}}, E]$. Thus, $\bar{\mathfrak{H}} = \bar{\mathfrak{H}}_1 \times \bar{\mathfrak{H}}_2$

and $\bar{\mathfrak{H}}_i \neq 1$, $i = 1, 2$. If $\mathfrak{H}_i$ is the inverse image of $\bar{\mathfrak{H}}_i$ in $\mathfrak{H}$, then $\mathfrak{E}\mathfrak{H}_i$ is of Frobenius type and satisfies Hypothesis 28.1. Two applications of Lemma 28.1 imply that $n_1 \geqq 4|\mathfrak{E} : \mathfrak{E}_0|$. Hence, (28.2) does not hold for any m, $1 < m \leqq k$. The proof is complete.

29. Characters of Subgroups of Type III and IV

The following notation will be used.

$\mathfrak{S} = \mathfrak{S}'\mathfrak{Q}^*$ is a subgroup of type II, III, or IV. $\mathfrak{Q}^*$ plays the role of $\mathfrak{W}_1$ in the definition of subgroups of type II, III, and IV given in Section 14. $\mathfrak{H}$, $\mathfrak{U}$, and $\mathfrak{W}_2$ have the same meaning as in these definitions. $\mathfrak{T} = \mathfrak{T}'\mathfrak{W}_2$ is a subgroup of type II, III, IV, or V whose existence follows from Theorem 14.1 (ii) (b), (e).

Let $\pi(\mathfrak{H}) = \{p_1, \cdots, p_t\}$ and for $1 \leqq i \leqq t$, let $\mathfrak{P}_i$ be the S_{p_i}-subgroup of $\mathfrak{H}$. Define

$$\mathfrak{C}_i = \mathfrak{U} \cap C(\mathfrak{P}_i)\,, \qquad 1 \leqq i \leqq t\,,$$

$$\mathfrak{C} = \bigcap_{i=1}^{t} \mathfrak{C}_i\,.$$

Let $|\mathfrak{H}| = h$, $|\mathfrak{U}| = u$, $|\mathfrak{Q}^*| = q$, $|\mathfrak{C}_i| = c_i$, $1 \leqq i \leqq t$, and $|\mathfrak{C}| = c$. By definition, q is a prime.

$\mathscr{S}_0$ is the set of characters of $\mathfrak{S}$ which are induced by nonprincipal irreducible characters of $\mathfrak{S}'/\mathfrak{H}$.

$\mathscr{S}$ is the set of characters of $\mathfrak{S}$ which are induced by irreducible characters of $\mathfrak{S}'$ that do not have $\mathfrak{H}$ in their kernel.

The purpose of this section is to prove the following result.

THEOREM 29.1.

(i) *If* $\mathfrak{S}$ *is of type* III *then* $\mathscr{S} \cup \mathscr{S}_0$ *is coherent except possibly if* $|\mathfrak{H}| = p^q$ *for some prime* p *and* $\mathfrak{C} = 1$.

(ii) *If* $\mathfrak{S}$ *is of type* IV, *then* $\mathscr{S} \cup \mathscr{S}_0$ *is coherent except possibly if* $|\mathfrak{H}| = p^q$ *for some prime* p, $\mathfrak{C} = \mathfrak{U}'$ *and* $\mathscr{S}_0$ *is not coherent.*

Hypothesis 29.1.

$\mathfrak{S}$ *is a subgroup of type* III *or* IV.

Throughout this section, Hypothesis 29.1 will be assumed. Thus, by Theorem 14.1 (ii) (d), $\mathfrak{T}$ is of type II. Consequently, $\mathfrak{W}_2$ has prime order p. Let $p = p_1$, $\mathfrak{P} = \mathfrak{P}_1$, and $\mathfrak{W}_2 = \mathfrak{P}^*$. Thus, by 3.16 (i), $\mathfrak{U} \subseteq C(\mathfrak{P}_i)$ for $2 \leqq i \leqq t$. Since $\mathfrak{U} \not\subseteq C(\mathfrak{H})$, this yields that $\mathfrak{U} \not\subseteq C(\mathfrak{P})$. As $\mathfrak{U}$ does not act trivially on $\mathfrak{P}/D(\mathfrak{P})$, Lemma 4.6 (i) implies that $C_{\mathfrak{U}}(\mathfrak{P}^*) = \mathfrak{C}_1 \subset \mathfrak{U}$.

For any subgroup $\mathfrak{H}_1$ of $\mathfrak{H}\mathfrak{C}$, let $\mathscr{S}(\mathfrak{H}_1)$ denote the set of characters in $\mathscr{S}_0 \cup \mathscr{S}$ which have the same degree and the same weight as some character in $\mathscr{S}_0 \cup \mathscr{S}$ that has $\mathfrak{H}_1$ in it kernel.

LEMMA 29.1. *Hypothesis* 11.1 *is satisfied if* $\mathscr{S}$ *in that hypothesis is replaced by* $\mathscr{S}_0 \cup \mathscr{S}$, $\mathfrak{H}$ *is replaced by* $\mathfrak{H}\mathfrak{C}$, $\mathfrak{H}_0$ *is taken as* $\langle 1\rangle$, $\mathfrak{L}$ *is replaced by* $\mathfrak{S}$, $\hat{\mathfrak{L}}$ *and* $\mathfrak{K}$ *are replaced by* $\mathfrak{S}'$, *and* $d = 1$.

Proof. By Theorem 14.2, Condition (i) is satisfied. Condition (ii) follows from the fact that $\mathfrak{S}$ is a three step group. Condition (iii) is immediate and Condition (vi) is simply definition (consistent with the present definition). Since $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group, $\mathscr{S}_0$ contains an irreducible character of degree q. Hence, Condition (iv) is satisfied. The group $\mathfrak{S}$ satisfies Hypothesis 13.2. Hence, by Theorem 14.2, Hypothesis 13.3 is satisfied with $\mathfrak{L} = \mathfrak{S}$, $\mathfrak{X} = \mathfrak{G}$, and $\hat{\mathfrak{L}} = \mathfrak{K} = \mathfrak{S}'$, and with $\mathscr{S}$ replaced by $\mathscr{S}_0 \cup \mathscr{S}$. By Lemmas 13.7, 13.9, and 13.10, Condition (v) of Hypothesis 11.1 is satisfied. The proof is complete.

LEMMA 29.2. *If* $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ *is coherent, then* $\mathscr{S}_0 \cup \mathscr{S}$ *is coherent.*

Proof. As $\mathfrak{U} \not\subseteq C(\mathfrak{P})$, $\mathfrak{U}$ does not act trivially on $\mathfrak{P}/D(\mathfrak{P})$. Since $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group, 3.16 (iii) yields that $|\mathfrak{P} : D(\mathfrak{P})| \geqq p^q$. As either $p \geqq 3$ and $q \geqq 5$ or $p \geqq 5$ and $q \geqq 3$, (5.9) yields that

$$\mathfrak{H}\mathfrak{C} : (\mathfrak{H}\mathfrak{C})' \,| \geqq |\, \mathfrak{P} : D(\mathfrak{P}) \,| \geqq p^q > 4q^2 + 1 = 4\,|\,\mathfrak{S} : \mathfrak{S}'\,|^2 + 1 \,.$$

Hence, (11.5) is satisfied with $\mathfrak{H}_1 = (\mathfrak{H}\mathfrak{C})$.' By Lemma 29.1, Theorem 11.1 may be applied. This implies the required result.

LEMMA 29.3. *If* $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ *is not coherent, then* $\mathfrak{S}'' = \mathfrak{H}\mathfrak{C}$.

Proof. Let $b = |\,\mathfrak{H}\mathfrak{C} : \mathfrak{S}''\,|$. We have $\mathfrak{P}^* \subseteq \mathfrak{S}''$, as $\mathfrak{P}^* \subseteq \mathfrak{S}'$ and $\mathfrak{Q}^*$ centralizes $\mathfrak{P}^*$. Hence, $\mathfrak{S}/\mathfrak{S}''$ is a Frobenius group. Let $d_1 < \cdots < d_k$ be all the degrees of characters in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ and let $\ell_m = d_m/q$ for $1 \leqq m \leqq k$. Then for each m, ℓ_m is an integer and $\ell_1 = 1$. Every character of $\mathfrak{S}/\mathfrak{S}''$ is a constituent of a character induced by a linear character of $\mathfrak{H}\mathfrak{C}$. Thus, $\ell_m \leqq u/c$ for $1 \leqq m \leqq k$. There are at least

$$\frac{\left(\dfrac{u}{c} b - 1\right)}{q}$$

irreducible characters of degree q in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$. Thus, if $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent, inequality (10.2) must be violated for some m. In particular, this implies that

$$\frac{\left(\dfrac{u}{c} b - 1\right)}{q} \leqq 2\ell_m \leqq 2\,\frac{u}{c}$$

Therefore, $b-(c/u)\leqq 2q$, so $b<2q+1$, since $c<u$. As $\mathfrak{H}\mathfrak{C}/\mathfrak{S}''$ is a normal subgroup of the Frobenius group $\mathfrak{S}/\mathfrak{S}''$, we have $b\equiv 1 \pmod q$. Since b and q are both odd, this implies that $b=1$ as required.

LEMMA 29.4. *If $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent, then $\mathfrak{H}=\mathfrak{P}$, $\mathfrak{P}'=D(\mathfrak{P})$, $|\mathfrak{P}:\mathfrak{P}'|=p^q$, $\mathfrak{P}^*\cap D(\mathfrak{P})=1$ and $\mathfrak{C}=\mathfrak{U}'$.*

Proof. By Lemma 29.3, $\mathfrak{S}''=\mathfrak{H}\mathfrak{C}$. If $2\leqq i\leqq t$, then $\mathfrak{U}\mathfrak{H}\subseteq\mathfrak{P}_iC(\mathfrak{P}_i)$, so that $p_i \,|\, |\mathfrak{S}':\mathfrak{S}''|$. Hence, $t=1$ and $\mathfrak{H}=\mathfrak{P}$. $\mathfrak{C}=\mathfrak{U}'$ follows directly from the fact that $\mathfrak{H}\mathfrak{C}=\mathfrak{S}''\subseteq\mathfrak{H}\mathfrak{U}'$. If $|\mathfrak{P}:D(\mathfrak{P})|>p^q$, then since $C_{\mathfrak{P}}(\mathfrak{Q}^*)=\mathfrak{P}^*$ is cyclic, Lemma 4.6 (i) implies that some non identity element of $\mathfrak{P}/D(\mathfrak{P})$ is in the center of $\mathfrak{P}\mathfrak{U}/D(\mathfrak{P})$. Thus, p divides $|\mathfrak{U}\mathfrak{H}:\mathfrak{S}''|$ which is not the case. Since $\mathfrak{U}$ does not act trivially on $\mathfrak{P}/D(\mathfrak{P})$, 3.16 (iii) now implies that $|\mathfrak{P}:D(\mathfrak{P})|=p^q$. Since $\mathfrak{P}^*$ has prime order and lies outside $D(\mathfrak{P})$, we get that $D(\mathfrak{P})\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group. Hence, by 3.16 (i), $D(\mathfrak{P})\mathfrak{U}$ is nilpotent. Consequently, $D(\mathfrak{P})/\mathfrak{P}'$ is in the center of $\mathfrak{P}\mathfrak{U}/\mathfrak{P}'$. As the fixed points of $\mathfrak{U}$ on $\mathfrak{P}/\mathfrak{P}'$ are a direct factor of $\mathfrak{P}/\mathfrak{P}'$, and since $\mathfrak{U}$ has no fixed points on $\mathfrak{P}/D(\mathfrak{P})$, we have $\mathfrak{P}'=D(\mathfrak{P})$. The lemma is proved.

LEMMA 29.5. *If $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent then $\mathfrak{P}$ is an elementary abelian p-group of order p^q.*

Proof. In view of Lemma 29.4 it suffices to show that $\mathfrak{P}'=1$. By 3.16 (i), $\mathfrak{U}\subseteq C(\mathfrak{P}')$. Thus, if $\mathfrak{P}'\neq 1$, there exists a subgroup $\mathfrak{P}_0$ of $\mathfrak{P}'$ such that $\mathfrak{P}_0\lhd\mathfrak{P}\mathfrak{U}$ and $|\mathfrak{P}':\mathfrak{P}_0|=p$. If $\mathfrak{U}$ acts irreducibly on $\mathfrak{P}/\mathfrak{P}'$, then $\mathfrak{P}'/\mathfrak{P}_0=Z(\mathfrak{P}/\mathfrak{P}_0)$. Hence, $\mathfrak{P}/\mathfrak{P}_0$ is an extra special p-group and $|\mathfrak{P}:\mathfrak{P}'|=p^{2b}$ for some integer b contrary to Lemma 29.4.

Suppose that $\mathfrak{U}$ acts reducibly on $\mathfrak{P}/\mathfrak{P}'$. Since the irreducible constituents of this representation are conjugate under the action of $\mathfrak{Q}^*$, all constituents have the same dimension. As $|\mathfrak{P}:\mathfrak{P}'|=p^q$ and q is a prime, this yields that they must all be one dimensional. Therefore, there exist elements $P_1,\cdots,P_q$ in $\mathfrak{P}$ such that

$$\mathfrak{P}/\mathfrak{P}'=\langle P_1\mathfrak{P}'/\mathfrak{P}'\rangle\times\cdots\times\langle P_q\mathfrak{P}'/\mathfrak{P}'\rangle$$

and

$$U^{-1}P_i\mathfrak{P}'U=P_i^{s_i(U)}\mathfrak{P}'\,,\quad U\in\mathfrak{U}\,,\quad 1\leqq i\leqq q\,,$$

where $s_1,\cdots,s_q$ are linear characters of $\mathfrak{U}$ (mod p) with $s_{i+1}(U)=s_1(Q^{-i}UQ^i)$ for $U\in\mathfrak{U}$ and a suitably chosen generator Q of $\mathfrak{Q}^*$. Since $|\mathfrak{Q}^*\mathfrak{U}|$ is odd, $s_is_j\neq 1$ for any i,j with $1\leqq i,\ j\leqq q$. Hence, if i,j are given, there exists $U\in\mathfrak{U}$ such that $s_i(U)s_j(U)\neq 1$. For $1\leqq k\leqq q$ let P_k' be an element of $\mathfrak{P}'$ such that

$$U^{-1}P_k\mathfrak{P}_0U = P_k^{s_k(U)}P_k'\mathfrak{P}_0\ .$$

Since $\mathfrak{P}'/\mathfrak{P}_0 \subseteq Z(\mathfrak{P}/\mathfrak{P}_0)$, we get that

$$\begin{aligned}[P_i, P_j] &\equiv U^{-1}[P_i, P_j]U \equiv [P_i^{s_i(U)}P_i', P_j^{s_j(U)}P_j'] \\ &\equiv [P_i^{s_i(U)}, P_j^{s_j(U)}] \equiv [P_i, P_j]^{s_i(U)s_j(U)} (\mathrm{mod}\ \mathfrak{P}_0)\ .\end{aligned}$$

Since $s_i(U)s_j(U) \neq 1$, this yields that $[P_i, P_j] \in \mathfrak{P}_0$ for $1 \leqq i,\ j \leqq q$. Since $\mathfrak{P} = \langle P_1, \cdots, P_q\rangle$, we get that $\mathfrak{P}' \subseteq \mathfrak{P}_0$ contrary to construction. Thus, $\mathfrak{P}' = 1$ as required.

LEMMA 29.6. *If $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent and $\mathfrak{C} \neq 1$, then $\mathscr{S}_0$ is not coherent.*

Proof. Suppose that $\mathfrak{C} \neq 1$. Assume that $\mathscr{S}_0$ is coherent. Let $\mathscr{S}_1 = \mathscr{S}_0$. Let $\mathscr{S}_2, \cdots, \mathscr{S}_k$ be the equivalence classes of $\mathscr{S}((\mathfrak{H}\mathfrak{C})') - \mathscr{S}_0$ chosen so that every character in $\mathscr{S}_m$ has degree $\mathscr{l}_m q$ for $2 \leqq m \leqq k$, and $\mathscr{l}_2 \leqq \cdots \leqq \mathscr{l}_k$. Suppose $\bigcup_{i=1}^k \mathscr{S}_i$ is not coherent. By Hypothesis 11.1, and Lemma 29.1, all parts of Hypothesis 10.1 are satisfied except possibly inequality (10.2). Since $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent, inequality (10.2) must be violated for some m.

Every character in $\bigcup_{i=1}^k \mathscr{S}_i$ is a constituent of a character induced by a linear character of $\mathfrak{H}\mathfrak{C}$. Thus $\mathscr{l}_m \leqq (u/c)$ for $1 < m \leqq k$. Hence, violation of inequality (10.2) yields that

$$\frac{u-1}{q} \leqq 2\mathscr{l}_m \leqq 2\frac{u}{c}\ .$$

Since $c \equiv 1\ (\mathrm{mod}\ 2q)$ and $c \neq 1$, this implies that

$$u - 1 \leqq 2q\frac{u}{c} = \frac{(2q+1)}{c}u - \frac{u}{c} \leqq u - \frac{u}{c} < u - 1\ .$$

Hence $\bigcup_{i=1}^k \mathscr{S}_i$ is coherent. Since $\mathscr{S}((\mathfrak{H}\mathfrak{C})') = \bigcup_{i=1}^k \mathscr{S}_i$, the proof is complete.

The proof of Theorem 29.1 is now immediate. Lemmas 29.2, 29.4 and 29.5 imply statement (i). Lemmas 29.2, 29.4, 29.5, and 29.6 imply statement (ii).

30. Characters of Subgroups of Type II, III and IV

The notation introduced at the beginning of Section 29 is used in this section. The main purpose of this section is to prove the following result.

THEOREM 30.1. *Let $\mathfrak{S}$ be a subgroup of type* II, III *or* IV. *Then $\mathscr{S}$ is coherent except possibly if $\mathfrak{S}$ is of type* II, *$\mathfrak{H}$ is a non abelian 3-group, $\mathfrak{H}\mathfrak{U}/\mathfrak{C}$ is a Frobenius group with Frobenius kernel $\mathfrak{H}\mathfrak{C}/\mathfrak{C}$, $u < 3^{q/2}$, $|\mathfrak{H}:\mathfrak{H}'| = 3^q$ and $\mathfrak{T}$ is a subgroup of type* V.

All lemmas in this section will be proved under the following assumption.

Hypothesis 30.1.
(i) *$\mathfrak{S}$ is a subgroup of type* II, III, *or* IV.
(ii) *$\mathscr{S}$ is not coherent except possibly if $\mathfrak{S}$ is of type* II.
(iii) *$\mathfrak{U}/\mathfrak{U}'$ has exponent a.*

For any subgroup $\mathfrak{H}_1$ of $\mathfrak{S}'$ let $\mathscr{S}(\mathfrak{H}_1)$ be the set of characters in $\mathscr{S}$ which have $\mathfrak{H}_1$ in their kernel. Notice that this notation differs from that used in Section 29.

LEMMA 30.1. *The degree of every character in $\mathscr{S}$ is divisible by aq.*

Proof. Every character in $\mathscr{S}$ is a constituent of a character of $\mathfrak{S}$ induced by a nonprincipal character of $\mathfrak{H}$. For any character θ of $\mathfrak{H}$ let $\tilde{\theta}$ be the character of $\mathfrak{H}\mathfrak{U}$ induced by θ. Set $\mathfrak{U}_1 = \mathfrak{J}(\theta) \cap \mathfrak{U}$. Let $|\mathfrak{U}:\mathfrak{U}_1| = b$. If $\mathfrak{S}$ is of type II or III, then by Lemma 4.5 it suffices to show that if $\theta \neq 1_{\mathfrak{H}}$, then $a \mid b$.

Let $\mathfrak{K}$ be the kernel of θ and let $H \in \mathfrak{H} - \mathfrak{K}$ such that $H\mathfrak{K} \in Z(\mathfrak{H}/\mathfrak{K})$. Then $\mathfrak{K} \lhd \mathfrak{H}\mathfrak{U}_1$ and $U^{-1}H\mathfrak{K}U = H\mathfrak{K}$ for $U \in \mathfrak{U}_1$. As $(u, h) = 1$, if $U \in \mathfrak{U}_1$, then U centralizes some element in $H\mathfrak{K}$. Hence, $\mathfrak{U}_1 \subseteq \hat{\mathfrak{S}}$. Let $\mathfrak{U}_0 = \{U^b \mid U \in \mathfrak{U}\}$. Then $\mathfrak{U}_0$ char $\mathfrak{U}$ and $\mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \hat{\mathfrak{S}}$.

Suppose $\mathfrak{U}_0 \neq 1$. If $\mathfrak{S}$ is of type II, then $\hat{\mathfrak{S}}$ is a T.I. set in $\mathfrak{G}$ by Theorem 14.2. Hence, $N(\mathfrak{U}) \subseteq N(\mathfrak{U}_0) \subseteq \mathfrak{S}$ contrary to definition. If $\mathfrak{S}$ is of type III, then by Theorem 29.1, $\mathfrak{U}\mathfrak{Q}^*$ is represented irreducibly on $\mathfrak{H}$. Since $\mathfrak{U}_0 \lhd \mathfrak{U}\mathfrak{Q}^*$, $\mathfrak{U}_0$ is in the kernel of this representation. Thus, $\mathfrak{U}_0 \subseteq C(\mathfrak{H})$ contrary to Theorem 29.1. Thus, $\mathfrak{U}_0 = 1$. Therefore $U^b = 1$ for $U \in \mathfrak{U}$ and so $a \mid b$ in case $\mathfrak{S}$ is of type II or III.

If $\mathfrak{S}$ is of type IV, we will show that Hypothesis 11.1 and (11.2) are satisfied with $\mathfrak{H}_0$ in that hypothesis being taken as our present $\mathfrak{H}$, $\mathfrak{L}$ being taken as $\mathfrak{S}/\mathfrak{H}$, $\mathfrak{H}$ and $\mathfrak{K}$ being taken as $\mathfrak{S}'/\mathfrak{H}$, and $\hat{\mathfrak{L}}_0$ being taken as $\mathfrak{S}'$. Certainly (i) is satisfied. Since $\mathfrak{S}/\mathfrak{H}$ is a Frobenius group with Frobenius kernel $\mathfrak{S}'/\mathfrak{H}$, (ii) and (11.2) are satisfied, and the remaining conditions follow immediately from the fact that $\mathfrak{S}/\mathfrak{H}$ is a

Frobenius group. The present $\mathscr{S}_0$ plays the role of $\mathscr{S}$ in Hypothesis 11.1 (iii).

Notice now that Hypothesis 11.2 is satisfied. By Lemma 11.2 and the fact that $\mathscr{S}_0$ is not coherent it follows that $\mathfrak{S}'/\mathfrak{H}$ is a non abelian r-group for some prime r whose derived group and Frattini subgroup coincide. But $\mathfrak{U} \cong \mathfrak{S}'/\mathfrak{H}$. Since $\mathfrak{C} = \mathfrak{U}'$, $\mathfrak{U}/\mathfrak{C}$ is of exponent r, so $a = r$. As $\mathfrak{U}$ has no fixed points on $\mathfrak{H}^{\sharp}$, it follows readily that every non linear character of $\mathfrak{S}'$ has degree divisible by r, as required.

LEMMA 30.2. *For* $1 \leqq i \leqq t$, $|\mathfrak{P}_i : D(\mathfrak{P}_i)| = p_i^q$ *and* $\mathfrak{U}/\mathfrak{C}_i$ *has exponent* a.

Proof. If $\mathfrak{S}$ is of type III or IV, the result follows from Theorem 29.1. Suppose $\mathfrak{S}$ is of type II. Then $\hat{\mathfrak{S}}$ is a T.I. set in $\mathfrak{G}$ by Theorem 14.2. Let a_i be the exponent of $\mathfrak{U}/\mathfrak{C}_i$ for $1 \leqq i \leqq t$. Let $\mathfrak{U}_i = \{U^{a_i} \mid U \in \mathfrak{U}\}$. Then $\mathfrak{U}_i \subseteq \mathfrak{C}_i \subseteq \hat{\mathfrak{S}}$ and $\mathfrak{U}_i$ char $\mathfrak{U}$. Thus, if $\mathfrak{U}_i \neq 1$, then $N(\mathfrak{U}) \subseteq N(\mathfrak{U}_i) \subseteq \mathfrak{S}$, contrary to definition of subgroups of type II.

Suppose $|\mathfrak{P}_i : D(\mathfrak{P}_i)| > p_i^q$ for some i with $1 \leqq i \leqq t$. Since $C_{\mathfrak{P}_i}(\mathfrak{Q}^*)$ is cyclic, this implies the existence of a subgroup $\mathfrak{H}_1$ with $\mathfrak{W}_2 \subseteq \mathfrak{H}_1 \subset \mathfrak{H}$ such that $\mathfrak{H}/\mathfrak{H}_1$ is a chief factor of $\mathfrak{S}$. By 3.16 (i), $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1$ is nilpotent. Thus, $\mathfrak{U} \subseteq \hat{\mathfrak{S}}$ and $N(\mathfrak{U}) \subseteq \mathfrak{S}$, contrary to definition.

LEMMA 30.3. *For* $1 \leqq i \leqq t$, *either* $a \mid (p_i - 1)$ *or* $a \mid (p_i^q - 1)$ *and* $(a, p_i - 1) = 1$. *In the first case,* $\mathfrak{P}_i/D(\mathfrak{P}_i)$ *is the direct product of* q *groups of order* p_i, *each of which is normalized by* $\mathfrak{U}$. *In the second case,* $\mathfrak{U}/\mathfrak{C}_i$ *is cyclic of order* a *and acts irreducibly on* $\mathfrak{P}_i/D(\mathfrak{P}_i)$.

Proof. By Lemma 30.2, $\mathfrak{U}\mathfrak{Q}^*$ is represented irreducibly on $\mathfrak{P}_i/D(\mathfrak{P}_i)$. As $\mathfrak{U} \lhd \mathfrak{U}\mathfrak{Q}^*$, the restriction of this representation to $\mathfrak{U}$ breaks up into a direct sum of irreducible representations all of which have the same degree d. By Lemma 30.2, $d \mid q$ and so $d = 1$ or $d = q$.

If $d = 1$, the order of every element in $\mathfrak{U}/\mathfrak{C}_i$ divides $(p_i - 1)$. Hence, by Lemma 30.2, $a \mid (p_i - 1)$.

If $d = q$, then $\mathfrak{U}$ acts irreducibly on $\mathfrak{P}_i/D(\mathfrak{P}_i)$. Thus, $\mathfrak{U}/\mathfrak{C}_i$ is cyclic. By Lemma 30.2, $|\mathfrak{U} : \mathfrak{C}_i| = a$. Therefore, $a \mid (p_i^q - 1)$. Let $\mathfrak{U}/\mathfrak{C}_i = \langle U \rangle$. Then the characteristic roots of U are algebraically conjugate over $GF(p)$. Hence, this is also the case for every power of U. If $(a, p_i - 1) \neq 1$, then some power $U_1 \neq 1$ of U has its characteristic roots in $GF(p)$ and thus is a scalar. This violates the fact that $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group.

LEMMA 30.4. *Suppose* $(a, p_i - 1) = 1$ *for some* i, $1 \leqq i \leqq t$. *Let*

$$\mathfrak{H}_1 = \mathfrak{P}_i' \prod_{j \neq i} \mathfrak{P}_j ,$$

and let $|\mathfrak{P}_i : \mathfrak{P}_i'| = p_i^{m_i'}$. *Then* $m_i' = m_i q$ *for some integer* m_i. *Furthermore,* $\mathscr{S}(\mathfrak{H}_1)$ *contains at least*

$$\frac{1}{q}\left\{\frac{(p_i^{q m_i} - 1)c_i}{a} - (p^{m_i} - 1)\right\}$$

irreducible characters of degree aq and at least $(p_i^{m_i} - 1)$ *characters of weight q and degree aq.*

Proof. By Lemma 30.3, $\mathfrak{U}/\mathfrak{C}_i$ is cyclic. By Theorem 29.1, $\mathfrak{S}$ is not of type IV, so $\mathfrak{U}$ is abelian. Hence, $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1\mathfrak{C}_i$ is a Frobenius group. By Lemma 30.2, $|\mathfrak{U} : \mathfrak{C}_i| = a$. Furthermore, since $\mathfrak{U}\mathfrak{Q}^*$ acts irreducibly on $\mathfrak{P}_i/D(\mathfrak{P}_i)$, $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H}_1$ is the direct product of q cyclic groups of the same order $p_i^{m_i}$. Thus, $qm_i = m_i'$, and $|C_{\bar{\mathfrak{H}}}(\mathfrak{Q}^*)| = p_i^{m_i}$. By 3.16 (iii) every non principal irreducible character of $\mathfrak{H}\mathfrak{C}_i/\mathfrak{H}_1\mathfrak{C}_i$ induces an irreducible character of $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1\mathfrak{C}_i$ of degree a. Since $\mathfrak{U}$ is abelian, this implies that every irreducible character of $\mathfrak{H}\mathfrak{C}_i/\mathfrak{H}_1$ which does not have $\bar{\mathfrak{H}}$ in its kernel induces an irreducible character of $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1$ of degree a. Hence, $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1$ has at least

$$\frac{(p_i^{m_i q} - 1)c_i}{a}$$

distinct irreducible characters of degree a.

Since $\mathfrak{S}/\mathfrak{H}_1$ satisfies Hypothesis 13.2, Lemma 13.7 implies that all but $p^{m_i} - 1$ non principal irreducible characters of $\mathfrak{H}\mathfrak{U}/\mathfrak{H}_1$ induce irreducible characters of $\mathfrak{S}$. The result now follows.

LEMMA 30.5. *Suppose that* $a \mid (p_i - 1)$ *for some* i *with* $1 \leqq i \leqq t$. *Let*

$$\mathfrak{H}_1 = \mathfrak{P}_i' \prod_{j \neq i} \mathfrak{P}_j$$

and let $|\mathfrak{P}_i : \mathfrak{P}_i'| = p_i^{m_i'}$. *Then* $m_i = m_i'/q$ *is an integer and* $\mathscr{S}(\mathfrak{H}_1)$ *contains at least*

$$\frac{(p_i^{m_i} - 1)}{a} \, \frac{u}{au'}$$

irreducible characters of degree aq, where $|\mathfrak{U}'| = u'$.

Proof. For any subgroup $\mathfrak{X}$ of $\mathfrak{S}$, let $\bar{\mathfrak{X}} = \mathfrak{X}\mathfrak{H}_1/\mathfrak{H}_1$. By Lemma 30.3, $\bar{\mathfrak{H}}$ contains a cyclic subgroup $\mathfrak{P}_{i1}$ which is normalized by $\mathfrak{U}$ such that

$$|\mathfrak{P}_{i1}| = p_i^{m_i}$$

and such that $\bar{\mathfrak{H}} = \mathfrak{P}_{i1} \times \mathfrak{H}_0$ for some subgroup $\mathfrak{H}_0$ which is normalized by $\mathfrak{U}$. Since $\mathfrak{U}\mathfrak{Q}^*$ acts irreducibly on $\mathfrak{P}_i/D(\mathfrak{P}_i)$, it follows that $m_i = m_i'/q$. Let $\mathfrak{U}_1$ be the kernel of the representation of $\mathfrak{U}$ on $\mathfrak{P}_{i1}$. Then $\mathfrak{U}/\mathfrak{U}_1$ is cyclic and so $|\mathfrak{U}:\mathfrak{U}_1| \leqq a$. There are at least

$$\frac{(p_i^{m_i}-1)}{u'}|\mathfrak{U}_1|$$

distinct linear characters of $\bar{\mathfrak{H}}\mathfrak{U}_1/\mathfrak{H}_0$ which do not have $\mathfrak{P}_{i1}$ in their kernel. Each of these induces an irreducible character of $\bar{\mathfrak{H}}\bar{\mathfrak{U}}$ of degree $|\mathfrak{U}:\mathfrak{U}_1|$. Thus, by Lemma 30.1, $|\mathfrak{U}:\mathfrak{U}_1| = a$ and there are at least

$$\frac{(p_i^{m_i}-1)\cdot u}{a\cdot a\cdot u'}$$

distinct irreducible characters of $\mathfrak{H}\mathfrak{U}$ of degree a which have $\mathfrak{H}_1$ in their kernel, and as characters of $\bar{\mathfrak{S}}'$ have $\mathfrak{H}_0$ in their kernel. If one of these induced a reducible character of $\mathfrak{S}$ or two of these induced the same character of $\mathfrak{S}$, then $\mathfrak{Q}^*$ would normalize $\mathfrak{H}_0$, contrary to the fact that $\mathfrak{U}\mathfrak{Q}^*$ acts irreducibly on $\mathfrak{P}_i/D(\mathfrak{P}_i)$.

LEMMA 30.6. *If $\mathscr{S}$ contains no irreducible character of degree aq, then $t = 1$, $\mathfrak{P}_1' = D(\mathfrak{P}_1)$, $a = u = (p_1^q - 1)/(p_1 - 1)$, and $c = c_1 = 1$. Furthermore, $\mathscr{S}(\mathfrak{H}')$ is coherent.*

Proof. By Lemmas 30.3 and 30.5, $(a, p_i - 1) = 1$ and a divides $(p_i^q - 1)/(p_i - 1)$ for $1 \leqq i \leqq t$. Suppose that for some i,

$$\frac{(p_i^{qm_i}-1)c_i}{a} - (p_i^{m_i}-1) \leqq 0 .$$

Then

$$\frac{(p_i^{qm_i}-1)}{(p_i^{m_i}-1)} c_i \leqq a .$$

Therefore, $c_i = 1$, $m_i = 1$, and $a = (p_i^q - 1)/(p_i - 1)$. Thus,

$$\frac{(p_i^{qm_i}-1)c_i}{a} - (p_i^{m_i}-1) = 0 . \tag{30.1}$$

Now Lemma 30.4 implies that (30.1) holds for $1 \leqq i \leqq t$. Thus, $t = 1$. Hence, $c = c_1 = 1$, $u = a = (p^q - 1)/(p - 1)$, $p = p_1$. Also, $m_1 = 1$, and so $\mathfrak{P}_1' = D(\mathfrak{P}_1)$.

If a character θ in $\mathscr{S} \cup \mathscr{S}_0$ is equivalent to a character in $\mathscr{S}(\mathfrak{H}')$, then its degree is prime to $|\mathfrak{H}|$, so $\mathfrak{H}' \subseteq \ker\theta$. Thus, the equivalence relation in Hypothesis 11.1 has the property that the present set $\mathscr{S}(\mathfrak{H}')$ is a union of equivalence classes. Therefore, $\mathscr{S}(\mathfrak{H}')$ consists of $(p-1)$ reducible characters of degree aq. Theorem 14.2 implies that Hypothesis 13.3 is satisfied. Hence, Lemma 13.9 implies that $\mathscr{S}(\mathfrak{H}')$ is coherent.

The remaining lemmas in this section will be proved under the following stronger assumption.

Hypothesis 30.2.
(i) *Hypothesis* 30.1 *is satisfied.*
(ii) $\mathscr{S}$ *is not coherent.*

LEMMA 30.7. *If $\mathscr{S}(\mathfrak{H}')$ is not coherent, then $\mathfrak{H} = \mathfrak{P}_1$, $\mathfrak{C}_1 = 1$, $a = (p-1)/2$, $p = p_1$, $u \neq a$, and $D(\mathfrak{P}_1) = \mathfrak{P}_1'$. The degree of every character in $\mathscr{S}(\mathfrak{H}')$ is either aq or uq, and $\mathscr{S}(\mathfrak{H}')$ contains exactly $2u/a$ irreducible characters of degree aq.*

Proof. Let $d_1 < \cdots < d_k$ be all the degrees of characters in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$. Define $\ell_i = d_i/aq$ for $1 \leqq i \leqq k$. By Lemmas 13.10, 30.1 and 30.6, all the assumptions of Theorem 10.1 are satisfied except possibly inequality (10.2). Every character in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is a constituent of a character of $\mathfrak{S}$ which is induced by a linear character of $\mathfrak{H}\mathfrak{C}$. Hence, $d_k \leqq qu/c$, and so $\ell_k \leqq u/ac$.

Choose the notation so that $a\,|\,(p_i - 1)$ for $1 \leqq i \leqq t_0$ and $(a, p_i - 1) = 1$ for $t_0 + 1 \leqq i \leqq t$. If $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is not coherent then inequality (10.2) is violated. Lemmas 30.2 and 30.3 imply that for $t_0 + 1 \leqq i \leqq t$, $c_i = u/a$. Thus by Lemmas 30.4 and 30.5, there exists m with $1 < m \leqq k$, such that

$$\sum_{i=1}^{t_0} \frac{u}{a} \cdot \frac{(p_i^{m_i} - 1)}{au'} + \sum_{i=t_0+1}^{t} \left\{\frac{u}{a} \cdot \frac{(p_i^{qm_i} - 1)}{qa} - \frac{(p_i^{m_i} - 1)}{q}\right\} + \sum_{i=t_0+1}^{t} \frac{(p_i^{m_i} - 1)}{q} \leqq 2\ell_m \leqq \frac{2u}{ca}\,.$$

Therefore,

$$\sum_{i=1}^{t_0} \frac{(p_i^{m_i} - 1)}{au'} + \sum_{i=t_0+1}^{t} \frac{(p_i^{qm_i} - 1)}{qa} \leqq 2\ell_m \frac{a}{u} \leqq \frac{2}{c} \leqq 2\,. \tag{30.2}$$

For

$$1 \leqq i \leqq t_0, \ \frac{(p_i^{m_i} - 1)}{a} \geqq 2p^{(m_i - 1)} .$$

By Theorem 29.1, $c \geqq u'$. Thus, (30.2) implies that

(30.3) $$t_0 \leqq 1. \text{ If } t_0 = 1, \text{ then } m_1 = 1, \ t = 1 .$$

Assume first that $t_0 = 0$. If $t = 1$, then since $q < p_1^q$ and $a \leqq (p_1^q - 1)/(p_1 - 1)$, (30.2) yields $m_1 = 1$. Thus, every character in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ has degree aq. Therefore the definition of subcoherence implies directly that $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is coherent contrary to assumption. Suppose now that $t \geqq 2$. Then (30.2) yields that $(p_1 - 1) + (p_2 - 1) \leqq 2q$. Therefore,

(30.4) $$p_i \not\equiv 1 \pmod q , \quad i = 1, 2 .$$

Further, (30.2) also implies that

(30.5) $$\frac{1}{a}\frac{(p_1^q - 1)}{(p_1 - 1)} + \frac{1}{a}\frac{(p_2^q - 1)}{(p_2 - 1)} \leqq q .$$

It follows from (30.4) that

(30.6) $$\frac{1}{a}\frac{(p_1^q - 1)}{(p_1 - 1)} \equiv \frac{1}{a} \equiv \frac{1}{a}\frac{(p_2^q - 1)}{(p_2 - 1)} \pmod{q} .$$

Each term on the left of (30.5) is an integer. Hence, if $p_1 > p_2$, (30.6) yields that

$$\frac{1}{a}\frac{(p_1^q - 1)}{(p_1 - 1)} \geqq q + \frac{1}{a}\frac{(p_2^q - 1)}{(p_2 - 1)} ,$$

contrary to (30.5). Consequently, $t_0 \neq 0$.

Now (30.2) and (30.3) imply that $t = 1$, so that $\mathfrak{H} = \mathfrak{P}_1$. We also conclude that $m_1 = 1$, so that $D(\mathfrak{P}_1) = \mathfrak{P}_1'$. Furthermore, $c = c_1 = u'$, and $(p_1 - 1)/a \leqq 2$. Since ap_1 is odd, we have $p_1 - 1 = 2a$. Finally we get that $\ell_m = u/ac$ and so $m = k$. If $k = m > 2$, or if $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ contains more than $2u/a$ irreducible characters of degree qa, then (30.2) is replaced by a strict inequality which is impossible as $(p_1 - 1)/a = 2$. Thus, $k = m = 2$, and so $d_2 = uq/c$ and the degree of a character in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is either aq or uq/c. If $\mathfrak{S}$ is of type II or III, then $(\mathfrak{H}\mathfrak{C})' = \mathfrak{H}'$ and the result is proved.

Suppose that $\mathfrak{S}$ is of type IV. Since the degree of any character in $\mathscr{S}((\mathfrak{H}\mathfrak{C})')$ is either aq or uq/c, $\mathfrak{U}/\mathfrak{C}$ is generated by two elements. Since $\mathfrak{C} = \mathfrak{U}'$, $\mathfrak{U}$ is generated by two elements. Thus, if we set $\mathfrak{H}_0 = \mathfrak{H}$, replace $\mathfrak{H}$ and $\mathfrak{K}$ by $\mathfrak{S}'/\mathfrak{H}$, and replace $\mathfrak{L}$ by $\mathfrak{S}$ in Hypothesis 11.2, then by Lemma 29.1, Hypothesis 11.2 holds and by Lemma 11.3 and Theorem 29.1, we conclude that $\mathscr{S} = \mathscr{S}(\mathfrak{H}')$ is coherent, contrary to assumption.

LEMMA 30.8. *$\mathscr{S}(\mathfrak{H}')$ is coherent.*

Proof. By Lemma 30.7, it may be assumed that $\mathfrak{H} = \mathfrak{P}$ is a p-group for some prime p, that $D(\mathfrak{P}) = \mathfrak{P}'$, and that $\mathfrak{C} = 1$. Suppose that $\mathscr{S}(\mathfrak{H}')$ is not coherent. Let $\mathscr{S}_1$ be the set of irreducible characters in $\mathscr{S}(\mathfrak{H}')$ of degree aq. Then by Lemma 30.7

$$(30.7) \qquad |\mathscr{S}_1| = \frac{2u}{a}, \qquad a = \frac{(p-1)}{2}.$$

Let $\mathscr{S}_2$ be the set of irreducible characters in $\mathscr{S}(\mathfrak{H}')$ of degree uq. The group $\mathfrak{S}/\mathfrak{H}'$ satisfies Hypothesis 13.2. Hence, by Lemmas 13.5, 13.7 and 30.7, there are $(p-1)$ reducible characters in $\mathscr{S}$ of weight q and degree uq which have $\mathfrak{H}'$ in their kernel. As the sum of the squares of degrees of irreducible characters of $\mathfrak{S}/\mathfrak{H}'$ is $p^q uq$, we get that

$$(30.8) \qquad uq + |\mathscr{S}_1| q^2a^2 + (p-1)qu^2 + |\mathscr{S}_2| q^2u^2 = p^q uq.$$

Since $\mathfrak{U}$ is abelian and is generated by two elements, we also have

$$(30.9) \qquad u \leqq a^2.$$

Now (30.7), (30.8) and (30.9) yield that

$$(30.10) \qquad |\mathscr{S}_2| \geqq \frac{p^q - (p-1)u - 2qa - 1}{uq} \geqq \frac{1}{a^2q}\left\{(p^q - 1) - (p-1)q - \frac{(p-1)^3}{4}\right\}.$$

Hence, by (5.8), $\mathscr{S}_2$ is non empty.

Let $\mathscr{S}_i = \{\lambda_{is} \mid 1 \leqq s \leqq n_i\}$ for $i = 1, 2$. The character λ_{11} is induced by a linear character of some subgroup $\mathfrak{S}_0$ of index a in $\mathfrak{S}'$. Define

$$(30.11) \qquad \alpha = (\tilde{1}_{\mathfrak{S}_0} - \lambda_{11}),$$

where $\tilde{1}_{\mathfrak{S}_0}$ is the character of $\mathfrak{S}$ induced by $1_{\mathfrak{S}_0}$. Since $\mathfrak{S}_0 \lhd \mathfrak{S}'$, it follows that $1_{\mathfrak{S}_0}$ induces $\rho_{\mathfrak{S}'/\mathfrak{S}_0}$ on $\mathfrak{S}'$. Since $\mathfrak{Q}^*$ does not normalize $\mathfrak{S}_0$, (30.11) is seen to imply that

$$\|\alpha\|^2 = a + 1 + (q-1)\frac{a^2}{u}.$$

Since $\hat{\mathfrak{S}}$ is tamely imbedded in $\mathfrak{G}$ and α vanishes on $\mathfrak{S} - \hat{\mathfrak{S}}$, we get that

(30.12) $$\|\alpha^\tau\|^2 = \|\alpha\|^2 = a + 1 + (q-1)\frac{a^2}{u}.$$

Furthermore,

$$(\alpha^\tau, \lambda_{2i}^\tau - \lambda_{2j}^\tau) = (\alpha, \lambda_{2i} - \lambda_{2j}) = 0$$

for all values of i and j.

Suppose that $(\alpha^\tau, \lambda_{2i}) \neq 0$ for some i. Then $(\alpha^\tau, \lambda_{2i}^\tau) \neq 0$ for all i. Hence (30.10) and (30.12) imply that

$$\frac{p^q-1}{qa^2} - \frac{(p-1)}{a^2} - \frac{(p-1)}{q} \leqq a + 1 + (q-1)\frac{a^2}{u}$$
$$= \frac{p-1}{2} + 1 + (q-1)\frac{a^2}{u}.$$

Thus

(30.13) $$2\{1 + \cdots + p^{q-1}\} = \frac{p^q-1}{a}$$
$$\leqq q\frac{(p-1)}{2}\left\{\frac{2}{a} + \frac{p-1}{q} + \frac{p-1}{2} + 1 + (q-1)\frac{a^2}{u}\right\}$$
$$\leqq q\frac{(p-1)}{2}\left(p + q\frac{a^2}{u}\right)$$
$$< q\frac{(p-1)}{2}\left(p + q\frac{p}{2}\right).$$

Therefore

$$4p^{q-2} < 4\frac{p^{q-1}-1}{p-1} < pq\left(1 + \frac{q}{2}\right).$$

Hence

$$3^{q-2} < 4p^{q-3} < q\left(1 + \frac{q}{2}\right) < q^2.$$

Thus $q = 3$ by (5.1). Now (30.13) becomes

$$2(1 + p + p^2) \leqq \frac{3}{2}(p-1)\left\{\frac{4}{p-1} + \frac{5}{6}(p-1) + 1 + \frac{2a^2}{u}\right\}.$$

Thus

$$\frac{4}{3}(1 + p + p^2) \leqq 4 + p - 1 + \frac{5}{6}(p-1)^2 + \frac{2a^2}{u}(p-1).$$

This implies that

$$\frac{4}{3}p^2 \leqq p + \frac{5}{6}p^2 + \frac{2a^2}{u}p.$$

Therefore $(1/2)p^2 \leqq p(1 + (2a^2/u))$, or equivalently $(1/2)p \leqq 1 + (2a^2/u)$. Thus (30.7) yields that

$$u \leqq \frac{2a^2}{\frac{1}{2}p - 1} = \frac{4a^2}{p-2} \leqq \frac{(p-1)^2}{(p-2)} < p + 1 < 3a .$$

This is impossible since $a \mid u$, $a \neq u$ and both a and u are odd. Thus,

$$(30.14) \qquad (\alpha^\tau, \lambda_{2i}^\tau) = 0 \quad \text{for} \quad \lambda_{2i} \in \mathscr{S}_2 .$$

Define $\beta = (u/a)\lambda_{11} - \lambda_{21} \in \mathscr{I}_0(\mathscr{S})$. Suppose that $(\beta^\tau, \lambda_{11}^\tau) = (u/a) - b$. As τ is an isometry on $\mathscr{I}_0(\mathscr{S})$, this yields that

$$(\beta^\tau, \lambda_{1i}^\tau) = \frac{u}{a}\delta_{i1} - b \quad \text{for all } i .$$

Therefore,

$$(30.15) \qquad \beta^\tau = \left(\frac{u}{a} - b\right)\lambda_{11}^\tau - b\sum_{i\neq 1}\lambda_{1i}^\tau + \Gamma + \Delta ,$$

where Γ is a linear combination of elements in $\mathscr{S}_2^{\,\tau}$ and Δ is orthogonal to $\mathscr{S}_1^{\,\tau} \cup \mathscr{S}_2^{\,\tau}$. Since $(\beta^\tau, \lambda_{21}^\tau - \bar{\lambda}_{21}^\tau) \neq 0$, it follows that $\|\Gamma\|^2 \geqq 1$. Since

$$(30.16) \qquad \|\beta^\tau\|^2 = \|\beta\|^2 = \left(\frac{u}{a}\right)^2 + 1 ,$$

(30.7) and (30.16) yield

$$\|\Delta\|^2 + \left(\frac{u}{a} - b\right)^2 + \left(2\frac{u}{a} - 1\right)b^2 \leqq \left(\frac{u}{a}\right)^2 .$$

This implies that

$$\|\Delta\|^2 + 2\frac{u}{a}b^2 - 2\frac{u}{a}b \leqq 0 ,$$

or $b^2 \leqq b$. Since b is an integer, $b = 0$ or 1 and $\Delta = 0$.

Suppose $b = 1$. Then (30.15) becomes

$$(30.17) \qquad \beta^\tau = \left(\frac{u}{a} - 1\right)\lambda_{11}^\tau - \sum_{i\neq 1}\lambda_{1i}^\tau + \Gamma .$$

As α, β vanish on $\mathfrak{S} - \hat{\mathfrak{S}}$, we have

$$(30.18) \qquad (\alpha^\tau, \beta^\tau) = (\alpha, \beta) = -\frac{u}{a} .$$

Since $(\alpha^\tau, \lambda_{11}^\tau - \lambda_{1i}^\tau) = -1$, we get that

(30.19) $$\alpha^{\tau} = (x-1)\lambda_{11}^{\tau} + x \sum_{i \neq 1} \lambda_{1i}^{\tau} + \Delta_0 ,$$

for some integer x and some Δ_0 which is orthogonal to $\mathscr{S}_1$. Now (30.14), (30.17), (30.18) and (30.19) yield that

$$-\frac{u}{a} = \left(\frac{u}{a} - 1\right)(x-1) - x\left(2\frac{u}{a} - 1\right).$$

Reading this equality mod u/a, we get

$$0 \equiv -(x-1) + x \equiv 1 \left(\bmod \frac{u}{a}\right).$$

Thus $u = a$, contrary to Lemma 30.7. Hence, $b = 0$. Consequently $\beta^{\tau} = (u/a)\lambda_{11}^{\tau} + \Gamma$, and so $\Gamma = \pm\lambda_{2j}^{\tau}$ for some j. Since $(\beta^{\tau}, \lambda_{21}^{\tau} - \bar{\lambda}_{21}^{\tau}) \neq 0$, $\lambda_{2j} = \lambda_{21}$ or $\bar{\lambda}_{21}$. This implies directly that $\mathscr{S}_1 \cup \mathscr{S}_2$ is coherent. Lemma 13.10 and Theorem 10.1 now yield that $\mathscr{S}(\mathfrak{H}')$ is coherent. The proof is complete.

LEMMA 30.9. *$\mathfrak{S}$ is of type* II.

Proof. If $\mathfrak{S}$ is of type III or IV, then Theorem 29.1 yields that $\mathfrak{H}' = 1$. Thus, by Lemma 30.8, $\mathscr{S}$ is coherent. Hence, Hypothesis 30.2 implies that $\mathfrak{S}$ is of type II.

LEMMA 30.10. *If $\mathscr{S}$ contains an irreducible character of degree aq, then Hypothesis* 11.1 *is satisfied with $\mathfrak{H}_0 = 1$, $\mathfrak{L} = \mathfrak{S}$, $\hat{\mathfrak{L}} = \hat{\mathfrak{S}}$, $\mathfrak{K} = \mathfrak{S}'$ and $d = a$.*

Proof. By Theorem 14.2, Condition (i) is satisfied. Condition (ii) follows from the definition of three step group. Conditions (iii) and (vi) are immediate, while Condition (iv) holds by assumption. The group $\mathfrak{S}$ satisfies Hypothesis 13.2. Hence, by Theorem 14.2 Hypothesis 13.3 is satisfied with $\mathfrak{X} = \mathfrak{G}$, $\mathfrak{L} = \mathfrak{S}$, $\hat{\mathfrak{L}} = \hat{\mathfrak{S}}$ and $\mathfrak{K} = \mathfrak{S}'$. By Lemmas 13.7, 13.9 and 13.10, Hypothesis 10.1 is satisfied. Thus, Lemma 10.1 yields that Condition (v) of Hypothesis 11.1 is satisfied. The proof is complete.

LEMMA 30.11. *If $\mathscr{S}$ contains an irreducible character of degree aq, then*

$$|\mathfrak{H} : \mathfrak{H}'| \leqq 4a^2q^2 + 1 .$$

Proof. By Hypothesis 30.2, $\mathscr{S}$ is not coherent. Thus, Lemmas 30.8, 30.9, and 30.10, together with Theorem 11.1 yield the result.

LEMMA 30.12. *For* $1 \leqq i \leqq t$, $(a, p_i - 1) = 1$ *and* $\mathfrak{P}_i\mathfrak{U}/\mathfrak{C}_i$ *is a Frobenius group.*

Proof. Suppose that $a \mid (p_i - 1)$ for some i. Then Lemmas 30.2 and 30.11 yield that $p_i^q \leqq 4a^2q^2 + 1 \leqq (p_i - 1)^2q^2 + 1$. Thus, $p_i^{q-2} < q^2$. Therefore, (5.1) implies that $q = 3$. Hence, $p_i = 5$ or 7. Thus, a divides 4 or 6. As a is odd and $(a, q) = 1$, this implies that $a = 1$ which is not the case. Therefore, by Lemma 30.3, $\mathfrak{U}/\mathfrak{C}_i$ is cyclic of order a for $1 \leqq i \leqq t$. If $\mathfrak{P}_i\mathfrak{U}/\mathfrak{C}_i$ were not a Frobenius group, then for some $b < a$, $\{U^b \mid U \in \mathfrak{U}\} = \mathfrak{U}_0$ would lie in $\hat{\mathfrak{S}}$. Since $\mathfrak{U}_0 \neq 1$ and $\mathfrak{U}_0$ char $\mathfrak{U}$, this implies that $N(\mathfrak{U}) \subseteqq N(\mathfrak{U}_0) \subseteqq \mathfrak{S}$, contrary to Lemma 30.9.

LEMMA 30.13. $t = 1$, $p_1 = 3$, $a < 3^{q/2}$ *and* $\mathfrak{P}_1' = D(\mathfrak{P}_1)$.

Proof. By Lemma 30.8, $\mathfrak{H}' \neq 1$. Choose the notation so that $\mathfrak{P}_1' \neq 1$. Let $\mathfrak{P}_1 = \mathfrak{P}_{11} \supset \mathfrak{P}_{12} \cdots \supset \mathfrak{P}_{1n} = \mathfrak{P}_1' \supset \mathfrak{P}_{1,n+1}$, where $\mathfrak{P}_{1i}/\mathfrak{P}_{1,i+1}$ is a chief factor of $\mathfrak{S}$ for $1 \leqq i \leqq n$. Thus, $\mathfrak{P}_1/\mathfrak{P}_{1,n+1}$ is of class two and so is a regular p-group. By Lemma 4.6 (i) $\mathfrak{Q}^*$ centralizes an element of $\mathfrak{P}_{1i} - \mathfrak{P}_{1,i+1}$ for $1 \leqq i \leqq n$. Since $C_{\mathfrak{P}_1}(\mathfrak{Q}^*)$ is cyclic, this implies that $\mathfrak{P}_1/\mathfrak{P}_{1,n+1}$ has exponent p^n. Let $\mathfrak{U}/\mathfrak{C}_i = \langle U \rangle$. Then the regularity of $\mathfrak{P}_1/\mathfrak{P}_{1,n+1}$ yields that U has the same minimal polynomial on $\mathfrak{P}_1/D(\mathfrak{P}_1)$ as on $\mathfrak{P}_1'/\mathfrak{P}_{1,n+1}$. Hence, by Lemma 6.2, $a < 3^{q/2}$. Now Lemma 30.11 implies that if $|\mathfrak{P}_1 : \mathfrak{P}_1'| = p_1^{mq}$, then

$$(30.20) \qquad p_1^{mq} \prod_{i=2}^{t} p_i^q \leqq 4.3^q q^2 + 1\,.$$

Since $3 \leqq p_1$, (30.20) implies that

$$p_1^{(m-1)q} \prod_{i=2}^{t} p_i^q \leqq 4q^2 + 1\,.$$

Hence, by (5.9), $m = 1$ and $t = 1$. Thus, (30.20) becomes

$$(30.21) \qquad p_1^q \leqq 4.3^q q^2 + 1\,.$$

If $p_1 \geqq 11$, (30.21) implies that

$$3^q < \left(\frac{p_1}{3}\right)^q \leqq 4q^2 + 1\,.$$

Thus, $3^{q-2} < q^2$ and so $q < 5$ by (5.1). Hence $q = 3$ and (30.21) yields $1331 = 11^3 < 4.3^5 + 1 < 1000$, which is not the case. If $p_1 = 7$, then (30.21) and (5.6) imply that $q < 7$. Thus, $q = 5$ or $q = 3$. If $q = 3$, then

$$\frac{p_1^q - 1}{p_1 - 1} = 57$$

and $a < 3^{q/2} < 9$. Since $(q, a) = 1$ and $a \mid 57$, this cannot be the case. If $q = 5$, then

$$\frac{p_1^q - 1}{p_1 - 1} = 2801$$

is a prime. Thus $2801 = a < 3^{q/2} < 27$. Suppose now that $p_1 = 5$. Then by (5.7), $q < 13$. Thus, $q = 3, 7$, or 11. Let r be a prime factor of a. Then $r < 3^{q/2}$ and $5^q \equiv 1 \pmod r$. Thus, $r \equiv 1 \pmod{2q}$. If $q = 3$, then $r \equiv 1 \pmod 6$ and $r < 3^{3/2}$, which is impossible. If $q = 7$, then $r < 3^{7/2} < 50$ and $r \equiv 1 \pmod{14}$. Thus $r = 29$ or 43. Since $5^7 \equiv -1 \pmod{29}$ and $5^7 \equiv -6 \pmod{43}$, these cases cannot occur. If $q = 11$, then $r < 3^{11/2} < 437$ and $r \equiv 1 \pmod{22}$. Thus, $r = 23, 67, 89, 199, 331, 353, 397$, or 419. Since $5^{11} \equiv 1 \pmod r$, the quadratic reciprocity theorem implies that $(r \mid 5) = 1$, so that $r \equiv \pm 1 \pmod 5$. Thus, $r = 89, 199, 331$ or 419. Since $5^{11} \equiv 55 \pmod{89}$, $5^{11} \equiv 92 \pmod{199}$, $5^{11} \equiv -2 \pmod{331}$, $5^{11} \equiv -40 \pmod{419}$, these cases cannot occur. Hence, $p_1 = 3$, and the lemma is proved.

If $\mathscr{S}$ is not coherent, then Lemmas 30.8 and 30.12 imply that $|\mathfrak{W}_2|$ is not a prime. Hence, $\mathfrak{T}$ is of Type V. The other statements in Theorem 30.1 follow directly from Lemmas 30.9 and 30.13.

31. Characters of Subgroups of Type V

In this section $\mathfrak{T} = \mathfrak{T}'\mathfrak{W}_2$ is a subgroup of type V. Let $\mathfrak{S}$ be the subgroup of $\mathfrak{G}$ which satisfies condition (ii) of Theorem 14.1. By Theorem 14.1 (ii) (d) $\mathfrak{S}$ is of type II. The notation introduced at the beginning of Section 29 will be used.

$\mathscr{T}$ is the set of all characters of $\mathfrak{T}$ which are induced by non principal irreducible characters of $\mathfrak{T}'$. For any class function α of $\mathfrak{T}'$ let $\tilde{\alpha}$ be the class function of $\mathfrak{T}$ induced by α.

For $0 \leqq i \leqq q-1$, $0 \leqq j \leqq w_2 - 1$ let η_{ij} be the generalized characters of $\mathfrak{G}$ defined by Lemma 13.1 and let ν_{ij} be the characters of $\mathfrak{T}$ defined by Lemma 13.3.

Hypothesis 13.2 is satisfied with $\mathfrak{L} = \mathfrak{T}$, $\mathfrak{K} = \mathfrak{T}'$ and $\mathfrak{W}_1$ replaced by $\mathfrak{W}_2$. By Lemma 13.7 $\mathfrak{T}'$ has exactly q irreducible characters which induce reducible characters of $\mathfrak{T}$. Denote these by ν_i for $0 \leqq i \leqq q-1$, where $\nu_0 = 1_{\mathfrak{T}'}$. Let $\zeta_i = \tilde{\nu}_i$ for $0 \leqq i \leqq q-1$. Since q is a prime the characters ν_i are algebraically conjugate for $1 \leqq i \leqq q-1$. Therefore

$$\nu_i(1) = \nu_1(1) \quad \text{for } 1 \leqq i \leqq q-1 .$$

LEMMA 31.1. *$\mathscr{S}(\mathfrak{H}')$ contains an irreducible character of $\mathfrak{S}$ except possibly if w_2 is a prime and $\mathfrak{H}\mathfrak{U}$ is a Frobenius group.*

Proof. If $\mathfrak{S}'$ is not a Frobenius group then there are strictly more than w_2 classes of $\mathfrak{S}'/\mathfrak{H}'$ whose order is not relatively prime to $|\mathfrak{H}|$. The result now follows from Lemma 13.7.

Suppose that $\mathfrak{S}'$ is a Frobenius group. By Lemma 6.2 and 3.16 (iii) $\mathfrak{H}$ is abelian and $|\mathfrak{H}| = w_2^q$ if the result is false. Then Lemma 13.7 implies that $\mathfrak{S}'$ contains exactly $w_2 - 1$ conjugate classes which are in $\mathfrak{H}^\#$. Therefore

$$\frac{|\mathfrak{H}| - 1}{u} = w_2 - 1 .$$

Hence

$$u = \frac{|\mathfrak{H}| - 1}{w_2 - 1} = \frac{|\mathfrak{H}| - 1}{|\mathfrak{H}|^{1/q} - 1} > \sqrt{|\mathfrak{H}|} .$$

This implies that $\mathfrak{H}$ is an elementary abelian p-group for some prime p. Since $\mathfrak{W}_2$ is cyclic w_2 is a prime as required.

LEMMA 31.2. *Let*

$$a_{ij} = ((\nu_1(1)\tilde{1}_{\mathfrak{T}'} - \zeta_i)^\tau, \eta_{0j}) .$$

Then $a_{ij} \neq 0$ for $1 \leqq i \leqq q - 1$, $0 \leqq j \leqq w_2 - 1$.

Proof. Lemma 10.3 implies that by Lemma 9.4

$$(31.1) \qquad (\nu_1(1)\tilde{1}_{\mathfrak{T}'} - \zeta_i, \eta_{0j|\mathfrak{T}}) = ((\nu_1(1)\tilde{1}_{\mathfrak{T}'} - \zeta_i)^\tau, \eta_{0j}) = a_{ij} .$$

Since η_{01} is rational on $\mathfrak{T}'$ by Lemma 13.1, $a_{ij} = a_j$ is independent of i. Thus (31.1) implies that

$$(31.2) \qquad \eta_{0j|\mathfrak{T}'} = b\rho_{\mathfrak{T}'} - a_j \sum_{i=1}^{q-1} \nu_{i0|\mathfrak{T}'} + \alpha_{|\mathfrak{T}'} ,$$

for some integer b, where α is an integral linear combination of irreducible characters of $\mathfrak{T}$ each of which vanishes on $\hat{\mathfrak{W}}$.

Let $Q \in \mathfrak{Q}^{*\#}$. Let p be a prime dividing w_2, let P be an element of order p in $\mathfrak{W}_2$ and let $\mathfrak{p}$ be a prime divisor of p in the ring of integers of $\mathscr{Q}_{|\mathfrak{G}|}$. Let ω_{ij} have the same meaning as in Hypothesis 13.1. Thus by Lemmas 13.1 and 13.3

$$(31.3) \qquad \eta_{0j}(PQ) = \omega_{0j}(PQ) , \quad \alpha(PQ) = 0 , \quad \nu_{i0}(PQ) = \varepsilon\omega_{i0}(PQ) ,$$

where $\varepsilon = \pm 1$ is independent of i. Therefore

$$(31.4) \qquad \sum_{i=1}^{q-1} \nu_{i0}(PQ) = \varepsilon \sum_{i=1}^{q-1} \omega_{i0}(PQ) = \varepsilon \sum_{i=1}^{q-1} \omega_{i0}(Q) = -\varepsilon .$$

In view of Lemma 4.2 (31.3) and (31.4) imply that

$$\eta_{0j}(Q) \equiv \eta_{0j}(PQ) \equiv \omega_{0j}(PQ) \equiv \omega_{0j}(Q) \equiv 1 \pmod{\mathfrak{p}}$$

$$(31.5) \qquad \sum_{i=1}^{q-1} \nu_{i0}(Q) \equiv -\varepsilon \pmod{\mathfrak{p}}$$

$$\alpha(Q) \equiv \alpha(PQ) \equiv 0 \pmod{\mathfrak{p}} .$$

Thus (31.2) and (31.5) yield that $1 \equiv \varepsilon a_j \pmod{\mathfrak{p}}$. Thus $a_j \neq 0$ as required.

The main purpose of this section is to prove that $\mathscr{T}$ is coherent. Theorem 12.1 will play an important role in the proof of this fact. The lemmas in this section will from now on satisfy the following assumption.

Hypothesis 31.1.
$\mathscr{T}$ is not coherent.

By Grün's theorem $\mathfrak{T}/\mathfrak{T}''$ is a Frobenius group. Hence by Lemma 11.2 $\mathfrak{T}' = \mathfrak{Q}$ is a q-group. Define

$$(31.6) \qquad |\mathfrak{Q} : \mathfrak{Q}'| = q^b , \qquad |\mathfrak{T} : \mathfrak{Q}| = w_2 = e .$$

Let $1 = q^{f_0} < q^{f_1} < \cdots$ be all the integers which are degrees of irreducible characters of $\mathfrak{Q}$. Let

$$(31.7) \qquad \nu_1(1) = q^{f_n} , \qquad n > 0 .$$

By Lemma 13.10 Hypothesis 12.1 is satisfied. Let $\mathscr{T}_s$ be defined by (12.3) for $0 \leqq s \leqq t$.

LEMMA 31.3. *Suppose that $b = 2c$ for some integer c. Then e is not a prime power.*

Proof. Suppose that $e = p^h$ for some prime p. Then by Lemma 11.5 $q^c + 1 = 2p^h$, $f_1 = c$ and $\mathfrak{Q}$ contains a subgroup $\mathfrak{Q}_1$ which is normal in $\mathfrak{T}$ and satisfies $|\mathfrak{Q}' : \mathfrak{Q}_1| = q$ and $\mathfrak{Q}^* \subseteqq \mathfrak{Q} - \mathfrak{Q}_1^\sharp$. Therefore $n = 1$ and $\mathscr{T}$ contains $2(q^c - 1)$ irreducible characters $\lambda_1, \lambda_2, \cdots$ of degree e. Define

$$\alpha = \widetilde{1}_{\mathfrak{Q}} - \lambda_1 , \qquad \beta = q^c\lambda_1 - \zeta_1 .$$

By Lemma 9.4 we have that

$$(31.8) \qquad ||\alpha^\tau||^2 = e + 1 , \quad ||\beta^\tau||^2 = q^{2c} + e , \quad (\alpha^\tau, \beta^\tau) = -q^c .$$

Furthermore

$$(31.9)\qquad \begin{aligned}(\alpha^\tau, \lambda_i^\tau - \lambda_j^\tau) &= \delta_{j1} - \delta_{i1}\,,\\ (\beta^\tau, \lambda_i^\tau - \lambda_j^\tau) &= q^c(\delta_{i1} - \delta_{j1})\,.\end{aligned}$$

Suppose that $(\alpha^\tau, \lambda_i^\tau) \neq 0$ for some i with $2 \leqq i \leqq 2(q^c - 1)$. Then (31.8) and (31.9) imply that

$$\frac{q^c + 1}{2} + 1 = e + 1 = \|\alpha^\tau\|^2 \geqq 1 + 2(q^c - 1) - 1\,.$$

Hence $q^c + 3 \geqq 4q^c - 4$, or $7 \geqq 3q^c$ which is not the case. Therefore

$$(31.10)\qquad \alpha^\tau = 1_{\mathfrak{S}} - \lambda_1^\tau + \Gamma\,,\quad (\Gamma, \lambda_i^\tau) = 0 \quad \text{for } 1 \leqq i \leqq 2(q^c - 1)\,.$$

Equation (31.9) also yields that for some integer x

$$(31.11)\qquad \beta^\tau = q^c\lambda_1^\tau - x\sum_{i=1}^{2(q^c-1)} \lambda_i^\tau + \Delta\,,\qquad (\lambda_i^\tau, \Delta) = 0 \text{ for } 1 \leqq i \leqq 2(q^c - 1)\,.$$

Furthermore Lemma 13.8 implies that for $2 \leqq s \leqq q - 1$,

$$(31.12)\qquad (\Delta, \zeta_s^\tau - \zeta_1^\tau) = (\beta^\tau, \zeta_s^\tau - \zeta_1^\tau) = (\beta, \zeta_s - \zeta_1) = e\,.$$

Since β^τ vanishes on $\hat{\mathfrak{W}}$ and $(\beta^\tau, 1_{\mathfrak{S}}) = 0$ Lemma 13.2 yields that

$$(31.13)\qquad \Delta = \sum_{i=1}^{q-1} a_{i0} \sum_{j=0}^{e-1} \eta_{ij} + \sum_{j=1}^{e-1} a_{0j} \sum_{i=0}^{q-1} \eta_{ij} + \Delta_0\,,$$

where $(\Delta_0, \eta_{ij}) = 0$ for $0 \leqq i \leqq q - 1$, $0 \leqq j \leqq e - 1$. Now (31.12) and (31.13) imply that

$$a_{s0} - a_{10} = \pm 1 \quad \text{for } 2 \leqq s \leqq q - 1\,.$$

Define $a = a_{20}$. Then (31.13) implies that

$$(31.14)\qquad (a \pm 1)^2 + (q - 2)a^2 + \sum_{j=1}^{e-1} a_{0j}^2 + \sum_{j=1}^{e-1} \{(a \pm 1 + a_{0j})^2 + (q - 2)(a + a_{0j})^2\} \leqq \|\Delta\|^2\,.$$

For any value of j the term in the last summation in (31.14) is non zero. Furthermore $(a \pm 1)^2 + (q - 2)a^2 \neq 0$. Thus (31.14) implies that if there are exactly k values of j with $a_{0j} \neq 0$, then

$$(31.15)\qquad k + e \leqq \|\Delta\|^2\,,\quad k \text{ is even.}$$

The last statement follows from the fact that $(\eta_{0j}, \Delta) = (\bar{\eta}_{0j}, \Delta)$ since β^τ and thus Δ has its values in $\mathscr{Q}_{|\mathfrak{Q}|}$. By definition

$$(q^c\tilde{1}_{\mathfrak{Q}} - \zeta_1)^\tau = q^c(\tilde{1}_{\mathfrak{Q}} - \lambda_1)^\tau + (q^c\lambda_1 - \lambda_1)^\tau = q^c\alpha^\tau + \beta^\tau\,.$$

Lemma 31.2 implies that for any value of j with $1 \leqq j \leqq e - 1$

$$(31.16)\qquad (\alpha^\tau, \eta_{0j}) \neq 0 \quad \text{or} \quad (\beta^\tau, \eta_{0j}) \neq 0 .$$

Now (31.8), (31.11) and (31.15) yield that

$$(q^c - x)^2 + x^2\{2(q^c - 1) - 1\} \leqq q^{2c} ,$$

or

$$2(q^c - 1)x^2 \leqq 2q^c x .$$

Therefore

$$0 \leqq x \leqq \frac{q^c}{q^c - 1} < 2 .$$

Suppose that $x \neq 0$, then $x = 1$. Now (31.8) and (31.11) imply that $\|\Delta\|^2 \leqq q^{2c} + e - \{(q^c - 1)^2 + 2(q^c - 1) - 1\} = e + 2$. By (31.15) this implies that $k = 0$ or $k = 2$. Assume first that $k = 0$, then (31.10) implies that $\|\Gamma\|^2 \leqq e - 1$. Hence by (31.16)

$$\Gamma = \sum_{j=1}^{e-1} \pm \eta_{0j} .$$

This implies that $(\beta^\tau, \Gamma) = 0$. Consequently (31.8), (31.10) and (31.11) yield that

$$-q^c = (\alpha^\tau, \beta^\tau) = (-\lambda_1^\tau, \beta^\tau) = x - q^c = 1 - q^c$$

which is not the case.

Assume now that $k = 2$. Choose $1', 2'$ with $1 \leqq 1' < 2' \leqq e - 1$ so that $a_{0j} \neq 0$ for $j = 1', 2'$. Thus $\eta_{01'} = \overline{\eta_{02'}}$, $a_{01'} = a_{02'} = \pm 1$ and by (31.16)

$$\alpha^\tau = 1_{\mathfrak{G}} - \lambda_1^\tau + \sum_{j \neq 1', 2', 0} \pm \eta_{0j} + \Gamma_0 , \quad \|\Gamma_0\|^2 = 2 .$$

Since β^τ has its values in $\mathcal{Q}_{|\mathfrak{Q}|}$ and $\eta_{01'}$ has its values in $\mathcal{Q}_e$, $(\eta_{0j}, \beta^\tau) \neq 0$ for any algebraic conjugate η_{0j} of $\eta_{01'}$. By Lemma 13.1 $\eta_{01'}$ has at least $(p - 1)$ algebraic conjugates. Hence $p = 3$, therefore $q \neq 3$. Since α^τ vanishes on $\hat{\mathfrak{W}}$ Lemma 13.1 implies that for $1 \leqq s \leqq q - 1$

$$0 = (\alpha^\tau, 1_{\mathfrak{G}} - \eta_{s0} - \eta_{01'} + \eta_{s1'}) = 1 + (\Gamma_0, - \eta_{s0} + \eta_{s1'}) - (\Gamma_0, \eta_{01'}) .$$

Hence if $(\Gamma_0, \eta_{01'}) = 0$ then

$$2 = \|\Gamma_0\|^2 \geqq (q - 1) > 2 .$$

Therefore $(\Gamma_0, \eta_{01'}) \neq 0$. Hence

$$\Gamma = \sum_{j=1}^{e-1} \pm \eta_{0j} .$$

Consequently (31.8), (31.10) and (31.11) yield that

$$-q^c = (\alpha^\tau, \beta^\tau) = (-\lambda_1^\tau, \beta^\tau) \pm 2 = x - q^c \pm 2 = 1 - q^c \pm 2 .$$

The assumption that $x \neq 0$ has led to a contradiction in all cases. Therefore (31.8), (31.11) and (31.15) imply that

$$\beta^\tau = q^c\lambda_1^\tau + \Delta , \qquad \|\Delta\|^2 = e .$$

Thus $a_{0j} = 0$ for $1 \leqq j \leqq e - 1$. Thus (31.14) implies that

$$(a \pm 1)^2 e + (q - 2)a^2 e \leqq e .$$

Hence $a = 0$ or $q = 3$ and $a \pm 1 = 0$. Thus $\beta^\tau = q^c\lambda_1^\tau - \zeta_1^\tau$ or $q = 3$ and $\beta^\tau = q^c\lambda_1^\tau + \zeta_2^\tau$. In either case this implies that the set of characters consisting of λ_i, $1 \leqq i \leqq 2(q^c - 1)$ and ζ_s, $1 \leqq s \leqq q - 1$ is coherent. This includes all characters in $\mathscr{T}$ which have $\mathfrak{Q}_1$ in their kernel. Since $|\mathfrak{Q} : \mathfrak{Q}_1| = q^{2c+1} > 4p^{2b}$ the result now follows from Theorem 11.1 with $\mathfrak{H} = \hat{\mathfrak{L}} = \mathfrak{K} = \mathfrak{Q}$, $\mathfrak{H}_1 = \mathfrak{Q}_1$ and $\mathfrak{L} = \mathfrak{T}$.

LEMMA 31.4. *$\mathscr{S}$ is coherent.*

Proof. By Theorem 30.1 w_2 is a power of 3 if $\mathscr{S}$ is not coherent. By Lemma 31.3 b is odd. Thus the lemma follows from Lemma 11.6.

LEMMA 31.5. *For $0 \leqq i \leqq n - 1$ let λ_i be an irreducible character of $\mathfrak{T}$ with $\lambda_i(1) = eq^{f_i}$. Let $\mathfrak{Q}_0$ be the normal closure of $\mathfrak{Q}^*$ in $\mathfrak{T}$. Let $1 = q^{g_0} < \cdots < q^{g_m}$ be all the degrees of irreducible characters of $\mathfrak{Q}/\mathfrak{Q}_0$. Then $\mathfrak{T}/\mathfrak{Q}_0$ is a Frobenius group. For any value of j with $0 \leqq j \leqq m$ let θ_j be an irreducible character of $\mathfrak{T}/\mathfrak{Q}_0$ of degree eq^{g_j}. Define*

$$\begin{aligned}
\alpha &= \tilde{1}_{\mathfrak{Q}} - \lambda_0 , \\
\beta_i &= q^{f_i - f_{i-1}}\lambda_{i-1} - \lambda_i \quad \text{for } 1 \leqq i \leqq n - 1 , \\
\gamma_j &= q^{g_j - g_{j-1}}\theta_{j-1} - \theta_j \quad \text{for } 1 \leqq j \leqq m .
\end{aligned}$$

Then

$$\begin{aligned}
(\beta_i^\tau, \eta_{0t}) &= 0 \quad \text{for } 0 \leqq t \leqq e - 1,\ 1 \leqq i \leqq n - 1 , \\
(\gamma_j^\tau, \eta_{0t}) &= 0 \quad \text{for } 0 \leqq t \leqq e - 1,\ 1 \leqq j \leqq m .
\end{aligned}$$

Furthermore if e is a prime then one of the following possibilities must occur:

$$\alpha^\tau = 1_{\mathfrak{G}} - \lambda_0^\tau + \sum_{t=1}^{e-1} \eta_{0t} ,$$

$$\alpha^\tau = 1_{\mathfrak{G}} + \bar{\lambda}_0^\tau + \sum_{t=1}^{e-1} \eta_{0t} \text{ and } 2e + 1 = |\mathfrak{Q} : \mathfrak{Q}'| ,$$

$$\alpha^\tau = 1_{\mathfrak{G}} + \sum_{s=1}^{q-1} \eta_{s0} + \Gamma ,$$

with $(\Gamma, \eta_{st}) = 0$ *for* $0 \leqq s \leqq q-1$, $0 \leqq t \leqq e-1$.

Proof. For $1 \leqq i \leqq n-1$, $1 \leqq j \leqq m$ let

$$\alpha^\tau = \Gamma_{00} + \Delta_{00}\,, \quad \beta_i^\tau = \Gamma_{i0} + \Delta_{i0}\,, \quad \gamma_j^\tau = \Gamma_{0j} + \Delta_{0j}\,,$$

where each Δ_{ij} is a linear combination of the generalized characters η_{st} and each Γ_{ij} is orthogonal to each of these generalized characters. Since for $1 \leqq s \leqq q-1$, $(\zeta_s - \zeta_1)^\tau$ is orthogonal to α^τ, β_i^τ and γ_j^τ and all of these vanish on $\hat{\mathfrak{W}}$, Lemma 13.2 implies that

$$\Delta_{ij} = a_{00}1_{\mathfrak{G}} + a\sum_{s=1}^{q-1}\sum_{t=0}^{e-1}\eta_{st} + \sum_{t=1}^{e-1}a_{0t}\sum_{s=0}^{q-1}\eta_{st} - a_{00}\sum_{s=1}^{q-1}\sum_{t=1}^{e-1}\eta_{st}\,, \tag{31.17}$$

where $\{a\} \cup \{a_{st}\}$ is a set of integers depending on (i, j). Since $(\lambda_0^\tau - \bar{\lambda}_0^\tau, \alpha^\tau) \neq 0$, $\|\Delta_{00}\|^2 \leqq e$. Since $(\lambda_i^\tau - \bar{\lambda}_i^\tau, \beta_i^\tau) \neq 0$, $(\theta_j^\tau - \bar{\theta}_j^\tau, \gamma_j^\tau) \neq 0$, Theorem 12.1 implies that

$$\|\Delta_{ij}\|^2 \leqq e \quad \text{for all } (i, j)\,. \tag{31.18}$$

Assume first that $(i, j) \neq (0, 0)$. Then $a_{00} = 0$. Thus (31.17) and (31.18) imply that

$$(q-1)a^2 + (q-1)\sum_{t=1}^{e-1}(a + a_{0t})^2 + \sum_{t=1}^{e-1}a_{0t}^2 \leqq e\,.$$

If $a \neq 0$ then for each value of t either $a_{0t} \neq 0$ or $a + a_{0t} \neq 0$. Thus $(q-1)a^2 \leqq 1$ which is not the case. Hence $a = 0$ and so

$$\Delta_{ij} = \sum_{t=1}^{e-1}a_{0t}\sum_{s=0}^{q-1}\eta_{st}\,. \tag{31.19}$$

As $\mathscr{I}_0(\mathscr{S})^\tau$ is orthogonal to $\mathscr{I}_0(\mathscr{T})^\tau$ Lemma 31.4 yields that for all (i, j)

$$(\xi_k(1)\xi_{k'}^\tau - \xi_{k'}(1)\xi_k^\tau, \Delta_{ij}) = 0 \quad \text{for } 1 \leqq k,\ k' \leqq e-1\,.$$

By (31.19) $(\Delta_{ij}, \xi_k^\tau) = \pm a_{0k}q$. Hence

$$\xi_k(1)a_{0k'} - \xi_{k'}(1)a_{0k} = 0\,.$$

Suppose now that $a_{0t} \neq 0$ for some t. Then $a_{0t} \neq 0$ for all t with $1 \leqq t < e$. Hence (31.18) and (31.19) imply that

$$q(e-1) \leqq q\sum_{t=1}^{e-1}a_{0t}^2 \leqq e$$

which is not the case. The result is proved in case $(i, j) \neq (0, 0)$.

Let $(i, j) = (0, 0)$. Then $a_{00} = 1$. By assumption $\xi_k(1) = \xi_1(1)$ for $1 \leqq k \leqq e-1$, since e is a prime. By (31.17)

$$(\Delta_{00}, \xi_k^\tau) = \pm\{a(q-1) + a_{0k}q - a_{00}(q-1)\}\,, \quad \text{for } 1 \leqq k \leqq e-1$$

where the sign is independent of k. Since $(\Delta_{00}, \xi_k^\tau - \xi_1^\tau) = 0$ this yields that $a_{0k} = a_{01}$ for $1 \leqq k \leqq e-1$. Hence (31.17) and (31.18) imply that

$$(q-1)a^2 + (e-1)a_{01}^2 + (e-1)(q-1)(a + a_{01} - 1)^2 \leqq e - 1\,.$$

If $a_{01} \neq 0$ this yields that $a = 0$ and $a_{01} = 1$ and the result follows. If $a_{01} = 0$ then we get that

$$(q-1)a^2 + (e-1)(q-1)(a-1)^2 \leqq e-1\,.$$

Hence $a = 1$ and the result is proved also in this case.

LEMMA 31.6. *Let $\lambda = \lambda_{n-1}$ have the same meaning as in Lemma* 31.5. *Define*

$$\beta_n = \beta = q^{f_n - f_{n-1}}\lambda - \zeta_1\,.$$

Then $(\beta^\tau, \eta_{0t}) = 0$ for $0 \leqq t \leqq e-1$.

Proof. Let $\mathscr{T}_b$ be the equivalence class in $\mathscr{T}$ defined by (12.3) which contains λ. If ζ_1 is in $\mathscr{T}_b$ then the result follows from the coherence of $\mathscr{T}_b$. For any i, let a_i/e be the number of characters of degree $q^{f_i}e$ in $\mathscr{T}_b$ and define c as in (12.4) by

$$c = \sum_i a_i q^{2(f_i - f_m)}\,, \tag{31.20}$$

where $q^{f_m}e$ is the minimum degree of any character in $\mathscr{T}_b$.

Let

$$\beta^\tau = \Delta_0 + \Delta + \Gamma\,, \tag{31.21}$$

where $\Delta_0 \in \mathscr{I}(\mathscr{T}_b^\tau)$, Δ is an integral linear combination of the generalized characters η_{st} and Γ is orthogonal to $\mathscr{T}_b^\tau$ and to every η_{st}. Theorem 12.1 yields that

$$\|\Delta\|^2 + \|\Gamma\|^2 \leqq 2e\,. \tag{31.22}$$

β^τ vanishes on $\hat{\mathfrak{W}}$ and $(\beta^\tau, 1_{\mathfrak{G}}) = 0$. Furthermore $(\zeta_s^\tau - \zeta_1^\tau, \Delta) = e$ for $2 \leqq s \leqq q-1$. Therefore Lemma 13.2 implies that

$$\Delta = \varepsilon \sum_{t=0}^{e-1} \eta_{1t} + a_{10} \sum_{s=1}^{q-1}\sum_{t=0}^{e-1} \eta_{st} + \sum_{t=1}^{e-1} a_{0t} \sum_{s=0}^{q-1} \eta_{st}\,, \tag{31.23}$$

where $\varepsilon = \pm 1$.

Since $\mathscr{I}_0(\mathscr{S})^\tau$ is orthogonal to $\mathscr{I}_0(\mathscr{T})^\tau$ Lemma 31.4 yields that

$$(\xi_k(1)\xi_{k'}^\tau - \xi_{k'}(1)\xi_k^\tau, \Delta) = 0 \quad \text{for } 1 \leqq k,\ k' \leqq e-1\,.$$

By (31.23)

$$(\xi_k^\tau, \Delta) = \pm\{\varepsilon + (q-1)a_{10} + qa_{0k}\} ,$$

where the sign is independent of k. Therefore

$$\xi_k(1)\{\varepsilon + (q-1)a_{10} + qa_{0k'}\} = \xi_{k'}(1)\{\varepsilon + (q-1)a_{10} + qa_{0k}\}$$

for $1 \leqq k,\ k' \leqq e$. By (31.22) and (31.23) we see that

$$(31.24) \qquad \sum_{t=1}^{e-1} a_{0t}^2 + (a_{10}+\varepsilon)^2 + (q-2)a_{10}^2 + \sum_{t=1}^{e-1}(\varepsilon + a_{10} + a_{0t})^2 + (q-2)\sum_{t=1}^{e-1}(a_{10}+a_{0t})^2 = \|\Delta\|^2 \leqq 2e .$$

If $a_{10} \neq 0$ and $a_{10} + \varepsilon \neq 0$ then for each t at most one of $a_{0t}, a_{10} + a_{0t}$ $\varepsilon + a_{10} + a_{0t}$ vanishes. Hence (31.24) yields that

$$(a_{10}+\varepsilon)^2 + (q-2)a_{10}^2 \leqq 2 .$$

This is impossible as either a_{10} or $a_{10}+\varepsilon$ is even. If $a_{10} \neq 0$ then (31.24) implies that

$$2\sum_{t=1}^{e-1} a_{0t}^2 + (q-2) + (q-2)\sum_{t=1}^{e-1}(a_{0t}-\varepsilon)^2 \leqq 2e .$$

If $q \neq 3$, then $2\,a_{0t}^2 + (q-2)(a_{0t}-\varepsilon)^2 \geqq 2$ for $1 \leqq t < e$. Hence $q - 2 \leqq 2$ which is not the case. Thus $a_{10} = 0$ or $q = 3$ and $a_{10} + \varepsilon = 0$. Thus we get

$$(31.25) \qquad \begin{aligned} &(\xi_k, \Delta) = \pm\{\pm\varepsilon + qa_{0k}\} \\ &\xi_k(1)\{\pm\varepsilon + qa_{0k'}\} = \xi_{k'}(1)\{\pm\varepsilon + qa_{0k}\} \quad \text{for } 1 \leqq k,\ k' < e . \end{aligned}$$

Assume that the result is false. Then $a_{0t} \neq 0$ for some value of t. We will next show that $a_{0t} \neq 0$ for $1 \leqq t < e$. If this is false then there exists j such that $a_{0j} = 0$. If γ is any character in $\mathscr{S}$ then $(\gamma(1)\xi_j^\tau - \xi_j(1)\gamma^\tau,\ \Delta + \Gamma) = 0$. Thus (31.25) implies that

$$(31.26) \qquad (\gamma^\tau, \Delta + \Gamma) = \frac{\pm\gamma(1)}{\xi_j(1)} .$$

Thus $\xi_j(1) \mid \gamma(1)$ for every γ in $\mathscr{S}$. Let a be the exponent of $\mathfrak{U}$. By Lemmas 30.1, 30.4 and 30.5 $\xi_j(1) = aq$. Thus $\mathfrak{H}'$ is in the kernel of ξ_j. Define

$$\sigma = \{t \mid 1 \leqq t < e,\ \xi_t(1) \neq \xi_j(1)\} .$$

By (31.25)

$$a_{0t} = \frac{\pm\{\xi_t(1) - \xi_j(1)\}}{q\xi_j(1)} \quad \text{for } 1 \leqq t < e .$$

Thus (31.22), (31.23) and (31.26) yield that

$$2e(aq)^2 \geqq \Sigma\, \gamma(1)^2 + \frac{1}{q^2}\sum_{t=1}^{e-1}\{\xi_t(1) - \xi_j(1)\}^2 \geqq \sum \gamma(1)^2 + \frac{x}{q^2}\sum_{t\in\sigma}\xi_t(1)^2 ,$$

where $x = 4/9$ if $q \neq 3$ and $x = 16/25$ if $q = 3$, and γ ranges over the irreducible characters in $\mathscr{S}$. By Lemma 13.7 there exist irreducible characters μ_t of $\mathfrak{S}'$ which induce the characters ξ_t for $1 \leqq t < e$. Consequently

$$2ea^2q \geqq \sum \chi(1)^2 + x\sum_{t\in\sigma}\mu_t(1)^2 \geqq x\left\{\sum \chi(1)^2 + \sum_{t\in\sigma}\mu_t(1)^2\right\}$$

where χ ranges over the irreducible characters of $\mathfrak{S}'$ which are distinct from all μ_t and do not have $\mathfrak{H}$ in their kernel. Therefore $C(\mathfrak{H}) \subseteqq \mathfrak{H}$ otherwise since $|\mathfrak{G}|$ is odd there are at least $2eq$ characters χ of degree at least a. Furthermore

$$2ea^2q \geqq x\{u(h-1) - a^2(e-1)\} .$$

This implies that

$$yeqa^2 \geqq \left\{\frac{2eq}{x} + e - 1\right\}a^2 \geqq u(h-1) , \tag{31.27}$$

where $y = 4$ if $q = 3$ and $y = 5$ otherwise. Let $1 \subset \mathfrak{H}_1 \subset \mathfrak{H}$, where $\mathfrak{H}_1 \lhd \mathfrak{S}$. Let $h_1 = |\mathfrak{H}_1|$, $h_2 = |\mathfrak{H} : \mathfrak{H}_1|$, $e_1 = |C_{\mathfrak{H}_1}(\mathfrak{Q}^*)|$ and $e_2 = |C_{\mathfrak{H}/\mathfrak{H}_1}(\mathfrak{Q}^*)|$. Since $\mathfrak{S}$ is of type II $ae_1 < 2h_1$ and $a \leqq u$. Thus (31.27) implies that $h_2 - 1 \leqq 2yqe_2$. Since $h_2 \geqq p^{q-1}e_2$ for some prime p dividing h_2 we get that $p^{q-1} \leqq 2yq$. Thus $q = 3$ by (5.1). Hence $p^2 \leqq 24$ which is not the case as $p \geqq 5$. Hence no such group $\mathfrak{H}_1$ exists. Thus $\mathfrak{H}$ is an elementary abelian p-group for some prime. Therefore $e = p$ is a prime and $\xi_t(1) = \xi_1(1)$ for $1 \leqq t < e$. Consequently $a_{0t} = a_{0j} = 0$ for $1 \leqq t < e$ contrary to assumption.

Returning to (31.24) we see that

$$\sum_{t=1}^{e-1} a_{0t}^2 \leqq e + 1 .$$

Therefore $a_{0t}^2 = 1$ for $1 \leqq t \leqq e - 1$. Thus

$$a_{0t} = \pm 1 \quad \text{for } 1 \leqq t \leqq e - 1 . \tag{31.28}$$

Now (31.24) implies that

$$\begin{aligned}(a_{10} + \varepsilon)^2 &+ (q-2)a_{10}^2 \\ &+ (e-1)\{(a_{10} + \varepsilon + a_{01})^2 + (q-2)(a_{10} + a_{01})^2\} \\ &\leqq e + 1 ,\end{aligned} \tag{31.29}$$

Suppose that $q \neq 3$. Thus $q \geqq 5$ and $a_{10} = 0$. Then (31.29) implies that $(e-1)(q-2) \leqq e+1$. As $q \geqq 5$ this implies that $3e - 3 \leqq e + 1$ or $e \leqq 2$ which is not the case. Therefore

(31.30) $$q = 3\ .$$

By (31.29) either $a_{10} = 0$, $a_{01} = -(a_{10} + \varepsilon)$ or $a_{10} + \varepsilon = 0$, $a_{01} = -a_{10}$. Now (31.23) and (31.28) imply that

$$\varDelta = \pm\left\{\sum_{t=0}^{e-1} \eta_{1t} - \sum_{t=1}^{e-1}\sum_{s=0}^{2} \eta_{st}\right\}$$

or

$$\varDelta = \pm\left\{\sum_{t=0}^{e-1} \eta_{2t} - \sum_{t=1}^{e-1}\sum_{s=0}^{2} \eta_{st}\right\}.$$

This is equivalent to

$$\varDelta = \pm\left\{\eta_{10} - \sum_{t=1}^{e-1} (\eta_{0t} + \eta_{2t})\right\}$$

(31.31) or

$$\varDelta = \pm\left\{\eta_{20} - \sum_{t=1}^{e-1} (\eta_{0t} + \eta_{1t})\right\}.$$

Since $(\beta^\tau - \bar{\beta}^\tau, \varGamma) = 0$, $\varGamma$ is a real valued generalized character. Thus $\|\varGamma\|^2 \neq 1$. By (31.31) $\|\varDelta\|^2 = 2e - 1$, hence by (31.22) $\varGamma = 0$. Now (31.21) implies that

(31.32) $$\beta^\tau = q^{f_n - f_{n-1}}\lambda^\tau - x\sum_{i=m}^{n-1}\sum_{j=1}^{a_i/e} q^{f_i - f_m}\lambda_{ij}^\tau + \varDelta\ ,$$

where for $m \leqq i \leqq n-1$, λ_{ij} ranges over the characters of degree eq^{f_i} in $\mathscr{T}_b$.

Suppose that $\mathscr{S}$ contains an irreducible character γ. Then by Lemma 31.4

$$(\gamma(1)\xi_t^\tau - \xi_t(1)\gamma^\tau, \beta^\tau) = 0 \quad \text{for } 1 \leqq t \leqq e-1\ .$$

As γ^τ is rational valued on elements of $\mathfrak{Q}$, $\gamma^\tau \neq \lambda_{ij}^\tau$ for all i, j. Thus (31.31) and (31.32) imply that

$$\pm 2\gamma(1) = (\gamma(1)\xi_t^\tau, \beta^\tau) = (\xi_t(1)\gamma^\tau, \beta^\tau) = 0\ .$$

Therefore $\mathscr{S}$ contains no irreducible characters. Hence by Lemma 31.1

(31.33) $$e = p\ , \quad p \text{ a prime.}$$

Now Lemma 31.3 implies that b is odd, where b is defined in (31.6). As $\|\varDelta\|^2 = 2p - 1 > 2p - 2$ Theorem 12.1 implies that if c is defined in (31.20) then

$$c \equiv 0 \pmod{q} \quad \text{or } c \geqq p^2 . \tag{31.34}$$

Assume first that $m \neq 0$ in (31.32). Let α be defined as in Lemma 31.5. Suppose that

$$\alpha^\tau = 1_{\mathfrak{G}} \pm \lambda_0^\tau + \sum_{t=1}^{p-1} \eta_{0t} .$$

Then (31.31) and (31.32) yield that

$$0 = (\alpha^\tau, \beta^\tau) = \pm(p-1) .$$

Thus by Lemma 31.5

$$\alpha^\tau = 1_{\mathfrak{G}} \pm \lambda_0^\tau + \sum_{s=1}^{2} \eta_{s0} + \Gamma_0 , \quad \|\Gamma_0\|^2 \leqq p - 3 . \tag{31.35}$$

Then

$$\Gamma_0 = \Gamma_{00} + y \sum_{i=m}^{n-1} \sum_{j=1}^{a_i/p} q^{f_i - f_m} \lambda_{ij}^\tau , \tag{31.36}$$

where $(\Gamma_{00}, \lambda_{ij}^\tau) = 0$ for $m \leqq i \leqq n-1$, $1 \leqq j \leqq (a_i/p)$. Suppose that $y = 0$. Then (31.31), (31.32) and (31.36) yield that $0 = (\alpha^\tau, \beta^\tau) = \pm 1$. Hence $y \neq 0$. Thus by (31.35) and (31.36)

$$(p-3) \geqq y^2 \frac{c}{p} \geqq \frac{c}{p} .$$

Thus (31.34) yields that

$$c \equiv 0 \pmod{q} . \tag{31.37}$$

Equations (31.31), (31.32), (31.35) and (31.36) imply that

$$0 = (\alpha^\tau, \beta^\tau) = \pm 1 + yq^{f_n - f_{n-1}} q^{f_{n-1} - f_m} - xy \frac{c}{p} .$$

Hence (31.37) implies that $0 \equiv \pm 1 \pmod{q}$. This contradiction arose from assuming $m \neq 0$.

Assume now that $m = 0$. Then

$$c = q^b - 1 + \sum_{i=1}^{n-1} a_i q^{2f_i} .$$

Hence $c \not\equiv 0 \pmod{q}$. Thus (31.34) and 3.15 imply that

$$c \geqq p^2 , \quad c + 1 \equiv 0 \pmod{q^{2f_n}} . \tag{31.38}$$

Now (31.31) and (31.32) yield that

$$q^{2(f_n - f_{n-1})} + p = \| \beta^\tau \|^2 = q^{2(f_n - f_{n-1})} - 2xq^{f_n} + x^2 \frac{c}{p} + 2p - 1 .$$

Therefore

$$x^2c + p(p-1) = 2xq^{f_n}p . \tag{31.39}$$

By (31.38), $(c+1) > pq^{f_n}$. Thus (31.39) yields that

$$f(x) = x^2(pq^{f_n} - 1) - 2xq^{f_n}p + p(p-1) < 0 .$$

It is easily verified that $f(x)$ is a monotone increasing function for $x \geqq 2$ and $f(2) = p(p-1) - 4 > 0$. Thus $x < 2$. By (31.39) $x > 0$. Hence $x = 1$. Now (31.39) becomes

$$c + p(p-1) = 2q^{f_n}p ,$$

or equivalently

$$p^2 - p(1 + 2q^{f_n}) + c = 0 . \tag{31.40}$$

Therefore $(1 + 2q^{f_n})^2 - 4c \geqq 0$, hence

$$4c \leqq 4q^{2f_n} + 4q^{f_n} + 1 < 8q^{2f_n} .$$

Thus $c < 2q^{2f_n}$. As c is even, (31.38) now yields that $c = q^{2f_n} - 1$. Now (31.40) becomes

$$q^{2f_n} - 2q^{f_n}p + p^2 - p - 1 = 0 ,$$

or

$$(q^{f_n} - p - 1)(q^{f_n} - p + 1) = p .$$

As p is a prime one of the factors is ± 1 and the other is $\pm p$. As the factors differ by 2 this implies that $p \pm 1 = 2$. Hence $p = 3$. Since $p \neq q$ (31.30) implies that $p \neq 3$. This contradiction establishes the lemma in all cases.

THEOREM 31.1. *$\mathscr{T}$ is coherent.*

Proof. Suppose that $\mathscr{T}$ is not coherent so that Hypothesis 31.1 is assumed. Let $\alpha, \beta_i, \gamma_j, \lambda_i, \theta_j$ have the same meaning as in Lemmas 31.5 and 31.6. Choose $\lambda_0 = \theta_0$. Then

$$(q^{f_n}\widetilde{1}_{\mathfrak{Q}} - \zeta_1)^\tau = q^{f_n}\alpha^\tau + \sum_{i=1}^{n} q^{f_n - f_i}\beta_i^\tau . \tag{31.41}$$

$$(q^{f_n}\lambda_0 - \zeta_1)^\tau = \sum_{i=1}^{n} q^{f_n - f_i}\beta_i^\tau . \tag{31.42}$$

$$(q^{g_j}\theta_0 - \theta_j)^\tau = \sum_{s=1}^{j} q^{g_j - g_s}\gamma_s^\tau \quad \text{for } 1 \leqq j \leqq m . \tag{31.43}$$

Lemmas 31.2, 31.5 and 31.6 together with (31.41) imply that

$$\alpha^\tau = 1_{\mathfrak{S}} - \lambda_0^\tau + \sum_{t=1}^{e-1} \eta_{0t}$$

or

$$\alpha^\tau = 1_{\mathfrak{S}} + \bar{\lambda}_0^\tau + \sum_{t=1}^{e-1} \eta_{0t}$$

and $2e + 1 = |\mathfrak{Q} : \mathfrak{Q}'|$. If the latter possibility occurs then by Lemma 10.1 it may be assumed after changing notation that in any case

(31.44) $$\alpha^\tau = 1_{\mathfrak{S}} - \lambda_0^\tau + \sum_{t=1}^{e-1} \eta_{0t} \, .$$

Now Lemma 31.5, (31.43) and (31.44) imply that

(31.45) $$\begin{aligned} -q^{g_s} &= (\alpha^\tau, (q^{g_s}\theta_0 - \theta_s)^\tau) \\ &= (-\theta_0^\tau, (q^{g_s}\theta_0 - \theta_s)^\tau) \, , \quad \text{for } 1 \leqq s \leqq m \, . \end{aligned}$$

Since $\| (q^{g_s}\theta_0 - \theta_s)^\tau \|^2 = q^{2g_s} + 1$ and $((q^{g_s}\theta_0 - \theta_s)^\tau, (\theta_s^\tau - \bar{\theta}_s^\tau)) = -1$, (31.45) implies that

(31.46) $$(q^{g_s}\theta_0 - \theta_s)^\tau = q^{g_s}\theta_0^\tau - \theta_s^\tau \quad \text{for } 1 \leqq s \leqq m \, .$$

Lemmas 31.2 and 31.5 and equations (31.42) and (31.44) yield that

(31.47) $$-q^{f_n} = ((q^{f_n}\lambda_0 - \zeta_1)^\tau, \alpha^\tau) = ((q^{f_n}\lambda_0 - \zeta_1)^\tau, -\lambda_0^\tau) \, .$$

By Lemma 13.10 $\{\zeta_i \mid 1 \leqq i \leqq q - 1\}$ is subcoherent in $\mathscr{T}$. Since $\| (q^{f_n}\lambda_0 - \zeta_1)^\tau \|^2 = q^{2f_n} + e$ it follows from (31.47) that

(31.48) $$(q^{f_n}\lambda_0 - \zeta_1)^\tau = q^{f_n}\lambda_0^\tau - \zeta_1^\tau \, .$$

Let $\mathfrak{Q}_0$ have the same meaning as in Lemma 31.5. Then there exists a subgroup $\mathfrak{Q}_1$ of $\mathfrak{Q}_0$ such that $\mathfrak{Q}_0/\mathfrak{Q}_1$ is a chief factor of $\mathfrak{T}$ and $|\mathfrak{Q}_0 : \mathfrak{Q}_1| = q$. Let $\mathscr{T}(\mathfrak{Q}_0)$ be the irreducible characters of $\mathfrak{T}$ of degree eq^{g_j}, $0 \leqq j \leqq m$. Then (31.46) implies directly that $\mathscr{T}(\mathfrak{Q}_0)$ is coherent. Hypothesis 11.1 is satisfied with $\mathfrak{H} = \hat{\mathfrak{L}} = \mathfrak{K} = \mathfrak{Q}$ and $\mathfrak{T} = \mathfrak{L}$. If $\mathscr{T}$ is not coherent then Theorem 11.1 implies that $|\mathfrak{Q} : \mathfrak{Q}_0| < 4e^2 + 1$. As $\mathfrak{T}/\mathfrak{Q}_0$ is a Frobenius group this implies that $\mathfrak{Q}_0 = \mathfrak{Q}'$. Therefore $\mathfrak{Q}/\mathfrak{Q}_1$ is an extra special q-group. Thus $|\mathfrak{Q} : \mathfrak{Q}'| = q^{2c}$ for some integer c. Define

$$\mathscr{T}(\mathfrak{Q}_1) = \mathscr{T}(\mathfrak{Q}_0) \cup \{\zeta_i \mid 1 \leqq i \leqq q - 1\} \, .$$

Then $\mathscr{T}(\mathfrak{Q}_1)$ consists of all characters in $\mathscr{T}$ having the same weight and degree as some character in $\mathscr{T}$ which has $\mathfrak{Q}_1$ in its kernel. By (31.48) $\mathscr{T}(\mathfrak{Q}_1)$ is coherent. Thus if $\mathscr{T}$ is not coherent Theorem 11.1 implies that

(31.49) $$q^{2c+1} = |\mathfrak{Q} : \mathfrak{Q}_1| \leqq 4e^2 + 1 .$$

Lemma 13.6 applied to the group $\mathfrak{W}_2\mathfrak{Q}/\mathfrak{Q}_1$ implies that $e \mid q^c + 1$ or $e \mid q^c - 1$ and $|\mathfrak{W}_2| = e$. As e is odd this yields that $2e \leqq q^c + 1$ in any case. Thus by (31.49)

$$q^{2c+1} \leqq 4e^2 + 1 \leqq (q^c + 1)^2 + 1 < 2q^{2c} .$$

This contradiction suffices to prove Theorem 31.1.

COROLLARY 31.1.1. *If λ_0 is an irreducible character of $\mathfrak{T}$ of degree w_2 then*

$$(\tilde{1}_{\mathfrak{T}'} - \lambda_0)^\tau = 1_{\mathfrak{G}} - \lambda_0^\tau + \sum_{t=1}^{w_2-1} \eta_{0t} .$$

Proof. Let $\alpha = \tilde{1}_{\mathfrak{T}'} - \lambda_0$ and let $a_t = (\alpha^\tau, \eta_{0t})$. By Theorem 31.1

(31.50) $$\begin{aligned}(\nu_1(1)\tilde{1}_{\mathfrak{T}'} - \zeta_1)^\tau &= \nu_1(1)\alpha^\tau + (\nu_1(1)\lambda_0 - \zeta_1)^\tau \\ &= \nu_1(1)\lambda_0^\tau - \zeta_1^\tau + \nu_1(1)\alpha^\tau .\end{aligned}$$

As η_{0t} is rational on $\mathfrak{T}'$, $(\eta_{0t}, \lambda_0^\tau) = 0$. By Lemma 13.9 $(\eta_{0t}, \zeta_1^\tau) = 0$. Thus (31.50) implies that

$$((\nu_1(1)\tilde{1}_{\mathfrak{T}'} - \zeta_1)^\tau, \eta_{0t}) = a_t\nu_1(1) \quad \text{for } 1 \leqq t \leqq w_2 - 1 .$$

Hence by Lemma 31.2 $(\alpha^\tau, \eta_{0t}) \neq 0$ for $1 \leqq t \leqq w_2 - 1$. As $|\mathscr{T}| > 2$, $(\alpha^\tau, 1_{\mathfrak{G}}) = 1$, $(\alpha^\tau, \lambda_0^\tau - \bar{\lambda}_0^\tau) = -1$ and $\|\alpha^\tau\|^2 = w_2 + 1$ we get that

$$\alpha^\tau = 1_{\mathfrak{G}} - \lambda_0^\tau + \sum_{t=1}^{w_2-1} \pm\eta_{0t} .$$

As α^τ vanishes on $\hat{\mathfrak{W}}$ Lemma 13.2 now implies the required result.

COROLLARY 31.1.2. *$\mathfrak{S}'$ is a Frobenius group and w_2 is a prime.*

Proof. Suppose that $\mathscr{S}$ contains an irreducible character θ. Choose ξ_j in $\mathscr{S}(\mathfrak{H}')$. Then $(\theta(1)\xi_j^\tau - \xi_j(1)\theta^\tau) \in \mathscr{I}_0(\mathscr{S})$. If $\mathscr{S}$ is not coherent θ may be chosen in $\mathscr{S}(\mathfrak{H}')$ by Theorem 30.1 and Lemma 31.1. Hence by Corollary 31.1.1 and Lemmas 13.9 and 30.8,

$$\begin{aligned}0 &= (\theta(1)\xi_j^\tau - \xi_j(1)\theta^\tau, (\tilde{1}_{\mathfrak{T}'} - \lambda_0)^\tau) \\ &= \theta(1)\Big(\pm\sum_{i=0}^{q-1} \eta_{ij}, \sum_{t=1}^{w_2-1} \eta_{0t}\Big) = \pm\theta(1) .\end{aligned}$$

Therefore $\mathscr{S}$ contains no irreducible characters. Lemma 31.1 now implies that $\mathfrak{S}'$ is a Frobenius group and w_2 is a prime.

32. Subgroups of Type V

THEOREM 32.1. *$\mathfrak{G}$ contains no subgroup of type V.*

Proof. Suppose that the result is false and $\mathfrak{T}$ is a subgroup of type V. $\mathfrak{T}'$ is tamely imbedded in $\mathfrak{G}$ by Theorem 14.2. For $0 \leqq i \leqq n$ let $\mathfrak{L}_i$ have the same meaning as in Definition 9.1 and let $\mathfrak{A}_L$ be defined by (9.2). Let $\mathfrak{G}_1$ be the set of elements in $\mathfrak{G}$ which are conjugate to some element of $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. By Lemma 9.5

$$\text{(32.1)} \qquad \begin{aligned} \frac{1}{|\mathfrak{G}|} |\mathfrak{G}_1| &= \frac{1}{|\mathfrak{G}|} \Sigma_{\mathfrak{G}_1} 1_{\mathfrak{G}}(G) \\ &= \frac{1}{|\mathfrak{T}|} \Sigma_{\mathfrak{T}'^{\sharp}} 1_{\mathfrak{G}}(T) = \frac{1}{w_2}\left(1 - \frac{1}{|\mathfrak{T}'|}\right). \end{aligned}$$

Let λ be an irreducible character of degree w_2 in $\mathscr{T}$. By Theorem 31.1 and Lemmas 10.3 and 9.4

$$\text{(32.2)} \qquad \lambda^{\tau}(T) = a + \lambda(T) \quad \text{for } T \in \mathfrak{T}'^{\sharp},$$

where a is independent of T. Now Theorem 31.1 and Corollary 31.1.1 imply that $a = 0$ in (32.2). Thus $\lambda^{\tau}(T) = \lambda(T)$ for $T \in \mathfrak{T}'^{\sharp}$. Hence Theorem 31.1 and Lemmas 10.3 and 9.5 imply that

$$\text{(32.3)} \qquad \frac{1}{|\mathfrak{G}|} \Sigma_{\mathfrak{G}_1} |\lambda^{\tau}(G)|^2 = \frac{1}{|\mathfrak{T}|} \Sigma_{\mathfrak{T}'^{\sharp}} |\lambda(G)|^2 = 1 - \frac{w_2}{|\mathfrak{T}'|}.$$

Let $\mathfrak{W}$ be defined by Theorem 14.1 (ii) (a) and let $\hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{W}_2 - \mathfrak{Q}^*$. Define

$$\mathfrak{G}_2 = \bigcup_{G \in \mathfrak{G}} G^{-1} \hat{\mathfrak{W}} G.$$

Thus Theorem 14.2 (ii) (a) implies that

$$\text{(32.4)} \qquad \frac{1}{|\mathfrak{G}|} |\mathfrak{G}_2| = 1 - \frac{1}{w_2} - \frac{1}{q} + \frac{1}{q w_2}.$$

Let $\mathfrak{G}_3$ be the set of elements in $\mathfrak{G}$ which are conjugate to some element of $\mathfrak{H}^{\sharp}$. Since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$,

$$\text{(32.5)} \qquad \frac{1}{|\mathfrak{G}|} |\mathfrak{G}_3| = \frac{1}{q u |\mathfrak{H}|} (|\mathfrak{H}| - 1).$$

Define

$$\mathfrak{G}_0 = \mathfrak{G} - \mathfrak{G}_1 - \mathfrak{G}_2 - \mathfrak{G}_3.$$

Then (32.1), (32.4) and (32.5) imply that

$$\begin{aligned}\frac{1}{|\mathfrak{G}|}|\mathfrak{G}_0| &\geq 1-\left(1-\frac{1}{w_2}-\frac{1}{q}+\frac{1}{qw_2}\right)-\left(\frac{1}{w_2}-\frac{1}{w_2|\mathfrak{T}'|}\right)\\ &-\left(\frac{1}{qu}-\frac{1}{qu|\mathfrak{H}|}\right)=\frac{1}{q}-\frac{1}{w_2q}-\frac{1}{qu}+\frac{1}{w_2|\mathfrak{T}'|}\\ &+\frac{1}{qu|\mathfrak{H}|}>\frac{1}{q}-\frac{1}{3q}-\frac{1}{3q}=\frac{1}{3q}\,.\end{aligned} \tag{32.6}$$

By (32.3)

$$\frac{1}{|\mathfrak{G}|}\sum_{\mathfrak{G}_0}|\lambda^\tau(G)|^2 \leq 1-\left(1-\frac{w_2}{|\mathfrak{T}'|}\right)=\frac{w_2}{|\mathfrak{T}'|}\,. \tag{32.7}$$

By Corollary 31.1.2 w_2 is a prime and $\mathfrak{H}\mathfrak{U}$ is a Frobenius group. Hence by Lemma 13.1 $\eta_{01}, \cdots, \eta_{0,w_2-1}$ are algebraically conjugate characters whose values lie in $\mathcal{Q}_{w_2}$. Every element whose order is divisible by w_2 lies in $\mathfrak{G}_2 \cup \mathfrak{G}_3$. Thus $\eta_{0j}(G) = \eta_{01}(G)$ is a rational integer for $G \in \mathfrak{G}_0$ and $1 \leq j \leq w_2 - 1$. Now Corollary 31.1.1 implies that $1 - \lambda^\tau(G) + (w_2 - 1)\eta_{01}(G) = 0$ for $G \in \mathfrak{G}_0$. Hence $\lambda^\tau(G) \equiv 1 \pmod 2$ for $G \in \mathfrak{G}_0$. Therefore $|\lambda^\tau(G)| \geq 1$ for $G \in \mathfrak{G}_0$. Now (32.6) and (32.7) imply that

$$\frac{w_2}{|\mathfrak{T}'|} > \frac{1}{3q}$$

or

$$3qw_2 > |\mathfrak{T}'|\,. \tag{32.8}$$

Since $\mathfrak{T}'' \neq 1$, (32.8) yields that $3w_2 > |\mathfrak{T}' : \mathfrak{T}''|$ and $|\mathfrak{T}''| = q$. Thus, $\mathfrak{W}_2$ acts irreducibly on $\mathfrak{T}'/\mathfrak{T}''$. Therefore $\mathfrak{T}'$ is an extra special group. Let $|\mathfrak{T}' : \mathfrak{T}''| = q^{2c}$. Then by Lemma 13.6, $w_2 \leq (q^c + 1)/2$. Thus (32.8) implies that $q^{2c} < (3/2)(q^c + 1) < 2q^c$. Hence $q^c < 2$ which is not the case. The proof is complete.

COROLLARY 32.1.1. *Let $\mathfrak{S}$ be a subgroup of type* II, III *or* IV. *Let $\mathcal{S}$ have the same meaning as in Section 29. Then $\mathcal{S}$ is coherent.*

Proof. This is an immediate consequence of Theorems 30.1 and 32.1.

33. Subgroups of Type I

LEMMA 33.1. *Let $\mathfrak{L}$ be a maximal subgroup of $\mathfrak{G}$ and let $\hat{\mathfrak{L}}$ have the same meaning as in section* 14. *If $\mathfrak{L}$ is of type* I *with Frobenius* *kernel* $\mathfrak{H}$ *let* $\mathscr{S}$ *be the set of all irreducible characters of* $\mathfrak{L}$ *which do not have* $\mathfrak{H}$ *in their kernel. If* $\mathfrak{L}$ *is of type* II, III *or* IV *let* $\mathscr{S}$ *be the set of characters of* $\mathfrak{L}$ *each of which is induced by a non principal irreducible character of* $\mathfrak{L}'$ *which vanishes outside* $\hat{\mathfrak{L}}$. *Let* $\mathfrak{L}_i$ *have the same meaning as in section* 9 *and let* $\mathfrak{A}_L$ *be defined by* (9.2). *If* $\lambda \in \mathscr{S}$ *then* λ^τ *can be defined. Furthermore* λ^τ *is constant on* $\mathfrak{A}_L$ *for* $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$.

Proof. Since $|\mathfrak{G}|$ is odd Lemmas 10.1 and 13.9 imply that λ^τ can always be defined as $\{\lambda, \bar{\lambda}\}$ is coherent.

If $L \in \mathfrak{L}_0$ then $\mathfrak{A}_L = \{L\}$ and there is nothing to prove. If $L \in \mathfrak{L}_i$ with $i \neq 0$ let $\mathfrak{H}_i$ be a supporting subgroup of $\hat{\mathfrak{L}}$ such that $C(L) \subseteq \mathfrak{N}_i = N(\mathfrak{H}_i)$. If $\mathfrak{N}_i$ is of type I then the result follows from Lemmas 4.5 and 10.3. By definition $\mathfrak{N}_i$ cannot be of type III or IV. If $\mathfrak{N}_i$ is of type II then the result is a simple consequence of Corollary 32.1.1.

The main purpose of this section is to prove

THEOREM 33.1. *Every subgroup of type* I *is a Frobenius group.*

All the remaining lemmas in this section will be proved under the following assumption.

Hypothesis 33.1.
$\mathfrak{G}$ *contains a subgroup of type* I *which is not a Frobenius group.*

If Hypothesis 33.1 is satisfied the following notation will be used.

σ is a set of primes defined as follows: $p_i \in \sigma$ if and only if $\mathfrak{G}$ contains a subgroup $\mathfrak{M}_i$ of type I with Frobenius kernel $\mathfrak{K}_i$ such that a S_{p_i}-subgroup of $\mathfrak{M}_i/\mathfrak{K}_i$ is not cyclic.

$p = p_k$ is the smallest prime in σ. $\mathfrak{M} = \mathfrak{M}_k$; $\mathfrak{K} = \mathfrak{K}_k$.

$\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{M}$.

$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ with $\mathfrak{P}_0 \subseteq \mathfrak{P}$.

$\mathfrak{L}$ is a maximal subgroup of $\mathfrak{G}$ such that $N(\Omega_1(\mathfrak{P}_0)) \subseteq \mathfrak{L}$.

$\mathscr{S}$ has the same meaning as in Lemma 33.1.

If $\mathfrak{L}$ is of type I let $\mathfrak{U}$ be the Frobenius kernel of $\mathfrak{L}$. Let $\mathfrak{L} = \mathfrak{U}\mathfrak{E}$ with $\mathfrak{U} \cap \mathfrak{E} = 1$.

If $\mathfrak{L}$ is of type II, III or IV let $\mathfrak{H}$ be the maximal normal nilpotent S-subgroup of $\mathfrak{L}$. Let $\mathfrak{U}$ be a complement of $\mathfrak{H}$ in $\mathfrak{L}'$ and let $\mathfrak{W}_1$ be a complement of $\mathfrak{L}'$ in $\mathfrak{L}$ with $\mathfrak{W}_1 \subseteq N(\mathfrak{U})$.

LEMMA 33.2. $\mathfrak{L}$ *is the unique maximal subgroup of* $\mathfrak{G}$ *which contains* $N(\Omega_1(\mathfrak{P}_0))$. *Furthermore* $\mathfrak{L}$ *is either a Frobenius group or* $\mathfrak{L}$ *is of type* III *or* IV *and* $\mathfrak{P}$ *can be chosen to lie in* $\mathfrak{U}$.

Proof. By Theorem 32.1 $\mathfrak{L}$ is not of type V. If $\mathfrak{L}$ is of type II, III or IV then $\mathfrak{P}_0 \subseteq \mathfrak{L}'$ since $\mathfrak{P}_0$ is not cyclic. Since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$ it may be assumed that $\mathfrak{P}_0 \subseteq \mathfrak{U}$.

There exists $P \in \Omega_1(\mathfrak{P}_0)$ such that $C(P) \subseteq \mathfrak{M}$. Thus either $\mathfrak{P} = \mathfrak{P}_0$ or $Z(\mathfrak{P})$ is cyclic and $Z(\mathfrak{P}) \subseteq \mathfrak{P}_0$. If a S_p-subgroup of $\mathfrak{U}$ is abelian then $\mathfrak{P}_0$ is the S_p-subgroup of $\mathfrak{U}$. Hence $\Omega_1(\mathfrak{P}_0)$ char $\mathfrak{U}$ and so $N(\mathfrak{U}) \subseteq N(\Omega_1(\mathfrak{P}_0)) \subseteq \mathfrak{L}$. Therefore $\mathfrak{L}$ is of type III or IV and $\mathfrak{P} = \mathfrak{P}_0 \subseteq \mathfrak{U}$. By definition $\mathfrak{L}$ is the unique maximal subgroup which contains $N(\Omega_1(\mathfrak{P}_0))$. If the S_p-subgroup of $\mathfrak{U}$ is not abelian then $\mathfrak{L}$ is of type IV and it may be assumed that $\mathfrak{P} \subseteq \mathfrak{U}$. Then $\Omega_1(\mathfrak{P}_0) \subseteq \hat{\mathfrak{L}}$ and in this case also $\mathfrak{L}$ is the unique maximal subgroup of $\mathfrak{G}$ which contains $N(\Omega_1(\mathfrak{P}_0))$.

Suppose that $\mathfrak{L}$ is of type I. Let $\mathfrak{P}_1$ be a S_p-subgroup of $\mathfrak{L}$ with $\mathfrak{P}_0 \subseteq \mathfrak{P}_1$. If $p \in \pi(\mathfrak{E})$, then $\mathfrak{P}_1$ is abelian. Thus, $\mathfrak{P}_0 = \mathfrak{P}_1$ and so $\mathfrak{P}_0 = \mathfrak{P}$. Hence, $\mathfrak{P}$ is an abelian S_p-subgroup of $\mathfrak{G}$. By construction, $N(\mathfrak{P}) \subseteq \mathfrak{L}$. Hence, $\mathfrak{P} \subseteq \mathfrak{L}'$, by Burnside's transfer theorem. Since $|\mathfrak{L}|$ is odd, if an element of $N(\mathfrak{P})$ induces an automorphism of $\mathfrak{P}$ of prime order q, then $q < p$. By the minimal nature of p, a S_q-subgroup of $\mathfrak{L}$ is cyclic. Let $\mathfrak{P}^* = \mathfrak{P} \cap C(\mathfrak{U})$. Since $\mathfrak{L}$ is of type I, $\mathfrak{P}^*$ is cyclic. We can now find a prime q such that some element $N(\mathfrak{P})$ induces an automorphism of order q on $\mathfrak{P}/\mathfrak{P}^*$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{P}$. Since $q < p$, $\mathfrak{Q}$ normalizes $\mathfrak{P}$, and $\mathfrak{Q}$ is cyclic. Since $\mathfrak{Q}\mathfrak{U}$ is a Frobenius group, $\Omega_1(\mathfrak{Q})$ centralizes $\mathfrak{P}/\mathfrak{P}^*$. Let $\mathfrak{P}_0^* = C_{\mathfrak{P}}(\Omega_1(\mathfrak{Q}))$. Then $\mathfrak{P} = \mathfrak{P}^*\mathfrak{P}_0^*$, and $[\mathfrak{Q}, \mathfrak{P}_0^*] \not\subseteq \mathfrak{P}^*$.

Let $\mathfrak{L}^*$ be a maximal subgroup containing $N(\Omega_1(\mathfrak{Q}))$. The minimal nature of p implies that $\mathfrak{Q} \subseteq \mathfrak{L}^{*\prime}$. Hence, by Lemma 8.13, $\mathfrak{Q}$ centralizes every chief p-factor of $\mathfrak{L}^*$, so $\mathfrak{Q}$ centralizes $\mathfrak{P}_0^*$, which is not the case. We conclude that $p \notin \pi(\mathfrak{E})$. Therefore $p \in \pi(\mathfrak{U})$. Hence $\mathfrak{P} \subseteq \mathfrak{U}$. $\mathfrak{U}$ is not a T.I. set since $\mathfrak{P}$ is not a T.I. set in $\mathfrak{G}$. This yields that either $p \in \pi_1^*$ or $m(\mathfrak{U}) = 2$. In either case this implies that every prime divisor of $|\mathfrak{E}|$ is less than p. The minimal nature of p now implies that $\mathfrak{L}$ is a Frobenius group.

The previous parts of the lemma imply that if $\mathfrak{L}_1$ is a maximal subgroup of $\mathfrak{G}$ which contains $N(\Omega_1(\mathfrak{P}_0))$ then $\mathfrak{L}_1$ is a Frobenius group and p divides the order of the Frobenius kernel of $\mathfrak{L}_1$. If $\mathfrak{P}$ is abelian then $\mathfrak{P} = \mathfrak{P}_0$ and $\mathfrak{L} = \mathfrak{L}_1 = N(\Omega_1(\mathfrak{P}_0))$. If $\mathfrak{P}$ is non abelian then $\mathfrak{L} = \mathfrak{L}_1 = N(Z(\mathfrak{P}))$. The uniqueness of $\mathfrak{L}$ is proved.

LEMMA 33.3. *There exists an irreducible character $\lambda \in \mathscr{L}$ which does not have $\mathfrak{P}$ in its kernel such that $\lambda(1) \mid (p-1)$ or $\lambda(1) \mid (p+1)$.*

Proof. Let λ be a character of $\mathfrak{L}$ which does not have $\mathfrak{P}$ in its kernel and is induced by a linear character of $\mathfrak{U}$ if $\mathfrak{L}$ is a Frobenius group and by a linear character of $\mathfrak{L}'$ if $\mathfrak{L}$ is of type III or IV.

Either $\mathfrak{P} = \mathfrak{P}_0$ and so $m(\mathfrak{P}) = 2$, or $\boldsymbol{Z}(\mathfrak{P})$ is cyclic. In either case this implies that if $q \in \pi\,(\boldsymbol{N}(\mathfrak{P})/\boldsymbol{C}(\mathfrak{P}))$, $q \neq p$ then $q \mid (p+1)$ or $q \mid (p-1)$. If $\mathfrak{L}$ is of type III or IV then $\lambda(1) = |\mathfrak{W}_1|$ is a prime and the result follows. Suppose that $\mathfrak{L}$ is a Frobenius group. If $p \in \pi_1^*$ then $|\mathfrak{E}| = \lambda(1)$ has the required properties by assumption. If $p \notin \pi_1^*$ then $\mathfrak{H}$ is abelian since $\mathfrak{H}$ is not a T.I. set in $\mathfrak{G}$. Thus $\mathfrak{P} = \mathfrak{P}_0$ and $m(\mathfrak{P}) = 2$. Suppose that $q_1, q_2 \in \pi(\mathfrak{E})$ where $q_1 \mid (p-1)$ and $q_2 \mid (p+1)$. Then an element of $\mathfrak{E}$ of order q_1 acts as a scalar on $\mathfrak{P}$. There exists $P \in \mathfrak{P}^\sharp$ such that $N(\langle P \rangle) \subseteq \mathfrak{M}$. Thus $\mathfrak{M}$ contains a Frobenius group of order pq_1 which is not the case. Therefore every prime in $\pi(\mathfrak{E})$ divides $(p-1)$ or every prime in $\pi(\mathfrak{E})$ divides $(p+1)$. Since $(p+1, p-1) = 2$ this yields that $|\mathfrak{E}| \mid (p+1)$ or $|\mathfrak{E}| \mid (p-1)$. The lemma follows since $\lambda(1) = |\mathfrak{E}|$.

LEMMA 33.4. *Let λ be the character defined in Lemma 33.3. Then*

$$\lambda^\tau(L) = \lambda(L) \quad \textit{for } L \in \hat{\mathfrak{L}}^\sharp$$

Proof. Set $e = |\mathfrak{L} : \mathfrak{L}'|$. Observe that if $\mathfrak{L}$ is a Frobenius group, then since $p \in \pi^*$, it follows that $\mathfrak{L}' = \mathfrak{U}$, so that $\lambda(1) = e$. This equality also holds if $\mathfrak{L}$ is of type III or IV.

Set $\alpha = (\tilde{1}_{\mathfrak{L}'} - \lambda)$ so that $\alpha^\tau = 1_{\mathfrak{G}} - \lambda^\tau + \Delta$, where Δ is a generalized character of $\mathfrak{G}$ orthogonal to $1_{\mathfrak{G}}$. Let $\lambda = \lambda_1, \cdots, \lambda_f$ be the characters in $\mathscr{L}$ of degree e. Since e divides $(p+1)/2$ or $(p-1)/2$, it follows that $f > e + 1$, and so $(\Delta, \lambda_i^\tau) = 0$, $1 \leqq i \leqq f$.

We next show that $\mathscr{L}$ is coherent. If $\mathfrak{L}$ is a Frobenius group, the coherence of $\mathscr{L}$ follows from Lemma 11.1 and the fact that $\mathfrak{L}$ is of type I.

Suppose $\mathfrak{L}$ is of type III or IV. Then Hypothesis 11.1 and (11.2) are satisfied with the present $\mathfrak{L}$ in the role of $\mathfrak{L}_0$, $\mathfrak{H}$ in the role of $\mathfrak{H}_0$, and $\mathfrak{L}'/\mathfrak{H}$ in the role of $\mathfrak{H}$. By Lemma 11.1, we may assume that $|\mathfrak{L}' : \mathfrak{L}''| \leqq 4|\mathfrak{L} : \mathfrak{L}'|^2 + 1$. Hence, $|\mathfrak{L}' : \mathfrak{L}''| = p^2$ and $e = (p+1)/2$, so that $\mathfrak{P} = \mathfrak{U}$. If $\mathfrak{P}$ is non abelian, then e divides $(p-1)/2$. Hence, we may assume that $\mathfrak{P}$ is abelian of order p^2 and $\mathfrak{L}$ is of type III. By Theorem 29.1 (i), no element of $\mathfrak{P}^\sharp$ centralizes $\mathfrak{H}$. This implies that if $\mu_1, \cdots, \mu_{f'}$ are the characters in $\mathscr{L}$ of degree pe, then $f' \geqq 2p$. Hence, $(\Delta, \mu_j^\tau) = 0$, $1 \leqq j \leqq f'$.

Let $\beta = (p\lambda_1 - \mu_1)$, so that $\beta^\tau = p\lambda_1^\tau - x\sum_i \lambda_i^\tau - \mu_1^\tau + \Delta_1$, with $(\Delta_1, \lambda_i^\tau) = 0$. If $x = 0$, the coherence of $\mathscr{L}$ follows from Theorem 30.1. As $\|\beta^\tau\|^2 = p^2 + 1$, and $f = 2(p-1)$, it follows that $0 \leqq x < 2$, and $\|\Delta_1\|^2 \leqq 2$. Hence, $x = 1$ and $(\Delta_1, \mu_j^\tau) = 0$. But now $(\alpha^\tau, \beta^\tau) = (\alpha, \beta) = -p = -(p-1) + (\Delta, \Delta_1)$, so that $(\Delta, \Delta_1) = -1$. This is not the case as Δ and Δ_1 are real valued generalized characters of $\mathfrak{G}$

orthogonal to $1_{\mathfrak{G}}$. The coherence of $\mathscr{L}$ is proved in all cases.

Since $(\Delta, \lambda^\tau) = 0$, the lemma follows from Lemmas 9.4 and 33.1.

LEMMA 33.5. *Let* λ *be the character defined in Lemma* 33.3. *Then*

$$\frac{1}{|\mathfrak{M}|} \Sigma_{\mathfrak{K}^\sharp} |\lambda^\tau(K)|^2 < \frac{\lambda(1)^2}{|\mathfrak{L}|} .$$

Proof. Let $\mathfrak{G}_0$ be the set of all elements in $\mathfrak{G}$ which are conjugate to an element of $\mathfrak{A}_L$ for some $L \in \hat{\mathfrak{L}}^\sharp$. Let $\mathfrak{G}_1$ be the set of all elements in $\mathfrak{G}$ which are conjugate to an element of $\mathfrak{A}_K$ for some $K \in \mathfrak{K}^\sharp$. No subgroup of $\mathfrak{G}$ can be a supporting subgroup for both $\hat{\mathfrak{L}}$ and $\hat{\mathfrak{M}}$. If $\mathfrak{L}$ were a supporting subgroup of $\hat{\mathfrak{M}}$ then p would not be minimal in the set σ. Thus $\mathfrak{G}_0$ is disjoint from $\mathfrak{G}_1$. Therefore by Lemmas 9.5, 4.5, 10.3, 33.1 and 33.4

$$\begin{aligned} \frac{1}{|\mathfrak{M}|} \Sigma_{\mathfrak{K}^\sharp} |\lambda^\tau(K)|^2 &= \frac{1}{|\mathfrak{G}|} \Sigma_{\mathfrak{G}_1} |\lambda^\tau(G)|^2 < 1 - \frac{1}{|\mathfrak{G}|} \Sigma_{\mathfrak{G}_0} |\lambda^\tau(G)|^2 \\ &= 1 - \frac{1}{|\mathfrak{L}|} \Sigma_{\hat{\mathfrak{L}}^\sharp} |\lambda^\tau(G)|^2 = 1 - \frac{1}{|\mathfrak{L}|} \Sigma_{\hat{\mathfrak{L}}^\sharp} |\lambda(G)|^2 \\ &= 1 - \left(1 - \frac{\lambda(1)^2}{|\mathfrak{L}|}\right) = \frac{\lambda(1)^2}{|\mathfrak{L}|} . \end{aligned}$$

LEMMA 33.6. *Let* $\mathfrak{M} = \mathfrak{K}\mathfrak{F}$ *where* $\mathfrak{F} = \mathfrak{M} \cap \mathfrak{L}$. *Then there exists* F *in* $(\mathfrak{P}_0 \cap Z(\mathfrak{F}))^\sharp$ *such that* $C_{\mathfrak{K}}(F) \not\subseteq \mathfrak{K}'$. *Furthermore* $\mathfrak{M}$ *satisfies Hypothesis* 28.1.

Proof. If $\mathfrak{L}$ is of type I, then $\mathfrak{F} \subseteq \mathfrak{U}$. Thus, $\mathfrak{F}$ is nilpotent and hence abelian. The result follows from 3.16 (ii) and the fact that $\mathfrak{P}_0$ is not cyclic.

Suppose $\mathfrak{L}$ is not of type I. If $\mathfrak{F} \not\subseteq \mathfrak{U}\mathfrak{H}$, then we may assume that $\mathfrak{W}_1 \subseteq \mathfrak{F}$. Then $\mathfrak{W}_1\mathfrak{P}_0$ is a Frobenius group and $\mathfrak{W}_1\mathfrak{P}_0 \subseteq \mathfrak{F}$. By 3.16 (ii), $\mathfrak{W}_1$ centralizes an element of $\mathfrak{K}^\sharp$. Since $|\mathfrak{W}_1|$ is a prime, this contradicts the fact that $\mathfrak{M}$ contains a Frobenius group of order $|\mathfrak{W}_1\mathfrak{K}|$. Thus, $\mathfrak{F} \subseteq \mathfrak{U}\mathfrak{H}$. Let $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{H}$. Since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$, we get that $\mathfrak{F}_1$ is a cyclic normal S-subgroup of $\mathfrak{F}$. If $\mathfrak{F}_1 = 1$, then $\mathfrak{F}$ is abelian and the result follows from 3.16 (ii).

Assume now that $\mathfrak{F}_1 \neq 1$. We may assume that $\mathfrak{F} = \mathfrak{F}_1(\mathfrak{F} \cap \mathfrak{U})$. If $\Omega_1(\mathfrak{P}_0)$ does not centralize $\mathfrak{F}_1$, then there exists $\mathfrak{P}^* \subseteq \Omega_1(\mathfrak{P}_0)$ such that $\mathfrak{F}_1\mathfrak{P}^*$ is a Frobenius group. Hence, $C_{\mathfrak{K}}(\mathfrak{P}^*) \neq 1$ by 3.16 (ii). But in this case, $\mathfrak{P}^*$ lies in no normal abelian subgroup of $\mathfrak{F}$ contrary to the definition of groups of Frobenius type. Thus, $\Omega_1(\mathfrak{P}_0)$ centralizes $\mathfrak{F}_1$. Since $\mathfrak{F} \cap \mathfrak{U}$ is abelian and $\mathfrak{F} = \mathfrak{F}_1(\mathfrak{F} \cap \mathfrak{U})$, this implies that $\Omega_1(\mathfrak{P}_0) \subseteq$

$\boldsymbol{Z}(\mathfrak{F})$. The lemma now follows from 3.16 (ii).

LEMMA 33.7. *Let $\mathscr{M}$ be the set of all irreducible characters of $\mathfrak{M}$ which do not have $\mathfrak{K}$ in their kernel. Let λ be the character defined in Lemma 33.3. If $\mathscr{M}$ is coherent then λ^τ is constant on $\mathfrak{K}^\#$.*

Proof. Let $\mathfrak{H}_1, \cdots \mathfrak{H}_s$ be a set of supporting subgroups of $\hat{\mathfrak{M}}$ in $\mathfrak{G}$, and let $\mathfrak{N}_i = \boldsymbol{N}_{\mathfrak{G}}(\mathfrak{H}_i)$. By definition,

$$\hat{\mathfrak{M}} = \bigcup_{K \in \mathfrak{K}^\#} C_{\mathfrak{M}}(K)\ .$$

Suppose $M \in \hat{\mathfrak{M}}^\#$ and $C_{\mathfrak{G}}(M) \not\subseteq \mathfrak{M}$. We will show that $M \in \mathfrak{K}$. For otherwise, some power of M is $\mathfrak{M}$-conjugate to an element A of $\mathfrak{F}^\#$. Since $\mathfrak{K}$ is a supporting subgroup of some tamely imbedded subset of $\mathfrak{G}$, it follows that $C_{\mathfrak{G}}(A) \subseteq \mathfrak{M}$. Hence, M is in $\mathfrak{K}^\#$.

We next show that $\mathfrak{N}_i$ is of type I or II, $1 \leqq i \leqq s$. Suppose $\mathfrak{N}_i$ is not of type I. Then $\mathfrak{N}_i = \mathfrak{H}_i(\mathfrak{N}_i \cap \mathfrak{M})$, and we assume that $\mathfrak{N}_i \cap \mathfrak{M} = (\mathfrak{N}_i \cap \mathfrak{K})(\mathfrak{N}_i \cap \mathfrak{F})$. Since $\mathfrak{H}_i$ is a supporting subgroup of $\hat{\mathfrak{M}}$, we may choose M in $\hat{\mathfrak{M}}$ so that $C_{\mathfrak{G}}(M) \subseteq \mathfrak{N}_i$, $C_{\mathfrak{G}}(M) \not\subseteq \mathfrak{M}$. By the first paragraph, $M \in \mathfrak{K}^\#$. Hence, $\mathfrak{N}_i \cap \mathfrak{K} \neq 1$. If $\boldsymbol{N}_{\mathfrak{G}}(\mathfrak{N}_i \cap \mathfrak{K}) \subseteq \mathfrak{N}_i$, then by a well known property of nilpotent groups, we have $\mathfrak{K} = \mathfrak{N}_i \cap \mathfrak{K}$, so that $\mathfrak{M} \subseteq \mathfrak{N}_i$, which is not the case. Hence, $\boldsymbol{N}_{\mathfrak{G}}(\mathfrak{N}_i \cap \mathfrak{K}) \not\subseteq \mathfrak{N}_i$, so $\mathfrak{N}_i$ is not of type III or IV; $\mathfrak{N}_i$ is of type II.

Let a be the least common multiple of the orders of all elements of $\hat{\mathfrak{L}}$. We will show that $(a, |\mathfrak{K}|) = (a, |\mathfrak{H}_i|) = 1, 1 \leqq i \leqq s$. If $\mathfrak{L}$ is of type I, then $\mathfrak{L}$ is a Frobenius group, so a divides $|\mathfrak{U}|$, and we only need to verify that $\mathfrak{L}$ is not conjugate to $\mathfrak{M}$ or $\mathfrak{N}_i$, $1 \leqq i \leqq s$. As none of the groups $\mathfrak{M}, \mathfrak{N}_1, \cdots, \mathfrak{N}_s$ is a Frobenius group, this is clear. Suppose $\mathfrak{L}$ is of type III of IV, so that $\mathfrak{L} = \mathfrak{H}\mathfrak{U}\mathfrak{W}_1$, $\hat{\mathfrak{L}} = \mathfrak{H}\mathfrak{U}$. Since none of $\mathfrak{M}, \mathfrak{N}_1, \cdots, \mathfrak{N}_s$ is of type III or IV, we have $(|\mathfrak{H}|, |\mathfrak{K}|) = (|\mathfrak{H}|, |\mathfrak{H}_i|) = 1, 1 \leqq i \leqq s$. Since $\boldsymbol{N}_{\mathfrak{G}}(\mathfrak{U}) \subseteq \mathfrak{L}$, it is trivial that $(|\mathfrak{U}|, |\mathfrak{K}|) = (|\mathfrak{U}|, |\mathfrak{H}_i|) = 1$.

We appeal to Lemma 10.4 and conclude that λ^τ is rational on $\mathfrak{K}$ and on every supporting subgroup of $\hat{\mathfrak{M}}$.

Let $\mathfrak{H}_i$ be a supporting subgroup of $\hat{\mathfrak{M}}$ and let α be a character of $\mathfrak{H}_i$ with $(\alpha, 1_{\mathfrak{H}_i}) = 0$. Let μ_1, μ_2 be irreducible characters of $\mathfrak{N}_i$ with $\mu_{1|\mathfrak{H}_i} = \mu_{2|\mathfrak{H}_i} = \alpha$. Then $\|(\mu_1 - \mu_2)^*\|^2 = 2$ and no irreducible character of $\mathfrak{G}$ appearing in $(\mu_1 - \mu_2)^*$ is rational on $\mathfrak{H}_i$. Thus, $(\lambda^\tau, (\mu_1 - \mu_2)^*) = 0$. If $\mathfrak{N}_i$ is of type I, then Hypothesis 10.2 is satisfied with our present $\hat{\mathfrak{M}}$ in the role of $\mathfrak{L}$. If $\mathfrak{N}_i$ is of type II, then a complement to $\mathfrak{H}_i$ in $\mathfrak{N}_i'$ is abelian, and again Hypothesis 10.2 is satisfied. Hence, by Lemma 10.2, λ^τ is constant on the cosets of $\mathfrak{H}_i$ in $\mathfrak{N}_i - \mathfrak{H}_i$, and in particular is constant on all the sets $\mathfrak{A}_M$, $M \in \hat{\mathfrak{M}}$. As $\mathscr{M}$ is assumed coherent, an appeal to Lemma 10.5 completes the proof of this lemma.

Theorem 33.1 will now be proved by showing that Hypothesis 33.1 leads to a contradiction.

Choose $P \in \mathfrak{P}_0^\#$ and $K \in C(P) \cap \mathfrak{K}^\#$. By Lemmas 33.1 and 33.4

$$(33.2) \qquad \lambda^\tau(KP) = \lambda^\tau(P) = \lambda(P) .$$

Let $\mathfrak{p}$ be a prime divisor of p in $\mathscr{O}_{|\mathfrak{G}|}$. By Lemma 4.2

$$(33.3) \qquad \lambda^\tau(K) \equiv \lambda^\tau(PK) \pmod{\mathfrak{p}}$$

$$(33.4) \qquad \lambda(P) \equiv \lambda(1) \pmod{\mathfrak{p}} .$$

Now (33.2), (33.3) and (33.4) yield that

$$\lambda^\tau(K) \equiv \lambda^\tau(PK) \equiv \lambda(P) \equiv \lambda(1) \pmod{\mathfrak{p}} .$$

By Lemma 10.4 $\lambda^\tau(K)$ is rational. Thus

$$\lambda^\tau(K) \equiv \lambda(1) \pmod{p} .$$

Since $\lambda(1) \leqq (p+1)/2$ by Lemma 33.3, we get that

$$(33.5) \qquad |\lambda^\tau(K)| \geqq \lambda(1) - 1 \quad \text{for } K \in \mathfrak{K}^\#, \quad C_{\mathfrak{P}_0}(K) \neq 1 .$$

If every element in $\mathfrak{K}^\#$ commutes with an element of $\mathfrak{P}_0^\#$ then (33.5) implies that

$$(33.6) \qquad |\lambda^\tau(K)| \geqq \lambda(1) - 1 \quad \text{for } K \in \mathfrak{K}^\# .$$

If not every element in $\mathfrak{K}^\#$ commutes with an element of $\mathfrak{P}_0^\#$ then λ^τ is constant on $\mathfrak{K}^\#$ by Lemmas 28.2, 33.6 and 33.7. As (33.5) holds for at least one element in $\mathfrak{K}^\#$ we get that (33.6) holds in any case. Now Lemma 33.5 and (33.6) imply that

$$\frac{\lambda(1)^2}{|\mathfrak{L}|} > \frac{\{|\mathfrak{K}| - 1\}}{|\mathfrak{M}|} \{\lambda(1) - 1\}^2 .$$

This can be written as

$$(33.7) \qquad \frac{|\mathfrak{M} : \mathfrak{K}|}{|\mathfrak{L}|} > \frac{\{|\mathfrak{K}| - 1\}}{|\mathfrak{K}|} \left(\frac{e-1}{e}\right)^2 , \quad \text{where } e = \lambda(1) .$$

Since $|\mathfrak{L} : \mathfrak{L} \cap \mathfrak{M}| > 1$ and $\mathfrak{L} \cap \mathfrak{M}$ is a complement to $\mathfrak{K}$ in $\mathfrak{M}$, (33.7) yields that

$$\frac{1}{3} > \frac{\{|\mathfrak{K}| - 1\}}{|\mathfrak{K}|} \left(1 - \frac{1}{e}\right)^2 \geqq \frac{\{|\mathfrak{K}| - 1\}}{|\mathfrak{K}|} \left(\frac{2}{3}\right)^2 .$$

Hence $3|\mathfrak{K}|/4 > |\mathfrak{K}| - 1$ or $|\mathfrak{K}| < 4$. Thus $|\mathfrak{K}| = 3$ and a S_3-subgroup of $\mathfrak{G}$ is cyclic contrary to the simplicity of $\mathfrak{G}$ and the fact that $|\mathfrak{G}|$

is odd. This contradiction completes the proof of Theorem 33.1.

THEOREM 33.2. *$\mathfrak{G}$ contains a subgroup of type* II.

Proof. Suppose false. Then by Theorems 14.1 and 33.1, every maximal subgroup of $\mathfrak{G}$ is a Frobenius group. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$ and let $\mathfrak{E}$ be a complement to the Frobenius kernel of $\mathfrak{M}$. We will show that $\mathfrak{E}$ is abelian. Suppose false.

Let σ be the set of primes p such that for some maximal subgroup $\mathfrak{M}_1$ with Frobenius kernel $\mathfrak{H}_1$ and complement $\mathfrak{E}_1$, a S_p-subgroup of $\mathfrak{E}_1$ is not in $Z(\mathfrak{E}_1)$. Let p be the least prime in σ. We may suppose that a S_p-subgroup $\mathfrak{P}$ of $\mathfrak{E}$ is not contained in $Z(\mathfrak{E})$. Then $\mathfrak{P} \cap \mathfrak{E}' = 1$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\Omega_1(\mathfrak{P}))$. Since $\Omega_1(\mathfrak{P}) \subseteq Z(\mathfrak{E})$, $\mathfrak{E} \subseteq \mathfrak{M}_1$. If $\mathfrak{P}$ is contained in the Frobenius kernel $\mathfrak{K}$ of $\mathfrak{M}_1$, then so is $[\mathfrak{P}, \mathfrak{E}] \neq 1$. This is impossible as $\mathfrak{E}$ does not centralize $\mathfrak{P}$, while $\mathfrak{K}$ is nilpotent. Hence $\mathfrak{E} \cap \mathfrak{K} = 1$. Since $\mathfrak{M}_1'' \subseteq \mathfrak{K}$, it follows that $\mathfrak{P}$ is not contained in $\mathfrak{M}_1'$, and that a S_p-subgroup of $\mathfrak{M}_1$ is cyclic. Hence, by Burnside's transfer theorem, $\mathfrak{G}$ is not simple. Since this is not possible, $\mathfrak{E}$ is abelian.

Let $G \in \mathfrak{G}^\sharp$. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$ containing $C(G)$. It follows that $C(G)$ is nilpotent. Hence, $\mathfrak{G}$ is solvable by the main theorem of [10]. The proof is complete.

34. The Subgroups $\mathfrak{S}$ and $\mathfrak{T}$

By Theorems 32.1 and 33.2 $\mathfrak{G}$ contains two subgroups $\mathfrak{S}$ and $\mathfrak{T}$, each of which is of type II, III or IV and which satisfy Condition (ii)(b) of Theorem 14.1. The following notation will be used throughout the rest of this chapter. This differs slightly from that introduced previously.

$$\mathfrak{S} = \mathfrak{Q}^*\mathfrak{S}', \quad \mathfrak{T} = \mathfrak{P}^*\mathfrak{T}', \quad |\mathfrak{Q}^*| = q, \quad |\mathfrak{P}^*| = p .$$

Thus p and q are both primes. Let $\mathfrak{P}$ be the S_p-subgroup of $\mathfrak{S}$ and let $\mathfrak{Q}$ be the S_q-subgroup of $\mathfrak{T}$. Then $\mathfrak{P}^* \subseteq \mathfrak{P}, \mathfrak{Q}^* \subseteq \mathfrak{Q}$. Let

$$\mathfrak{W} = \mathfrak{P}^*\mathfrak{Q}^* , \qquad \hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{P}^* - \mathfrak{Q}^* .$$

Let $\mathfrak{U}$ be a complement of $\mathfrak{P}$ in $\mathfrak{S}'$ and let $\mathfrak{V}$ be a complement of $\mathfrak{Q}$ in $\mathfrak{T}'$. By 3.16 (i) $\mathfrak{U}$ and $\mathfrak{V}$ are nilpotent, thus

$$\bigcup_{P \in \mathfrak{P}^\sharp} C(P) = \hat{\mathfrak{S}} ,$$

if $\mathfrak{S}$ is of type II and

$$\bigcup_{Q \in \mathfrak{Q}^\sharp} C(Q) = \hat{\mathfrak{T}} ,$$

if $\mathfrak{T}$ is of type II. Let

$$\mathfrak{C} = C_{\mathfrak{U}}(\mathfrak{P}), \qquad \mathfrak{D} = C_{\mathfrak{V}}(\mathfrak{Q}).$$

If $\mathfrak{S}$ is of type III or IV let $\mathfrak{U}^* = \mathfrak{U}$. If $\mathfrak{S}$ is of type II then a maximal subgroup $\mathfrak{M}$ which contains $N(\mathfrak{U})$ is not conjugate to $\mathfrak{T}$ since $\mathfrak{M}$ is not q-closed. Hence by Theorem 33.1 $\mathfrak{M}$ is a Frobenius group. Let $\mathfrak{U}^*$ be the Frobenius kernel of $\mathfrak{M}$. Thus $\mathfrak{U} \subseteq \mathfrak{U}^*$. Define $\mathfrak{V}^*$ similarly. Let

$$|\mathfrak{C}| = c, \quad |\mathfrak{D}| = d, \quad |\mathfrak{U}| = uc, \quad |\mathfrak{V}| = vd,$$
$$|\mathfrak{U}^*| = u^*c, \quad |\mathfrak{V}^*| = v^*d, \quad |\mathfrak{G}| = g.$$

$\mathscr{S}$ is the set of characters of $\mathfrak{S}$ which are induced by irreducible characters of $\mathfrak{S}'$ which do not have $\mathfrak{P}$ in their kernel.

$\mathscr{T}$ is the set of characters of $\mathfrak{T}$ which are induced by irreducible characters of $\mathfrak{T}'$ which do not have $\mathfrak{Q}$ in their kernel.

The set $\mathscr{S}$ as defined here is a subset of the $\mathscr{S}$ as defined in Section 29. Thus by Corollary 32.1.1 $\mathscr{S}$ and $\mathscr{T}$ are coherent.

$\mathscr{U}_0$, $\mathscr{V}_0$ are the sets of irreducible characters of $N(\mathfrak{U}^*)$, $N(\mathfrak{V}^*)$ respectively which do not have $\mathfrak{U}^*$, $\mathfrak{V}^*$ respectively in their kernel.

For $0 \leqq i \leqq q-1$, $0 \leqq j \leqq p-1$, η_{ij} are the generalized characters of $\mathfrak{G}$ defined by Lemma 13.1; μ_{ij} are the characters of $\mathfrak{S}$ defined by Lemma 13.3; ν_{ij} are the characters of $\mathfrak{T}$ defined by Lemma 13.3. For $0 \leqq j \leqq p-1$, ξ_j is the character of $\mathfrak{S}$ defined by Lemma 13.5. For $0 \leqq i \leqq q-1$, ζ_i is the character of $\mathfrak{T}$ defined by Lemma 13.5.

If $\mathfrak{G}_1 \subseteq \mathfrak{G}_2 \subset \mathfrak{G}$, where $\mathfrak{G}_2$ is a maximal subgroup of $\mathfrak{G}$ and if α is a class function of $\mathfrak{G}_1$ then $\tilde{\alpha}$ denotes the class function of $\mathfrak{G}_2$ induced by α. Whenever this notation is used $\mathfrak{G}_2$ will be uniquely determined by the context.

Throughout this section no distinction is made between $\mathfrak{S}$ and $\mathfrak{T}$. Any result in this section about one of these groups is automatically valid for the other by symmetry.

LEMMA 34.1. *Either*

$$u \,\Big|\, \frac{p^q - 1}{p - 1}$$

and $\mathfrak{U}/\mathfrak{C}$ is cyclic or $\mathfrak{U}/\mathfrak{C}$ is the product of at most $q-1$ cyclic groups and $u \mid (p-1)^{q-1}$. For $1 \leqq j \leqq p-1$ ξ_j is induced by a linear character of $\mathfrak{P}\mathfrak{C}$, $\xi_j(1) = uq$. Either $\mathfrak{P}\mathfrak{U}$ is a Frobenius group with $|\mathfrak{P}| = p^q$ and

$$u = \frac{p^q - 1}{p - 1}$$

or $\mathscr{S}$ contains an irreducible character of degree uq which is induced by a linear character of $\mathfrak{P}\mathfrak{C}$.

Proof. If $\mathfrak{P}^* \subseteq D(\mathfrak{P})$ then by 3.16(i) $\mathfrak{P}\mathfrak{U}/D(\mathfrak{P})$ is nilpotent. Thus $\mathfrak{P}\mathfrak{U}$ is nilpotent contrary to assumption. Hence $\mathfrak{P}$ contains a subgroup $\mathfrak{P}_0$ such that $\mathfrak{P}^* \cap \mathfrak{P}_0 = 1$ and $\mathfrak{P}/\mathfrak{P}_0$ is a chief factor of $\mathfrak{S}$. Hence $\mathfrak{U}\mathfrak{Q}^*$ is represented on the elementary abelian group $\mathfrak{P}/\mathfrak{P}_0$. By 3.16 (i) $\mathfrak{P}_0\mathfrak{U}$ is nilpotent. Therefore $\mathfrak{U}\mathfrak{Q}^*/\mathfrak{C}$ is faithfully and irreducibly represented on $\mathfrak{P}/\mathfrak{P}_0$. By 3.16 (iii) $|\mathfrak{P} : \mathfrak{P}_0| = p^q$.

Let $\mathfrak{P}/\mathfrak{P}_0 = \mathfrak{P}_0\mathfrak{P}^*/\mathfrak{P}_0 \times \mathfrak{P}_1/\mathfrak{P}_0$, where $\mathfrak{Q}^* \subseteq N(\mathfrak{P}_1)$. By Lemma 4.6 (i) $N_{\mathfrak{U}}(\mathfrak{P}_1) \subseteq C_{\mathfrak{U}}(\mathfrak{P}_1/\mathfrak{P}_0)$. Thus $N_{\mathfrak{U}}(\mathfrak{P}_1) \subseteq C_{\mathfrak{U}}(\mathfrak{P}) = \mathfrak{C}$. Hence any non principal linear character of $\mathfrak{P}\mathfrak{C}/\mathfrak{P}_1\mathfrak{C}$ induces ξ_j for some j with $1 \leqq j \leqq p-1$. As p is a prime the characters ξ_j are algebraically conjugate for $1 \leqq j \leqq p-1$. Thus $\xi_j(1) = uq$ for $1 \leqq j \leqq p-1$. Let $\xi_j = \tilde{\psi}_j$ for ψ_j a linear character of $\mathfrak{P}\mathfrak{C}/\mathfrak{P}_1\mathfrak{C}$.

Suppose that $|\mathfrak{P}\mathfrak{C} : D(\mathfrak{P}\mathfrak{C})| > p^q$. Then $\mathfrak{P}\mathfrak{C}$ contains a subgroup $\mathfrak{H} \neq \mathfrak{P}_0\mathfrak{C}$ such that $\mathfrak{P}\mathfrak{C}/\mathfrak{H}$ is a chief factor of $\mathfrak{S}$. Let λ be a non principal linear character of $\mathfrak{P}\mathfrak{C}/\mathfrak{H}$. Then $\psi_1\lambda$ induces an irreducible character of $\mathfrak{S}$ of degree uq.

Suppose that $\mathfrak{U}$ is represented reducibly on $\mathfrak{P}/\mathfrak{P}_0$. Since $\mathfrak{U} \lhd \mathfrak{U}\mathfrak{Q}^*$ the irreducible constituents of this representation all have the same dimension. This dimension is 1 since q is a prime. Thus $\mathfrak{U}/\mathfrak{C}$ is the direct product of k cyclic subgroups for some integer k, each of which has order dividing $(p-1)$. No element of $\mathfrak{U}/\mathfrak{C}$ is represented as a scalar as $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group. Therefore $k < q$ and $u \mid (p-1)^{q-1}$. The irreducible constituents of the representation of $\mathfrak{U}/\mathfrak{C}$ on $\mathfrak{P}/\mathfrak{P}_0$ are distinct since $\mathfrak{U}\mathfrak{Q}^*$ is irreducibly represented on $\mathfrak{P}/\mathfrak{P}_0$. Let $\mathfrak{P}/\mathfrak{P}_0 = \mathfrak{P}_1 \times \cdots \times \mathfrak{P}_q$ where $\mathfrak{P}_{i+1} = Q^{-i}\mathfrak{P}_1Q^i$ for some generator Q of $\mathfrak{Q}^*$ and such that $\mathfrak{U}$ normalizes each $\mathfrak{P}_i$. Let

$$P = \prod_{i=1}^{q} P_i$$

with $P_1 \in \mathfrak{P}_1^\#$, $P_2 = Q^{-1}P_1^{-1}Q$ and $Q^{-i}P_1Q^i = P_{i+1}$ for $2 \leqq i \leqq q \quad 1$. Suppose $U \in \mathfrak{U}$ and UQ^j centralizes P for some j. Let $U^{-1}P_iU \quad P_i^{a_i}$ then

$$P \quad (UQ^j)^{-1}P(UQ^j) = Q^{-j}\prod_{i=1}^{q} P_i^{a_i}Q^j\,.$$

Then $Q^{-j}P_2^{a_2}Q^j = P_{2+j}$. If $j \neq q$ then P_{2+j} is conjugate to P_1. Hence $P_2^{a_j}$ is conjugate to P_2^{-1} which is impossible as $|\mathfrak{U}\mathfrak{Q}|$ is odd. Therefore $j = q$. Then $U^{-1}P_iU = P_i$ for $1 \leqq i \leqq q$ and so $U \in \mathfrak{C}$. This proves that no element of $(\mathfrak{U}\mathfrak{Q}/\mathfrak{C})^\#$ leaves P fixed. Let μ_1 be a non principal linear character of $\mathfrak{P}/\mathfrak{P}_0$ with $\ker \mu_1 = \mathfrak{P}_2 \times \cdots \times \mathfrak{P}_q$. Let $\mu_i = \mu_1^{Q^{i-1}}$; then $\mu = \mu_1\mu_2^{-1}\mu_3 \cdots \mu_q$ induces an irreducible character of $\mathfrak{S}$ of degree uq.

Assume now that $\mathfrak{U}$ is irreducibly represented on $\mathfrak{P}/\mathfrak{P}_0$. Then $\mathfrak{U}/\mathfrak{C}$ is cyclic since $\mathfrak{U}/\mathfrak{C}$ is abelian. If a subgroup of $\mathfrak{U}/\mathfrak{C}$ acts reducibly on $\mathfrak{P}/\mathfrak{P}_0$ then it is represented by scalar matrices. As $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group every non identity subgroup of $\mathfrak{U}/\mathfrak{C}$ acts irreducibly on $\mathfrak{P}/\mathfrak{P}_0$. Thus $\mathfrak{U}/\mathfrak{C}$ permutes the subgroups of order p in $\mathfrak{P}/\mathfrak{P}_0$ and no element of $(\mathfrak{U}/\mathfrak{C})^{\sharp}$ leaves any such subgroup fixed. Hence

$$u \Bigm| \frac{p^q - 1}{p - 1}.$$

Suppose now that $\mathscr{S}$ contains no irreducible character of degree uq. By an earlier part of the lemma this implies that $|\mathfrak{P}\mathfrak{C} : D(\mathfrak{P}\mathfrak{C})| = p^q$. Thus $\mathfrak{C} = 1$ and $|\mathfrak{P} : D(\mathfrak{P})| = p^q$. Since $D(\mathfrak{P}) \cap \mathfrak{P}^* = 1$, we must have $D(\mathfrak{P}) = \mathfrak{P}'$. By 3.16 (i) $\mathfrak{P}'\mathfrak{U}$ is nilpotent. If $\mathfrak{P}' \neq 1$ then there exists a subgroup $\mathfrak{P}_1$ of $\mathfrak{P}'$ such that $|\mathfrak{P}' : \mathfrak{P}_1| = p$. Hence $\mathfrak{P}'/\mathfrak{P}_1$ is the center of $\mathfrak{P}/\mathfrak{P}_1$ since $\mathfrak{U}$ acts irreducibly on $\mathfrak{P}/\mathfrak{P}'$. Thus $\mathfrak{P}/\mathfrak{P}_1$ is an extra special p-group. This implies that q is even which is not the case. Thus $\mathfrak{P}' = 1$. Hence $\mathfrak{P}\mathfrak{U}$ is a Frobenius group. Consequently $\mathfrak{P}\mathfrak{U}$ contains $(p^q - 1)/u$ irreducible characters of degree u. Lemma 13.7 now implies that

$$u = \frac{p^q - 1}{p - 1}.$$

LEMMA 34.2. *Either $\mathfrak{P}\mathfrak{U}$ is a Frobenius group with $|\mathfrak{P}| = p^q$ and*

$$u = \frac{p^q - 1}{p - 1}$$

or $\mathfrak{Q}\mathfrak{V}$ is a Frobenius group with $|\mathfrak{Q}| = q^p$ and

$$v = \frac{q^p - 1}{q - 1}.$$

Proof. If the result is false then Lemma 34.1 implies that $\mathscr{S}$ contains an irreducible character λ of degree uq and $\mathscr{T}$ contains an irreducible character θ of degree vp. Every character in $\mathscr{T}^{\tau}$ is rational valued on $\mathfrak{P}$ by Lemma 10.4. Since $|\mathfrak{G}|$ is odd this implies that every generalized character of weight 1 in $\mathscr{S}^{\tau}$ is orthogonal to $\mathscr{T}^{\tau}$. Define

$$\alpha = \lambda - \xi_1, \quad \beta = \theta - \zeta_1.$$

Then $\alpha(1) = \beta(1) = 0$ and $(\alpha^{\tau}, \beta^{\tau}) = 0$. Thus

$$0 = (\lambda^{\tau} - \xi_1^{\tau}, \theta^{\tau} - \zeta_1^{\tau}) = \left(\pm \sum_{i=0}^{q-1} \eta_{i1}, \pm \sum_{j=0}^{p-1} \eta_{1j}\right)$$
$$= \pm(\eta_{11}, \eta_{11}) = \pm 1.$$

This proves the lemma.

LEMMA 34.3. *For* $1 \leqq j \leqq p-1$

$$\sum_{X \in (\mathfrak{P}\mathfrak{C})^{\#}} |\eta_{0j}(X)|^2 \geqq uc\,|\mathfrak{P}| - u^2 .$$

Proof. Since $\mathfrak{P}\mathfrak{C}$ is a T.I. set in $\mathfrak{G}$ and $\mathscr{S}$ is coherent the Frobenius reciprocity theorem implies that for $1 \leqq j \leqq p-1$

$$\eta_{0j}(X) = \varepsilon(\mu_{0j}(X) + \alpha(X)) \quad \text{for } X \in (\mathfrak{P}\mathfrak{C})^{\#} ,$$

where α is a generalized character of $\mathfrak{S}'/\mathfrak{P}$, and $\varepsilon^2 = 1$. Therefore

$$\sum_{(\mathfrak{P}\mathfrak{C})^{\#}} |\eta_{0j}(X)|^2 = \sum_{(\mathfrak{P}\mathfrak{C})^{\#}} \{\mu_{0j}(X)\overline{\alpha(X)} + \overline{\mu_{0j}(X)}\alpha(X)\} + \sum_{(\mathfrak{P}\mathfrak{C})^{\#}} |\mu_{0j}(X)|^2 + \sum_{(\mathfrak{P}\mathfrak{C})^{\#}} |\alpha(X)|^2 .$$

This implies that

(34.1)
$$\sum_{(\mathfrak{P}\mathfrak{C})^{\#}} |\eta_{0j}(X)|^2 = -2\mu_{0j}(1)\alpha(1) + cu|\mathfrak{P}| - u^2 + |\mathfrak{P}| \sum_{C \in \mathfrak{C}} |\alpha(C)|^2 - \alpha(1)^2 .$$

By Lemma 34.1, $2u + 1 \leqq |\mathfrak{P}|$, thus

$$\begin{aligned} -2\mu_{0j}(1)\alpha(1) &+ |\mathfrak{P}| \sum_{\mathfrak{C}} |\alpha(C)|^2 - \alpha(1)^2 \\ &\geqq |\mathfrak{P}| \sum_{\mathfrak{C}} |\alpha(C)|^2 - (2u+1)\alpha(1)^2 \\ &\geqq |\mathfrak{P}| \sum_{\mathfrak{C}^{\#}} |\alpha(C)|^2 \geqq 0 . \end{aligned}$$

The result now follows from (34.1).

LEMMA 34.4. *For* $1 \leqq i \leqq q-1$

$$\sum_{X \in \mathfrak{P}\mathfrak{C} - \mathfrak{C}} |\eta_{i0}(X)|^2 \geqq \{|\mathfrak{P}| - 1\}\, c .$$

Proof. Since $\mathfrak{P}\mathfrak{C}$ is a T.I. set in $\mathfrak{G}$ the coherence of $\mathscr{S}$ and the Frobenius reciprocity theorem imply that $\eta_{i0}(X) = \alpha(X)$ for $X \in \mathfrak{P}\mathfrak{C} - \mathfrak{C}$, where α is a generalized character of $\mathfrak{S}'/\mathfrak{P}$. Therefore for $1 \leqq i \leqq q-1$

(34.2)
$$\sum_{X \in \mathfrak{P}\mathfrak{C} - \mathfrak{C}} |\eta_{i0}(X)|^2 = \sum_{X \in \mathfrak{P}\mathfrak{C} - \mathfrak{C}} |\alpha(X)|^2 = \{|\mathfrak{P}| - 1\} \sum_{\mathfrak{C}} |\alpha(C)|^2 .$$

If $P \in \mathfrak{P}^{**}$, $Q \in Q^{**}$ and $\mathfrak{q}$ is a prime divisor of q in $\mathscr{O}_{pq}$ then by Lemma 4.2

$$\eta_{i0}(P) \equiv \eta_{i0}(PQ) \equiv 1 \pmod{q} .$$

Thus the expression in (34.2) is non zero. The result now follows from the fact that

$$\sum_{\mathfrak{C}} |\alpha(C)|^2 \equiv 0 \pmod{c} .$$

LEMMA 34.5. *Suppose that $\mathscr{S}$ contains an irreducible character λ of degree uq which is induced by a character of $\mathfrak{P}\mathfrak{C}$. Then*

$$\sum_{X \in (\mathfrak{P}\mathfrak{C})^\#} |\lambda^\tau(X)|^2 > uqc\,|\mathfrak{P}| - (uq)^2 - 2uq^2 .$$

Proof. As $\mathfrak{P}\mathfrak{C}$ is a T.I. set in $\mathfrak{G}$ the coherence of $\mathscr{S}$ and the Frobenius reciprocity theorem imply that

$$\lambda^\tau(X) = \lambda(X) + \alpha(X) \quad \text{for } X \in (\mathfrak{P}\mathfrak{C})^\# ,$$

for some generalized character α of $\mathfrak{S}'/\mathfrak{P}$. Therefore

$$\begin{aligned} \sum_{(\mathfrak{P}\mathfrak{C})^\#} |\lambda^\tau(X)|^2 = \sum_{(\mathfrak{P}\mathfrak{C})^\#} |\lambda(X)|^2 + \sum_{(\mathfrak{P}\mathfrak{C})^\#} \{\lambda(X)\alpha(\bar{X}) + \lambda(\bar{X})\alpha(X)\} \\ + \sum_{(\mathfrak{P}\mathfrak{C})^\#} |\alpha(X)|^2 \geqq uqc\,|\mathfrak{P}| - (uq)^2 - 2\lambda(1)\alpha(1) \\ + \{|\mathfrak{P}| - 1\} \sum_{\mathfrak{C}} |\alpha(C)|^2 + \sum_{\mathfrak{C}^\#} |\alpha(C)|^2 . \end{aligned} \tag{34.3}$$

If $|\alpha(1)| \geqq q$ then by Lemma 34.1

$$2\lambda(1)|\alpha(1)| = 2uq|\alpha(1)| \leqq 2u\alpha(1)^2 \leqq \{|\mathfrak{P}| - 1\}\alpha(1)^2 .$$

Hence the result follows from (34.3) in this case. If $|\alpha(1)| < q$ then $2\lambda(1)|\alpha(1)| < 2uq^2$ thus (34.3) also implies the result in this case.

LEMMA 34.6. *Let $\mathfrak{G}_0$ be the set of elements in $\mathfrak{G}$ which are not conjugate to any element of $\mathfrak{P}\mathfrak{C}$, $\mathfrak{Q}$ or $\hat{\mathfrak{W}}$. Suppose that $\mathscr{S}$ contains an irreducible character λ of degree uq. Define*

$$\begin{aligned} \mathfrak{A}_1 &= \{G \mid G \in \mathfrak{G}_0, \lambda^\tau(G) \neq 0\} \\ \mathfrak{A}_2 &= \{G \mid G \in \mathfrak{G}_0, \eta_{10}(G) \neq 0\} \\ \mathfrak{A}_3 &= \{G \mid G \in \mathfrak{G}_0, \eta_{01}(G) \neq 0, \eta_{01}(G) \equiv 0 \pmod{(q-1)}\} . \end{aligned}$$

Then

$$\mathfrak{G}_0 = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 .$$

Proof. Suppose that $G \in \mathfrak{G}_0 - (\mathfrak{A}_1 \cup \mathfrak{A}_2)$. Let $\alpha = \xi_1 - \lambda$. Then $(\xi_1 - \lambda)^\tau(G) = 0$ and

$$(\xi_1 - \lambda)^\tau = \pm \sum_{i=0}^{q-1} \eta_{i1} - \lambda^\tau .$$

Since $G \in \mathfrak{G}_0$, $\eta_{i1}(G)$ is rational. Thus $\eta_{i1}(G) = \eta_{11}(G)$ for $1 \leqq i \leqq q-1$. As $G \notin \mathfrak{A}_1 \cup \mathfrak{A}_2$ we must have that

$$(34.4) \qquad 0 \equiv \sum_{i=0}^{q-1} \eta_{i1}(G) \equiv \eta_{01}(G) + (q-1)\eta_{11}(G) \equiv \eta_{01}(G) \pmod{(q-1)} .$$

Suppose that $\eta_{01}(G) = 0$. Then since $\alpha^\tau(G) = 0$ we must have that $\eta_{i1}(G) = 0$ for $0 \leqq i \leqq q-1$. Hence by Lemma 13.1

$$0 = (1_{\mathfrak{G}} - \eta_{10} - \eta_{01} + \eta_{11})(G) = 1 - \eta_{10}(G)$$

contradicting the fact that $G \notin \mathfrak{A}_2$. Hence $\eta_{01}(G) \neq 0$ and by (34.4) $G \in \mathfrak{A}_3$ as required.

LEMMA 34.7.
(i) *If* $q \geqq 5$ *then* $|\mathfrak{P}| = p^q$ *and* $u/c > 9p^{q-1}/20q$.
(ii) *If* $p, q \geqq 5$ *then* $c = 1$ *and* $u \geqq (13/20)\, p^{q-1}/q$.
(iii) *If* $p = 3$ *and* $c \neq 1$ *then* $u = 121, q = 5, c = 11$.
(iv) *If* $q = 3$ *then* $c = 1$ *or* $c = 7$. *Furthermore* $u > (p^2 + p + 1)/13$.
(v) *If* $q = 3$ *then* $\mathfrak{P}$ *is an elementary abelian p-group and* $|\mathfrak{P}| = p^3$ *or* $p = 7, c = 1$ *and* $|\mathfrak{P}| = 7^4$.
(vi) *If* $q = 3$ *and* $c = 7$ *then* $u > (p^2 + p + 1)/2$.

Proof. If $\mathfrak{P}\mathfrak{U}$ is a Frobenius group with $|\mathfrak{P}| = p^q$, $u = (p^q - 1)/(p-1)$ then all the statements in the lemma are immediate. Suppose that this is not the case. Then by Lemma 34.1 $\mathscr{S}$ contains an irreducible character λ which is induced by a linear character of $\mathfrak{P}\mathfrak{C}$. By Lemma 34.2 $\mathfrak{Q}\mathfrak{V}$ is a Frobenius group with $|\mathfrak{Q}| = q^p$, $v = (q^p - 1)/(q-1)$, $d = 1$.

$\mathfrak{P}\mathfrak{C}$, $\mathfrak{Q}$ and $\hat{\mathfrak{W}}$ are T.I. sets. Let $\mathfrak{G}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ have the same meaning as in Lemma 34.6. Then

$$(34.5) \qquad \begin{aligned} \frac{1}{g}|\mathfrak{G}_0| &= 1 - \left(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}\right) \\ &\quad - \frac{1}{quc|\mathfrak{P}|}\{|\mathfrak{P}|c - 1\} - \frac{1}{pv|\mathfrak{Q}|}\{|\mathfrak{Q}| - 1\} \\ &= \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} - \frac{1}{qu} - \frac{1}{pv} + \frac{1}{quc|\mathfrak{P}|} + \frac{1}{pvq^p} . \end{aligned}$$

Since λ^τ is rational valued on $\mathfrak{G}_0$ by Lemma 10.4, Lemma 34.5 implies that

$$(34.6) \qquad \begin{aligned} \frac{1}{g}|\mathfrak{A}_1| \leqq \frac{1}{g}\sum_{\mathfrak{A}_1} |\lambda^\tau(X)|^2 &\leqq 1 - \frac{1}{|\mathfrak{P}|ucq} \sum_{(\mathfrak{P}\mathfrak{C})^\#} |\lambda^\tau(X)|^2 \\ &< \frac{uq}{|\mathfrak{P}|c} + \frac{2q}{|\mathfrak{P}|c} . \end{aligned}$$

If Lemma 34.3 is applied to $\mathfrak{T}$ then Lemmas 13.1 and 34.4 yield that

$$\frac{1}{g}|\mathfrak{A}_2| \leqq \frac{1}{g}\sum_{\mathfrak{A}_2}|\eta_{10}(X)|^2 .$$

$$(34.7)\quad \leqq 1-\left(1-\frac{1}{p}-\frac{1}{q}+\frac{1}{pq}\right)-\frac{1}{pvq^p}\{vq^p-v^2\}-\frac{1}{|\mathfrak{P}|uqc}\{|\mathfrak{P}|-1\}c$$

$$=\frac{1}{q}-\frac{1}{pq}+\frac{v}{pq^p}-\frac{1}{uq}+\frac{1}{|\mathfrak{P}|uq}.$$

Lemmas 13.1 and 34.3 also imply that

$$\frac{1}{g}|\mathfrak{A}_3| \leqq \frac{1}{(q-1)^2}\frac{1}{g}\sum_{\mathfrak{A}_3}|\eta_{01}(X)|^2$$

$$(34.8)\quad \leqq \frac{1}{(q-1)^2}\left\{1-\left(1-\frac{1}{p}-\frac{1}{q}+\frac{1}{pq}\right)\right.$$

$$\left.-\frac{1}{quc|\mathfrak{P}|}(uc|\mathfrak{P}|-u^2)\right\}$$

$$=\frac{1}{(q-1)^2}\left\{\frac{(q-1)}{pq}+\frac{u}{qc|\mathfrak{P}|}\right\}.$$

Lemma 34.6 and (34.5), (34.6), (34.7) and (34.8) now imply that

$$\frac{1}{p}+\frac{1}{quc|\mathfrak{P}|}-\frac{1}{pvq^p}(q^p-1)+\frac{1}{q}-\frac{1}{pq}-\frac{1}{qu}\leqq\frac{uq}{|\mathfrak{P}|c}$$

$$+\frac{2q}{|\mathfrak{P}|c}+\frac{1}{|\mathfrak{P}|uq}+\frac{v}{pq^p}+\frac{1}{q}-\frac{1}{pq}-\frac{1}{qu}$$

$$+\frac{1}{pq(q-1)}+\frac{u}{qc|\mathfrak{P}|(q-1)^2}.$$

Since $v=(q^p-1)/(q-1)$, this can be simplified to

$$\frac{1}{p}\leqq\frac{(u+2)q}{|\mathfrak{P}|c}+\frac{(c-1)}{|\mathfrak{P}|quc}+\frac{1}{p(q-1)}-\frac{1}{pq^p(q-1)}$$

$$+\frac{(q-1)}{pq^p}+\frac{1}{pq(q-1)}+\frac{u}{qc|\mathfrak{P}|(q-1)^2}$$

$$(34.9)\quad =\frac{(u+2)q}{|\mathfrak{P}|c}+\frac{(c-1)}{|\mathfrak{P}|quc}+\frac{u}{qc|\mathfrak{P}|(q-1)^2}$$

$$+\frac{(q+1)}{pq(q-1)}+\frac{(q-1)^2-1}{pq^p(q-1)}.$$

By Lemma 34.1 $u \leqq (p^q - 1)/(p-1)$ and $|\mathfrak{P}| \geqq p^q$; thus (34.9) implies that

$$\frac{1}{p} \leqq \frac{(u+2)q}{|\mathfrak{P}|c} + \frac{(q+1)}{pq(q-1)} + \frac{1}{c(p-1)\,q(q-1)^2} + \frac{1}{p^q q} + \frac{1}{pq^{p-1}}. \tag{34.10}$$

Let $|\mathfrak{P}| = p^q x$ then

$$x \equiv c \equiv 1 (\mathrm{mod}\ 2q). \tag{34.11}$$

Suppose first that $p, q \geqq 5$. Then (34.10) implies that

$$\frac{1}{p} \leqq \frac{uq}{p^q xc} + \frac{2q}{p^q xc} + \frac{3}{10p} + \frac{1}{80(p-1)} + \frac{2}{5^4 p}.$$

Hence by (5.2)

$$\frac{1}{p} < \frac{uq}{p^q xc} + \frac{1}{40p} + \frac{3}{10p} + \frac{3/2}{80p} + \frac{1/2}{80p}.$$

Therefore

$$\frac{q}{xc}\frac{u}{p^q} > \frac{13}{20p}. \tag{34.12}$$

Therefore

$$\frac{1}{xc}u > \frac{13p^{q-1}}{20q} > \frac{p^{q-1}}{2q}. \tag{34.13}$$

Suppose that $cx \neq 1$. Then by (34.11) $cx > 2q$. Thus (34.12) implies that

$$\frac{13}{20p} < \frac{1}{2}\frac{u}{p^q} < \frac{1}{2}\frac{1}{(p-1)}.$$

Thus $13(p-1) < 10p$ or $3p < 13$ which is not the case. Hence $c = x = 1$ and (34.13) completes the proof of statement (ii) of the lemma.

Suppose now that $p = 3$. Hence (34.10) yields that

$$\frac{1}{3} \leqq \frac{(u+2)q}{cx3^q} + \frac{(q+1)}{3q(q-1)} + \frac{1}{2q(q-1)^2} + \frac{1}{3^q q} + \frac{1}{3q^2}. \tag{34.14}$$

As $q \geqq 5$ this implies that

$$\frac{1}{3} \leqq \frac{q}{cx}\frac{u}{3^q} + \frac{1}{10} + \frac{1}{160} + \frac{(2q^2+1)}{3^q q} + \frac{1}{75}.$$

Hence by (5.3)

$$\frac{q}{cx}\frac{u}{3^q} \geqq \frac{1}{3} - \frac{1}{10} - \frac{1}{160} - \frac{1}{20} - \frac{1}{75}$$
$$> \frac{160 - 48 - 3 - 24 - 10}{480} = \frac{75}{480} > \frac{3}{20}.$$

Thus

$$\frac{u}{cx} > \frac{3}{20} \cdot \frac{3^q}{q}. \tag{34.15}$$

This yields that

$$\frac{2q}{cx} > \frac{9}{10} \cdot \frac{3^{q-1}}{u} > \frac{3}{5}.$$

Hence $4q > cx$.

Assume that $cx \neq 1$. Then (34.11) implies that

$$cx = 2q + 1. \tag{34.16}$$

Suppose first then $q \geqq 11$. Then (34.14) implies that

$$\frac{1}{3} \leqq \frac{q}{cx}\frac{u}{3^q} + \frac{2}{55} + \frac{1}{2.10^3} + \frac{1}{10.3^{10}} + \frac{2q}{cx}\frac{1}{3^{10}} + \frac{1}{300}.$$

Hence

$$\frac{q}{cx}\frac{u}{3^q} > \frac{1}{3} - \frac{2}{55} - \frac{1}{60} > \frac{1}{3} - \frac{3}{54} = \frac{5}{18}.$$

Therefore

$$\frac{q}{cx} > \frac{3^q}{u}\frac{5}{18} > \frac{2.5}{18} > \frac{1}{2}$$

contrary to (34.16). Suppose that $q = 7$. Then $cx = 15$ by (34.16). Thus $x = 3$ and $c = 5$ since x is a power of 3 and $(c, 3) = 1$. This contradicts (34.11). Hence $q = 5$. Thus by (34.16) $cx = 11$. Hence $x = 1$ and $c = 11$ since x is a power of 3. Thus statement (i) of the lemma follows from (34.15) and statement (ii). If $c \neq 1$ then $q = 5$ and $c = 11$. By (34.15)

$$u > \frac{11.3^5}{100} > 2^4 = (p - 1)^{q-1}. \tag{34.17}$$

Hence by Lemma 34.1 $u \mid (3^5 - 1)/2 = 121$. Thus $u = 121$ by (34.17). This completes the proof of statement (iii) of the lemma.

Assume now that $q = 3$. Let $y = (p^2 + p + 1)/u$. (y is not necessarily integral) Then (34.9) implies that

$$\frac{1}{p} < \frac{3(p^2+p+1)}{cxyp^3} + \frac{6}{cxp^3} + \frac{1}{3p^3u} + \frac{(p^2+p+1)}{12cxyp^3} + \frac{2}{3p} + \frac{1}{2p3^{p-1}}.$$

Therefore

$$\frac{1}{3p} < \frac{37(p^2+p+1)}{12cxyp^3} + \frac{6}{cxp^3} + \frac{1}{3p^3u} + \frac{1}{2p3^{p-1}},$$

or

$$1 < \frac{37(p^2+p+1)}{4cxyp^2} + \frac{18}{cxp^2} + \frac{1}{p^2u} + \frac{1}{2\cdot 3^{p-2}}. \tag{34.18}$$

Suppose that $cxy \geqq 13$. Then (34.18) implies that

$$\frac{37}{52}\frac{(p^2+p+1)}{p^2} > 1 - \frac{19}{p^2} - \frac{1}{52}.$$

Therefore $37(p^2+p+1) > 51p^2 - 52\cdot 19$, or

$$14p^2 - 37p - 52\cdot 19 - 37 < 0.$$

Therefore, $p < 11$. Hence $p = 5$ or $p = 7$. Since $(6, u) = 1$, Lemma 34.1 now implies that $u \mid p^2 + p + 1$. Thus $u \mid 31$ if $p = 5$ and $u \mid 57$ if $p = 7$. Hence one of the following must occur:

$$p = 5, \quad u = 31, \quad y = 1, \quad cx \geqq 13$$

or

$$p = 7, \quad u = 19, \quad y = 3, \quad cx \geqq 5.$$

By (34.11)

$$cx = 7, \quad p = 7 \quad \text{or } cx \geqq 13.$$

If $cx \geqq 13$ then by (34.18)

$$1 < \frac{37}{52}\frac{(p^2+p+1)}{p^2} + \frac{19}{13p^2} + \frac{1}{52}.$$

Hence $p < 5$, which is not the case. Therefore we have shown that either $cxy < 13$ or $p = 7, u = 19, y = 3$ and $cx = 7$. If $cxy < 13$, then $y < 13$, and by (34.11) $cx = 7$ or $cx = 1$. Thus in any case

$$u > \frac{p^2+p+1}{13}, cx = 1 \quad \text{or } cx = 7. \tag{34.19}$$

This proves statement (iv) of the lemma.

If $x \neq 1$ then (34.19) implies that $c = 1$ and $x = 7$, hence $p = 7$ and $|\mathfrak{P}| = 7^4$. Since $(u, 6) = 1$, Lemma 34.1 implies that $u \mid 57$, thus $u = 19$. If $D(\mathfrak{P}) \neq 1$ then $\mathfrak{U}$ acts irreducibly on $\mathfrak{P}/D(\mathfrak{P})$ and centralizes $D(\mathfrak{P})$. If $\mathfrak{P}$ is non abelian this implies that $D(\mathfrak{P}) = Z(\mathfrak{P})$. Hence $\mathfrak{P}$ is an extra special p-group contrary to the fact that $|\mathfrak{P}:D(\mathfrak{P})| = p^3$. Thus $\mathfrak{P}$ is abelian. Hence $|\mathfrak{P}:\Omega_1(\mathfrak{P})| \leqq p$. If $\Omega_1(\mathfrak{P}) \neq \mathfrak{P}$ this implies that $\mathfrak{U}\mathfrak{Q}$ is represented on $\Omega_1(\mathfrak{P})$ and so $\mathfrak{U}$ acts irreducibly on $\Omega_1(\mathfrak{P})$ contrary to $D(\mathfrak{P}) \subseteq \Omega_1(\mathfrak{P})$ and $\mathfrak{U} \subseteq C(D(\mathfrak{P}))$. Thus $\mathfrak{P}$ is elementary abelian. Statement (v) of the lemma is proved.

Suppose that $c = 7$ and $y \geqq 2$; then (34.18) implies that

$$1 < \frac{37}{56}\frac{(p^2+p+1)}{p^2} + \frac{19}{7p^2} + \frac{1}{54}.$$

Therefore, $p < 5$ which is impossible. Hence if $c = 7$ then $y < 2$. This proves statement (vi) of the lemma and completes the proof of Lemma 34.7.

LEMMA 34.8. *If $q \geqq 5$ then $\mathfrak{P}\mathfrak{U}/\mathfrak{C}$ is a Frobenius group and $u \mid (p^q - 1)/(p - 1)$.*

Proof. By Lemma 34.7 (i) $|\mathfrak{P}| = p^q$. Thus if $\mathfrak{P}\mathfrak{U}/\mathfrak{C}$ is not a Frobenius group then by Lemma 34.1 $u \mid [(p-1)/2]^{q-1}$. Thus by Lemma 34.7 (i)

$$\frac{p^{q-1}}{2^{q-1}} > u > \frac{9\cdot p^{q-1}}{20q}.$$

Therefore $q > 2^{q-3}\cdot(9/10)$ which is not the case, since $q \geqq 5$.

LEMMA 34.9. If *$p, q \geqq 5$ then $c=1$, $|\mathfrak{P}|=p^q$ and either $u=(p^q-1)/(p-1)$ or $p \equiv 1 \pmod q$ and $u = 1/q\,[(p^q-1)/(p-1)]$.*

Proof. By Lemma 34.7(ii) $c = 1$. Lemma 34.8 implies that $|\mathfrak{P}| = p^q$ and $u \mid (p^q-1)/(p-1)$. Let $ux = (p^q-1)/(p-1)$. If $p \not\equiv 1 \pmod q$ then

$$u \equiv \frac{p^q-1}{p-1} \equiv 1 \pmod{2q}.$$

Thus $x \equiv 1 \pmod{2q}$. If $p \equiv 1 \pmod q$ then $(p^q-1)/(p-1) \equiv 0 \pmod q$. Hence $x \equiv 0 \pmod q$ as $(u, q) = 1$. Thus in any case $x \geqq 2q$ if the result is false. Now Lemma 34.7 (ii) implies that

$$\frac{p^q-1}{p-1} = ux \geqq 2qu \geqq \frac{13}{10}p^{q-1}.$$

Hence

$$p^q > p^q - 1 \geqq \frac{13}{10}p^q - \frac{13}{10}p^{q-1}.$$

Thus $13 > 3p$ contrary to the fact that $p \geqq 5$.

LEMMA 34.10.

$$\begin{aligned} |N(\mathfrak{V}^*):\mathfrak{V}^*C(\mathfrak{V}^*)| &= p \text{ or } pq && \text{if } p, q \geqq 5 \text{ or } p = 3, q \geqq 7 \\ &= 3 \text{ or } 15 \text{ or } 33 && \text{if } p = 3, q = 5 \\ &= p, 3p \text{ or } 7p && \text{if } q = 3. \end{aligned}$$

Proof. Let $\mathfrak{E}$ be a complement of $\mathfrak{V}^*C(\mathfrak{V}^*)$ in $N(\mathfrak{V}^*)$ which contains $\mathfrak{P}^*$. Every Sylow subgroup of $\mathfrak{E}$ is cyclic and every subgroup of prime order is normal in $\mathfrak{E}$ by 3.16 (ii) and Theorem 33.1. Thus $\mathfrak{E} \subseteq N(\mathfrak{P}^*) = \mathfrak{Q}^*\mathfrak{P}\mathfrak{C}$. Hence $\mathfrak{E} = \mathfrak{P}^*$ or $|\mathfrak{E}| = pq$ or $\mathfrak{E} \subseteq \mathfrak{P}^*\mathfrak{C}$. The result now follows from Lemma 34.7.

By Theorem 33.1 $\mathfrak{U}^*$ is tamely imbedded in $\mathfrak{G}$ unless $\mathfrak{U}^* = \mathfrak{U}$ and $C_{\mathfrak{P}}(\mathfrak{U}) \neq 1$. By Lemma 34.7 this can only happen if $p = 7$ and $q = 3$. In that case let $\mathscr{U}$ be the set of characters of $\mathfrak{S}$ which are induced by non principal irreducible characters of $\mathfrak{S}'/\mathfrak{P}$. In all other cases let $\mathscr{U}_0 = \mathscr{U}$. Define $\mathscr{V}$ similarly. Then $\mathscr{I}_0(\mathscr{U})^\tau$ and $\mathscr{I}_0(\mathscr{V})^\tau$ are always defined.

LEMMA 34.11. *Suppose that $\mathscr{V}$ is coherent and $p > q$. If*

$$\frac{dv^* - 1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} > \frac{v - 1}{p}$$

and

$$\frac{dv^* - 1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} > \frac{u - 1}{q}$$

then $|N(\mathfrak{V}^*):\mathfrak{V}^*| \geqq pq$. If furthermore $|N(\mathfrak{V}^*):\mathfrak{V}^*| = pq$ then $1/p \leqq pq/v^*d$.

Proof. Let $e = |N(\mathfrak{V}^*):\mathfrak{V}^*|$. Let $\psi \in \mathscr{V}$ with $\psi(1) = e$. Let $\alpha = \tilde{1}_{\mathfrak{V}^*} - \psi$. Then $\|\alpha^\tau\|^2 = \|\alpha\|^2 = e + 1$. Define

$$\beta_{\mathfrak{S}} = \tilde{1}_{\mathfrak{Q}^*\mathfrak{P}\mathfrak{C}} - \mu_{01}, \quad \beta_{\mathfrak{T}} = \tilde{1}_{\mathfrak{P}^*\mathfrak{Q}\mathfrak{D}} - \nu_{10}.$$

$\beta_{\mathfrak{S}}, \beta_{\mathfrak{T}}$ vanish on $\mathfrak{S} - \hat{\mathfrak{S}}_1$, $\mathfrak{T} - \hat{\mathfrak{T}}_1$ respectively. As $\hat{\mathfrak{S}}_1$ and $\hat{\mathfrak{T}}_1$ are T.I. sets in $\mathfrak{G}$

$$(34.20)\quad \|\beta_{\mathfrak{S}}^*\|^2 = \|\beta_{\mathfrak{S}}\|^2 = \frac{u-1}{q} + 2, \quad \|\beta_{\mathfrak{T}}^*\|^2 = \|\beta_{\mathfrak{T}}\|^2 = \frac{v-1}{p} + 2.$$

Furthermore by Lemma 13.8

$$(34.21)\qquad \beta_{\mathfrak{S}}^* = 1_{\mathfrak{G}} \pm \eta_{01} + \Gamma_{\mathfrak{S}}, \qquad \beta_{\mathfrak{T}}^* = 1_{\mathfrak{G}} \pm \eta_{10} + \Gamma_{\mathfrak{T}}$$

where $\Gamma_{\mathfrak{S}}, \Gamma_{\mathfrak{T}}$ are real valued generalized characters of $\mathfrak{G}$ which are orthogonal to $1_{\mathfrak{G}}$. The assumed inequalities and (34.20) imply that $(\psi^\tau, \beta^*_{\mathfrak{S}}) = 0 = (\psi^\tau, \beta^*_{\mathfrak{T}})$. Thus if $\alpha^\tau = 1_{\mathfrak{G}} \pm \psi^\tau + \Gamma_{\mathfrak{W}}$ then

$$0 \equiv (\alpha^\tau, \beta^*_{\mathfrak{S}}) \equiv 1 + (\eta_{01}, \Gamma_{\mathfrak{W}}) \pmod 2$$
$$0 \equiv (\alpha^\tau, \beta^*_{\mathfrak{T}}) \equiv 1 + (\eta_{10}, \Gamma_{\mathfrak{W}}) \pmod 2 .$$

Since $\Gamma_{\mathfrak{W}}$ is rational valued on $\hat{\mathfrak{W}}$ this implies that

$$(\eta_{i0}, \Gamma_{\mathfrak{W}}) \equiv (\eta_{0j}, \Gamma_{\mathfrak{W}}) \equiv 1 \pmod 2$$

for $1 \leqq i \leqq q-1$, $1 \leqq j \leqq p-1$. Hence by Lemma 13.1

$$(1_{\mathfrak{G}} - \eta_{i0} - \eta_{0j} + \eta_{ij}, \alpha^\tau) \equiv 1 + (\eta_{i0}, \Gamma_{\mathfrak{W}}) + (\eta_{0j}, \Gamma_{\mathfrak{W}}) + (\eta_{ij}, \Gamma_{\mathfrak{W}}) \pmod 2 .$$

Thus $(\eta_{ij}, \Gamma_{\mathfrak{W}}) \neq 0$ for $1 \leqq i \leqq q-1$, $1 \leqq j \leqq p-1$. Hence

$$e + 1 = \|\alpha^\tau\|^2 \geqq pq + 1 .$$

Suppose now that $e = pq$ then

$$\alpha^\tau = 1_{\mathfrak{G}} \pm \psi^\tau \pm \sum_{i=1}^{q-1} \eta_{i0} \pm \sum_{j=1}^{p-1} \eta_{0j} \pm \sum_{i=1}^{q-1}\sum_{j=1}^{p-1} \eta_{ij} . \tag{34.22}$$

Let $\mathfrak{G}_0$ be the set of elements in $\mathfrak{G}$ which are conjugate to some element of $\mathfrak{A}_V$ with $V \in \mathfrak{V}^{**}$. Since $\mathscr{V}$ is coherent by assumption, (34.22) Lemmas 33.1 and 9.4 imply that $\psi^\tau(VC) = \psi(V)$ for $VC \in \mathfrak{A}_V$, $V \in \mathfrak{V}^{**}$. Furthermore Lemma 9.5 and (34.22) imply that

$$\frac{1}{g} \sum_{\mathfrak{G}_0} |\psi^\tau(G)|^2 = \frac{1}{pqv^*d} \sum_{\mathfrak{V}^{**}} |\psi(G)|^2 = 1 - \frac{pq}{v^*d} . \tag{34.23}$$

By Lemma 9.5

$$\frac{1}{g} |\mathfrak{G}_0| = \frac{1}{g} \sum_{\mathfrak{G}_0} 1_{\mathfrak{G}}(G) = \frac{1}{pqv^*d} \sum_{\mathfrak{V}^{**}} 1_{\mathfrak{G}}(G) = \frac{(dv^* - 1)}{dv^*pq} . \tag{34.24}$$

Let $\mathfrak{G}_1$ be the set of elements in $\mathfrak{G} - \mathfrak{G}_0$ which are not conjugate to any element of $\hat{\mathfrak{W}}$, $\mathfrak{P}\mathfrak{C}$ or $\mathfrak{Q}\mathfrak{D}$. Now (34.22) implies that if $G \in \mathfrak{G}_1$ then $\psi^\tau(G)$ is rational and

$$0 \equiv \alpha^\tau(G) \equiv 1 + \psi^\tau(G) \pmod 2 .$$

Thus $|\psi^\tau(G)|^2 \geqq 1$ for $G \in \mathfrak{G}_1$. Hence (34.23) implies that

$$\frac{pq}{v^*d} \geqq \frac{1}{g} |\mathfrak{G}_1| \geqq 1 - \frac{(vd^* - 1)}{pqv^*d} - \left(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}\right) - \frac{|\mathfrak{P}\mathfrak{C}| - 1}{qu|\mathfrak{P}|c} - \frac{|\mathfrak{Q}\mathfrak{D}| - 1}{pv|\mathfrak{Q}|d} .$$

Therefore

$$\frac{pq}{v^*d} \geqq \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} - \frac{1}{pq} + \frac{1}{pqv^*d} - \frac{1}{qu} + \frac{1}{qu\,|\mathfrak{P}|\,c} - \frac{1}{pv} + \frac{1}{pv\,|\mathfrak{Q}|\,d}\,.$$

Since $u > 2q$, $v > 2p$ and $p > q \geqq 3$

$$\frac{1}{pq} + \frac{1}{pq} + \frac{1}{qu} + \frac{1}{pv} < \frac{3}{q^2} \leqq \frac{1}{q}\,;$$

thus the required inequality follows.

LEMMA 34.12. *If* $\mathfrak{U}^*$ *is cyclic then* $\mathfrak{U}^*$ *is a* T.I. *set in* $\mathfrak{G}$ *unless* $\mathfrak{U}^* = \mathfrak{U}$ *and* $N(\mathfrak{U}) \subseteq \mathfrak{S}$.

Proof. Since $\mathfrak{U}^*$ is a cyclic S-subgroup in $N(\mathfrak{U}^*)$, $\mathfrak{U}^*$ is a S-subgroup of $\mathfrak{G}$. Suppose that $\mathfrak{U}^*$ is not a T.I. set in $\mathfrak{G}$ and let $1 \neq \mathfrak{U}^* \cap G^{-1}\mathfrak{U}^*G = \mathfrak{U}_0 \subseteq \mathfrak{U}^*$. Then $\{N(\mathfrak{U}^*), N(G^{-1}\mathfrak{U}^*G)\} \subseteq N(\mathfrak{U}_0)$. Since $N(\mathfrak{U}^*)$ is a maximal subgroup of $\mathfrak{G}$ this implies that $\{\mathfrak{U}^*, G^{-1}\mathfrak{U}^*G\} \subseteq N(\mathfrak{U}^*)$. Thus $G^{-1}\mathfrak{U}^*G = \mathfrak{U}^*$ and $\mathfrak{U}^*$ is a T.I. set in $\mathfrak{G}$.

35. Further Results About $\mathfrak{S}$ and $\mathfrak{T}$

The notation of Section 34 is used in this section. However we will destroy the symmetry of $\mathfrak{S}$ and $\mathfrak{T}$ by choosing the notation so that

$$q < p\,. \tag{35.1}$$

The next three lemmas are restatements of Lemmas 34.7, 34.8, 34.9 and 34.10.

LEMMA 35.1. *If* $q \geqq 5$ *then* $c = d = 1$, $v = (q^p - 1)/(q - 1)$, $|\mathfrak{P}| = p^q$ *and* $|\mathfrak{Q}| = q^p$. *Either* $u = (p^q - 1)/(p - 1)$ *or* $p \equiv 1 \pmod q$ *and* $u = 1/q\,[(p^q - 1)/(p - 1)]$. *Furthermore* $\mathfrak{P}\mathfrak{U}$ *and* $\mathfrak{Q}\mathfrak{V}$ *are Frobenius groups.*

$$|N(\mathfrak{U}^*) : \mathfrak{U}^*| = q \text{ or } pq \text{ and } |N(\mathfrak{V}^*) : \mathfrak{V}^*| = p \text{ or } pq\,.$$

LEMMA 35.2. *Suppose that* $q = 3$. *Then* $|\mathfrak{Q}| = 3^p$,

$$\frac{v}{d} > \frac{9}{20} \cdot \frac{3^{p-1}}{p}$$

and $\mathfrak{Q}\mathfrak{V}/\mathfrak{D}$ *is a Frobenius group with* $v\,|\,(3^p - 1)/2$. *Either* $d = 1$ *or* $d = 11$, $p = 5$ *and* $v = 121$. *Furthermore* $\mathscr{V} = \mathscr{V}_0$ *and*

$$|N(\mathfrak{V}^*) : \mathfrak{V}^*| = p,\, 3p \text{ or } 7p\,.$$

LEMMA 35.3. *Suppose that* $q = 3$. *Then*

$$|N(\mathfrak{U}^*):\mathfrak{U}^*C(\mathfrak{U}^*)| = 3 \text{ or } 3p \quad \text{if } p \geqq 7$$
$$= 3, 15 \text{ or } 33 \quad \text{if } p = 5 .$$

Furthermore one of the following possibilities occurs:

(i) $c = 1, u > (p^2 + p + 1)/13$, $\mathfrak{P}$ *is an elementary abelian p-group with* $|\mathfrak{P}| = p^3$ *or* $|\mathfrak{P}| = 7^4$.

(ii) $c = 7, u > (p^2 + p + 1)/2$, $\mathfrak{P}$ *is an elementary abelian p-group with* $|\mathfrak{P}| = p^3$.

LEMMA 35.4. *Either* $q = 3, p = 5, v = 11, u = 31$ *or*

$$\frac{v-1}{p} > \frac{u-1}{q} .$$

Proof. By (5.12)

$$q^2\frac{(q^{p-1}-1)}{q-1} > p^2\frac{(p^{q-1}-1)}{p-1} .$$

Therefore if $v = (q^p - 1)/(q - 1)$ then by Lemma 34.1

$$\frac{v-1}{p} = \frac{1 + \cdots + q^{p-1} - 1}{p} = \frac{q(q^{p-1}-1)}{p(q-1)}$$
$$> \frac{p(p^{q-1}-1)}{q(p-1)} = \frac{\frac{p^q-1}{p-1} - 1}{q} \geqq \frac{u-1}{q} .$$

Suppose now that $v \neq (q^p - 1)/(q - 1)$. Then $q = 3$ by Lemma 35.1. By Lemma 35.2 $v \mid (3^p - 1)/2$ and $v > 9/20 \cdot (3^{p-1}/p)$. Thus if $(v - 1)/p \leqq (u - 1)/q$ then by Lemma 34.2

$$\frac{\frac{9}{20}\cdot\frac{3^{p-1}}{p} - 1}{p} \leqq \frac{p^2 + p}{3} .$$

Hence $p < 11$. Thus $p = 5$ or $p = 7$. If $p = 7$ then $v \mid (3^7 - 1)/2 = 1093$. As 1093 is a prime this implies that $v = (3^7 - 1)/2$ and the result follows from the first part of the lemma. If $p = 5$ then $v \mid (3^5 - 1)/2 = 121$. Thus $v = 11$ and $u \mid 31$. Thus $u = 31$. The proof is complete.

LEMMA 35.5. $\mathscr{Y}$ *is coherent.*

Proof. Suppose that $\mathscr{Y}$ is not coherent. Then by Lemma 11.2 v^*d is a power of some prime r. As $\mathfrak{V}/\mathfrak{D}$ is cyclic $r \equiv 1 \pmod{p}$. Thus

$$r > 2p > 2q . \tag{35.2}$$

Let $|\mathfrak{V}^* : D(\mathfrak{V}^*)| = r^n$, then $n \geqq 3$ by Lemma 11.3. By Lemma 11.1

$$r^n \leqq 4|N(\mathfrak{V}^*) : \mathfrak{V}^*|^2 + 1 . \tag{35.3}$$

Suppose that $|N(\mathfrak{V}^*) : \mathfrak{V}^*| = 7p$. Then $p \neq 7$ and (35.2) and (35.3) imply that $r^n \leqq 200p^2 \leqq 50r^2$. If $n \geqq 4$ this yields that $r \leqq 7$. Then $p = 3$ by (35.2) which is not the case as $p > q$. Hence $n = 3$. Thus Lemma 11.4 implies that $r^3 \leqq 2r(7p) + 1$. Hence by (35.2) $r^2 \leqq 14p < 7r$ and so $r < 7$ which is impossible.

By Lemmas 35.1 and 35.2 we may assume now that $|N(\mathfrak{V}^*) : \mathfrak{V}^*| \leqq pq$. Thus (35.2) and (35.3) imply that

$$r^n \leqq 4p^2q^2 + 1 < (2p)^4 < r^4 ,$$

thus $n = 3$. Hence Lemma 11.4 implies that

$$r^3 \leqq 2rpq + 1 < \frac{r^3}{2} .$$

This completes the proof in all cases.

LEMMA 35.6. $d = 1$. *If* $|N(\mathfrak{V}^*): \mathfrak{V}^*| \leqq pq$ *then* $v^* = v$ *or* $p = 5$, $q = 3$, $v = 11$, $v^* = 121$.

Proof. If $|N(\mathfrak{V}^*) : \mathfrak{V}^*| > pq$ then $c \neq 1$. Hence $d = 1$ by Lemma 34.2. Assume now that $|N(\mathfrak{V}^*):\mathfrak{V}^*| \leqq pq$.

Assume first that $d \neq 1$. By Lemmas 35.1 and 35.2 $d = 11$, $q = 3$, $p = 5$ and $v = 121$. By Lemma 34.2 $u = (5^3 - 1)/(5 - 1) = 31$. Thus

$$\frac{dv^* - 1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} \geqq \frac{11^3 - 1}{15} > \frac{11^2 - 1}{5} = \frac{v - 1}{p}$$

and

$$\frac{dv^* - 1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} \geqq \frac{11^3 - 1}{15} > \frac{31 - 1}{3} = \frac{u - 1}{q} .$$

Hence by Lemmas 35.5 and 34.11 $1/p \leqq pq/v^*d$.
Thus

$$11^3 \leqq v^*d \leqq p^2q = 75 .$$

Therefore $d = 1$.

Assume now that $q = 3$, $p = 5$, $v = 11$, $u = 31$. Let $v^* = vx$. $x \equiv 1 \pmod{10}$ as $v \equiv v^* \equiv 1 \pmod{10}$. If $v^* \neq 11$ and $v^* \neq 121$, then $x \geqq 21$. Thus $v^* \geqq 21.11$.

$$\frac{v^*-1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} \geqq \frac{21.11-1}{15} > \frac{11-1}{5} = \frac{v-1}{p}$$

and

$$\frac{v^*-1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} \geqq \frac{21.11-1}{15} > \frac{31-1}{3} = \frac{u-1}{q}.$$

Thus Lemmas 35.5 and 34.11 imply that $1/p \leqq pq/v^*$. Thus $21.11 \leqq v^* \leqq p^2q = 75$ which is not the case. Therefore $v = v^* = 11$ or $v^* = 121$, and we are done in this case.

By Lemma 35.4 it may now be assumed that $(v-1)/p > (u-1)/q$. If $v^* = vx$, then $x \equiv 1 \pmod{2p}$ since $v^* \equiv v \equiv 1 \pmod{2p}$. Thus

$$v^* = xv, x > 2p > 2q \quad \text{if } x \neq 1. \tag{35.4}$$

Therefore

$$\frac{v^*-1}{|N(\mathfrak{V}^*):\mathfrak{V}^*|} \geqq \frac{v^*-1}{pq} > \frac{2vq-1}{pq} > \frac{v-1}{p} > \frac{u-1}{q}.$$

Hence by Lemmas 35.5 and 34.11 $1/p < pq/v^*$. Hence (35.4) and Lemmas 35.1 and 35.2 imply that

$$q^{p-1} < \frac{20}{9}pv \leqq \frac{10}{9}v^* \leqq \frac{10}{9}p^2q.$$

Thus $q^{p-2} < 2p^2$. Hence $p < 7$ by (5.4). Thus $p = 5$. Hence $x \geqq 11$, $q = 3$ and $v \mid 121$. By assumption $v \neq 11$, hence $v = 121$. Thus $11^3 \leqq v^* \leqq p^2q = 75$. This completes the proof in all cases.

LEMMA 35.7.

$$|N(\mathfrak{U}^*):\mathfrak{U}^*C(\mathfrak{U}^*)| = q \text{ or } pq.$$

Proof. This follows directly from Lemmas 35.1, 35.2, 35.3 and 35.6.

THEOREM 35.1. *If $N(\mathfrak{U}^*)$ is conjugate to $N(\mathfrak{V}^*)$ then the conclusions of Theorem* 27.1 *hold.*

Proof. By Lemma 35.6 if $\mathfrak{V}^* \neq \mathfrak{V}$ then $p = 5$, $q = 3$ and $v^* = 121$. Thus $u = 31$. Hence u does not divide v^*. Thus by Lemmas 35.1 and 35.2, $\mathfrak{V}^* = \mathfrak{V}$ is cyclic. By Theorem 33.1 $N(\mathfrak{V}^*)$ is a Frobenius group with Frobenius kernel $\mathfrak{V}^*$. Hence by Lemma 34.12 $\mathfrak{V}^*$ is a T.I. set in $\mathfrak{G}$. Since $\mathfrak{Q}^* \subseteq N(\mathfrak{U}^*)$ and $p \mid |N(\mathfrak{V}^*):\mathfrak{V}^*|$ Lemma 35.7 implies that $N(\mathfrak{U}^*)/\mathfrak{U}^*$ is a cyclic group of order pq. Thus condition (iv) of Theorem 27.1 holds. Since $\mathfrak{V}^*$ is cyclic so is $\mathfrak{U}$. Thus $\mathfrak{C}$ char $\mathfrak{U}$. Hence if $\mathfrak{C} \neq 1$ then $N(\mathfrak{U}) \subseteq \mathfrak{S}$ which is not the case. Hence

$c = 1$. By Lemma 35.6 $d = 1$. Thus $C(\mathfrak{Q}^*) = \mathfrak{Q}\mathfrak{P}^*$ and $C(\mathfrak{P}^*) = \mathfrak{P}\mathfrak{Q}^*$. Hence condition (iii) of Theorem 27.1 holds. If $|\mathfrak{P}| \neq p^q$ or $|\mathfrak{Q}| \neq q^p$, then $N(\mathfrak{U}) \subseteq \mathfrak{S}$ or $N(\mathfrak{V}) \subseteq \mathfrak{T}$ respectively. This implies that $\mathfrak{P}$ is elementary abelian of order p^q and $\mathfrak{Q}$ is elementary abelian of order q^p. Hence condition (i) of Theorem 27.1 holds.

Since $\mathfrak{U}$ is cyclic and $\mathfrak{C} = 1$, $\mathfrak{P}\mathfrak{U}$ and $\mathfrak{U}\mathfrak{Q}^*$ are Frobenius groups and $N(\mathfrak{P})' = \mathfrak{S}' = \mathfrak{P}\mathfrak{U}$. Since $\mathfrak{U}^*$ is cyclic every divisor x of $|\mathfrak{U}^*|$ satisfies $x \equiv 1 \pmod{pq}$. Thus $(|\mathfrak{U}|, p-1) = 1$. Hence by Lemma 34.1 $|\mathfrak{U}| \mid (p^q - 1)/(p-1)$. Let $(p^q - 1)(p-1) = y|\mathfrak{U}|$. Suppose that $p \not\equiv 1 \pmod q$. Then $y \equiv 1 \pmod{pq}$ since

$$\frac{p^q - 1}{p - 1} \equiv |\mathfrak{U}| \equiv 1 \pmod{pq} .$$

Thus if $y \neq 1$, then $y > 2pq$. Furthermore Lemma 35.1 implies that in this case $q = 3$. Thus by Lemma 35.3 (i)

$$13 > \frac{p^2 + p + 1}{|\mathfrak{U}|} = y > 2pq = 6p$$

which is impossible as $p > 3$. Thus $y = 1$ and so $|\mathfrak{U}| = (p^q - 1)/(p-1)$. Suppose that $p \equiv 1 \pmod q$. Then $q \mid (p^q - 1)/(p-1)$. Hence $u \mid 1/q\,[(p^q-1)/(p-1)]$ since $(u, q) = 1$. As $q < p$ and $u \equiv (p^q - 1)/(p-1) \equiv 1 \pmod p$ we see that $u \neq 1/q\,[(p^q - 1)/(p-1)]$. Thus if $y \neq 1$, Lemma 35.1 yields that $q = 3$. Since $c = 1$, Lemma 35.3 (i) implies that $u > (p^2 + p + 1)/13$. This is impossible since $u \equiv 1 \pmod{3p}$. This verifies condition (ii) of Theorem 27.1 and completes the proof of the theorem.

36. The Proof of Theorem 27.1

In this section the study of the groups $\mathfrak{S}$ and $\mathfrak{T}$ is continued. All the lemmas in this section will be proved under the following assumption.

Hypothesis 36.1

(i) $q < p$.

(ii) $N(\mathfrak{U}^*)$ *is not conjugate to* $N(\mathfrak{V}^*)$.

The following notation is used in addition to that introduced in Section 34.

$$\phi \in \mathscr{U} , \quad \psi \in \mathscr{V}$$

and

$$\phi(1) = |N(\mathfrak{U}^*) : \mathfrak{U}^* C(\mathfrak{U}^*)| , \quad \psi(1) = |N(\mathfrak{V}^*) : \mathfrak{V}^*| .$$

If $\phi_i \in \mathscr{U}$ then ϕ_i^τ is defined since $|\mathfrak{G}|$ is odd. Let $\mathscr{U}^\tau = \{\phi_i^\tau \mid \phi_i \in \mathscr{U}\}$. Then

$$(36.1)\qquad \begin{aligned}(\tilde{1}_{\mathfrak{U}^*} - \phi)^\tau &= 1_{\mathfrak{G}} - \phi^\tau + \Gamma_{\mathfrak{U}} + \Xi_{\mathfrak{U}}, \quad \text{if } \mathscr{U} = \mathscr{U}_0\\ (\tilde{1}_{\mathfrak{S}'} - \phi)^\tau &= 1_{\mathfrak{G}} - \phi^\tau + \Gamma_{\mathfrak{U}} + \Xi_{\mathfrak{U}} \quad \text{if } \mathscr{U} \neq \mathscr{U}_0\end{aligned}$$

$$(36.2)\qquad (\tilde{1}_{\mathfrak{V}^*} - \psi)^\tau = 1_{\mathfrak{G}} - \psi^\tau + \Gamma_{\mathfrak{V}} + \Xi_{\mathfrak{V}}$$

$$(36.3)\qquad (\tilde{1}_{\mathfrak{P}\mathfrak{S}\mathfrak{Q}^*} - \mu_{0j})^* = 1_{\mathfrak{G}} \pm \eta_{0j} + \Gamma_{\mathfrak{P}} + \Xi_{\mathfrak{P}} \quad \text{for } 1 \leqq j \leqq p-1\,,$$

$$(36.4)\qquad (\tilde{1}_{\mathfrak{Q}\mathfrak{P}^*} - \nu_{i0})^* = 1_{\mathfrak{G}} \pm \eta_{i0} + \Gamma_{\mathfrak{Q}} + \Xi_{\mathfrak{Q}} \quad \text{for } 1 \leqq i \leqq q-1\,,$$

where $\Xi_{\mathfrak{U}}, \Xi_{\mathfrak{V}}$ are in $\mathscr{I}(\mathscr{U}^\tau), \mathscr{I}(\mathscr{V}^\tau)$ respectively, $\Gamma_{\mathfrak{U}}, \Gamma_{\mathfrak{V}}$ are orthogonal to $\mathscr{U}^\tau, \mathscr{V}^\tau$ respectively. $\Xi_{\mathfrak{P}}, \Xi_{\mathfrak{Q}}$ are linear combinations of the generalized characters η_{st} and $\Gamma_{\mathfrak{P}}, \Gamma_{\mathfrak{Q}}$ are orthogonal to each η_{st}. Then $\Gamma_{\mathfrak{U}}, \Gamma_{\mathfrak{V}}, \Gamma_{\mathfrak{P}}$ and $\Gamma_{\mathfrak{Q}}$ are real valued generalized characters each of which is orthogonal to $1_{\mathfrak{G}}$. Thus

$$(36.5)\qquad (\Gamma_{\mathfrak{U}}, \eta_{01}) + (\Gamma_{\mathfrak{P}}, \phi^\tau) \not\equiv 0 \pmod 2\,,$$

$$(36.6)\qquad (\Gamma_{\mathfrak{V}}, \eta_{01}) + (\Gamma_{\mathfrak{P}}, \psi^\tau) \not\equiv 0 \pmod 2\,.$$

$$(36.7)\qquad (\Gamma_{\mathfrak{U}}, \eta_{10}) + (\Gamma_{\mathfrak{Q}}, \phi^\tau) \not\equiv 0 \pmod 2\,.$$

It is a simple consequence of Lemma 13.1 that

$$(36.8)\qquad (\Gamma_{\mathfrak{U}}, \eta_{01}) + (\Gamma_{\mathfrak{U}}, \eta_{10}) + (\Gamma_{\mathfrak{U}}, \eta_{11}) \not\equiv 0 \pmod 2\,.$$

$$(36.9)\qquad (\Gamma_{\mathfrak{V}}, \eta_{01}) + (\Gamma_{\mathfrak{V}}, \eta_{10}) + (\Gamma_{\mathfrak{V}}, \eta_{11}) \not\equiv 0 \pmod 2\,.$$

By Hypothesis 36.1 (ii) $\mathscr{U}^\tau$ is orthogonal to $\mathscr{V}^\tau$. Thus

$$(36.10)\qquad (\Gamma_{\mathfrak{U}}, \psi^\tau) + (\Gamma_{\mathfrak{V}}, \phi^\tau) \not\equiv 0 \pmod 2\,.$$

Since τ is an isometry (36.1), (36.2), (36.3) and (36.4) yield that

$$(36.11)\qquad \|\Gamma_{\mathfrak{U}}\|^2 \leqq |N(\mathfrak{U}^*) : \mathfrak{U}^* C(\mathfrak{U}^*)| - 1$$

$$(36.12)\qquad \|\Gamma_{\mathfrak{V}}\|^2 \leqq |N(\mathfrak{V}^*) : \mathfrak{V}^*| - 1$$

$$(36.13)\qquad \|\Gamma_{\mathfrak{P}}\|^2 \leqq \frac{u-1}{q}$$

$$(36.14)\qquad \|\Gamma_{\mathfrak{Q}}\|^2 \leqq \frac{v-1}{p}\,.$$

LEMMA 36.1. *$\mathscr{U}$ is coherent.*

Proof. If $\mathfrak{S}$ is of type IV then by Lemmas 35.2 and 35.3 $c = 1$ or 7 so by Lemma 11.1 the result follows from Theorem 29.1. If $\mathfrak{S}$ is of type III then $\mathfrak{U} = \mathfrak{U}^*$ is abelian and the result follows from Lemma 11.2. Suppose that $\mathscr{U}$ is not coherent. Then $\mathscr{U} = \mathscr{U}_0$ and by Lemma 11.2 $\mathfrak{U}^*$ is an r-group for some prime r. Furthermore $\mathfrak{S}$ is of type II. Let $e = |N(\mathfrak{U}^*) : \mathfrak{U}^*|$ then by Lemmas 11.1, 11.3 and

11.4 $\mathfrak{U}^{*\prime} = D(\mathfrak{U}^*) \neq 1$,

$$|\mathfrak{U}^*:\mathfrak{U}^{*\prime}| = r^n \quad \text{with } n \geqq 3\,, \tag{36.15}$$

$$r^n \leqq 4e^2 + 1,\, n \geqq 4 \quad \text{or } r^3 \leqq 2re + 1 \text{ and } n = 3\,. \tag{36.16}$$

Suppose first that $\mathfrak{U}$ is not cyclic. Then by Lemma 35.1 $q = 3$. If $c \neq 1$, then by Lemma 35.3 $\mathfrak{C}$ is cyclic and

$$u > \frac{p^2 + p + 1}{2} > \left(\frac{p-1}{2}\right)^2.$$

Thus by Lemma 34.1 $\mathfrak{U}/\mathfrak{C}$ is cyclic. Hence $\mathfrak{U}$ is generated by two elements. If $c = 1$ then Lemma 34.1 implies that $\mathfrak{U}$ is generated by two elements. Thus $\mathfrak{U} \neq \mathfrak{U}^*$. As $\mathfrak{S}$ is of type II $\hat{\mathfrak{S}}$ is a T.I. set in $\mathfrak{G}$. Consequently there exists an element R of order r such that $\mathfrak{U} = C_{\mathfrak{U}^*}(R)$. Thus $Z(\mathfrak{U}^*)$ is cyclic. Hence $r \equiv 1 \pmod{e}$. This contradicts (36.15) and (36.16).

Suppose now that $\mathfrak{U}$ is cyclic. Thus $r \equiv 1 \pmod{q}$. By (36.16) $N(\mathfrak{U}^*)/\mathfrak{U}^*$ is irreducibly represented on $\mathfrak{U}^*/D(\mathfrak{U}^*)$. Thus $\mathfrak{Q}^*$ acts as a group of scalar matrices on $\mathfrak{U}^*/D(\mathfrak{U}^*)$. Hence by Lemma 6.4 $\mathfrak{U}^*$ has prime exponent. Since $\mathfrak{U}$ is a cyclic subgroup of $\mathfrak{U}^*$ this implies that

$$|\mathfrak{U}| = r\,. \tag{36.17}$$

If $q > 3$ then Lemmas 35.1, 35.7 and (36.15) and (36.16) imply that

$$\left(\frac{p^{q-1}}{q}\right)^3 \leqq |\mathfrak{U}|^3 < 4e^2 + 1 \leqq 4p^2q^2 + 1\,.$$

Hence $p^{3q-5} \leqq 5q^5$ and so

$$5^{3q-10} \leqq q^{3q-10} < p^{3q-10} < 5\,.$$

Thus $3q - 10 < 1$ which is not the case.

Suppose that $q = 3$: If $n \geqq 4$ then (36.16) and Lemmas 35.3 and 35.7 imply that

$$\frac{(p^2 + p + 1)^4}{13^4} < |\mathfrak{U}|^4 \leqq 36p^2 + 1\,.$$

Hence

$$p^8 < (p^2 + p + 1)^4 < 13^4(36p^2 + 1) < 3.13^5p^2\,.$$

Thus $p^6 < 3.13^5$. Hence $p < 13$. If $n = 3$ then (36.16) and Lemmas 35.3 and 35.7 imply that

$$\frac{(p^2 + p + 1)^2}{13^2} < |\mathfrak{U}|^2 \leqq 6p\,.$$

Hence

$$p^4 < (p^2 + p + 1)^2 < 13^2 \cdot 6p < 13^3 p .$$

Therefore $p < 13$ in this case also. Thus $p = 5, 7$ or 11. By Lemma 34.1 and (36.17) either $|\mathfrak{U}| \mid (p - 1)$ or $|\mathfrak{U}| \mid p^2 + p + 1$. If $|\mathfrak{U}| \mid (p - 1)$ then $p = 11$ and $|\mathfrak{U}| = 5$ since $(|\mathfrak{U}|, 6) = 1$. However in this case

$$\frac{(p^2 + p + 1)}{13} > 10 > |\mathfrak{U}|$$

which is impossible by Lemma 35.1. Thus $|\mathfrak{U}| \mid p^2 + p + 1$. Hence by (36.17) if $p = 5$, $|\mathfrak{U}| = 31$, if $p = 7$, $|\mathfrak{U}| = 19$ and if $p = 11$ then $|\mathfrak{U}| = 7$ or $|\mathfrak{U}| = 19$. If $p = 5$ then (36.16) and (36.17) imply that

$$31^3 \leqq 36.25 + 1$$

which is not the case. If $p = 7$ then (36.16) and (36.17) imply that

$$19^3 < 36.49 + 1 < 1800 .$$

Thus $19^2 < 100$ which is not the case. If $p = 11$ and $|\mathfrak{U}| = 19$ then (36.16) and (36.17) imply that

$$15.360 < 19^3 < 36.121 + 1 < 4800$$

which is not the case.

Assume now that $p = 11$ and $|\mathfrak{U}| = r = 7$. Then (36.15) and (36.16) imply that

$$(36.18) \qquad 7^n \leqq 36.11^2 + 1 , \quad 7^n \equiv 1 \pmod{11} .$$

Since

$$7^5 > 10^4 > 5000 > 36.11^2 + 1$$

we must have $n \leqq 4$. However

$$7^2 \equiv 5, 7^3 \equiv 2, 7^4 \equiv 3 \pmod{11}$$

contrary to (36.18). The proof is complete.

LEMMA 36.2. $q = 3$.

Proof. Suppose that $q \neq 3$. Then by (36.10) either $(\Gamma_{\mathfrak{U}}, \psi^\tau) \neq 0$ or $(\Gamma_{\mathfrak{B}}, \phi^\tau) \neq 0$. If $u = 1/q\,[(p^q - 1)/(p - 1)]$, then $u \not\equiv 1 \pmod{p}$. Hence by Lemmas 35.1, 35.5 and 36.1,

$$\frac{\dfrac{q^p - 1}{q - 1} - 1}{pq} \leqq pq - 1 \quad \text{or} \quad \frac{\dfrac{p^q - 1}{p - 1} - q}{pq} \leqq pq - 1 .$$

Therefore by (5.11) $p^{q-1} < (p^q - 1)/(p - 1) < p^2q^2$. Hence $p^{q-3} < q^2 < p^2$

which is impossible for $q \geqq 5$.

LEMMA 36.3. *$c = 1, |N(\mathfrak{B}^*):\mathfrak{B}^*| = p$ or $3p$.*

Proof. If $c \neq 1$ then $c = 7$ and $u > (p^2 + p + 1)/2$ by Lemma 35.3. Since $[(p-1)/2]^2 < (p^2+p+1)/2$ Lemma 34.1 implies that $u \mid p^2+p+1$. Thus $u = p^2 + p + 1$. By Lemma 34.2 $v = (3^p - 1)/2$.

Suppose first that $|N(\mathfrak{U}^*):\mathfrak{U}^*| = 3$. Then by (36.8) $\Gamma_{\mathfrak{U}} = \pm(\eta_{10}+\eta_{20})$. Thus $(\Gamma_{\mathfrak{U}}, \eta_{01}) = 0$. Hence $(\Gamma_{\mathfrak{P}}, \phi^\tau) \neq 0$ by (36.5). Since $\mathscr{U}$ is coherent (36.13) implies that

$$\frac{7u^* - 1}{3} \leqq ||\Gamma_{\mathfrak{P}}||^2 \leqq \frac{u-1}{3} \leqq \frac{u^*-1}{3},$$

which is not the case.

Suppose now that $|N(\mathfrak{U}^*):\mathfrak{U}^*| \neq 3$. Then by Lemma 35.7 $|N(\mathfrak{U}^*):\mathfrak{U}^*| = 3p$. Let $cu^* = xu = x(1 + p + p^2)$. Then $x \equiv 1 \pmod{6p}$ since

$$cu^* \equiv u \equiv 1 \pmod{6p}.$$

As $1 < c \leqq x$ this implies that $x \geqq 6p + 1$. Hence by Lemma 35.2 and (36.12)

$$\frac{cu^* - 1}{3p} > \frac{6pu}{3p} \geqq 2u > 7p - 1 \geqq ||\Gamma_{\mathfrak{P}}||^2. \tag{36.19}$$

Since $\mathscr{U}$ is coherent this implies that $(\Gamma_{\mathfrak{P}}, \phi^\tau) = 0$. Thus by (36.10)

$$(\Gamma_{\mathfrak{U}}, \psi^\tau) \neq 0. \tag{36.20}$$

Since $\mathscr{U}$ is coherent (36.13) and (36.19) imply that $(\Gamma_{\mathfrak{P}}, \phi^\tau) = 0$. Thus by (36.5)

$$(\Gamma_{\mathfrak{U}}, \eta_{01}) \not\equiv 0 \pmod 2. \tag{36.21}$$

Since $\mathscr{V}$ is coherent (36.11), (36.20) and (36.21) imply that

$$(p-1) + \frac{v^* - 1}{|N(\mathfrak{B}^*):\mathfrak{B}^*|} \leqq 3p - 1.$$

Hence by Lemma 35.2

$$\frac{3^p - 1}{2} - 1 = v - 1 \leqq v^* - 1 \leqq 2p\,|N(\mathfrak{B}^*):\mathfrak{B}^*| \leqq 14p^2.$$

Therefore $3^p - 3 \leqq 28p^2$. Hence $p = 5$ by (5.5). Thus $u = 31$ and $v = 121$. If the S_7-subgroup of $\mathfrak{U}^*$ has order 7^n, then $7^n \equiv 1 \pmod 5$. Thus $n \geqq 4$. Therefore

$$\frac{u^*-1}{3p} \geqq \frac{7^4.31-1}{15} > 24 = \frac{v-1}{p}.$$

Thus the coherence of $\mathscr{U}$ implies that $(\Gamma_{\mathfrak{Q}}, \phi^\tau) = 0$. Hence (36.7) yields that $(\Gamma_{\mathfrak{U}}, \eta_{10}) \not\equiv 0 \pmod 2$. Therefore (36.8), (36.11) and (36.21) imply that

$$\Gamma_{\mathfrak{U}} = \pm \sum_{i=1}^{q-1} \eta_{i0} \pm \sum_{j=1}^{p-1} \eta_{0j} \pm \sum_{i=1}^{q-1}\sum_{j=1}^{p-1} \eta_{ij}$$

contrary to (36.20). Thus $c = 1$ and consequently $|N(\mathfrak{V}^*) : \mathfrak{V}^*| = p$ or $3p$.

LEMMA 36.4. $|N(\mathfrak{U}^*) : \mathfrak{U}^* C(\mathfrak{U}^*)| = 3p$.

Proof. If the result is false then $|N(\mathfrak{U}^*) : \mathfrak{U}^* C(\mathfrak{U}^*)| = 3$ by Lemma 35.7. Thus (36.8) implies that $\Gamma_{\mathfrak{U}} = \pm(\eta_{10} + \eta_{20})$. Therefore by (36.5) and (36.10) $(\Gamma_{\mathfrak{V}}, \phi^\tau) \neq 0$ and $(\Gamma_{\mathfrak{P}}, \phi^\tau) \neq 0$. Since $u^* \geqq u$ (36.13) implies that $u^* = u$ and

$$\Gamma_{\mathfrak{P}} = \pm \sum_i \phi_i^\tau , \tag{36.22}$$

where ϕ_i ranges over $\mathscr{U}$. Thus by (36.6) $(\Gamma_{\mathfrak{V}}, \eta_{01})$ is odd. Hence by Lemma 36.3 and (36.12)

$$\Gamma_{\mathfrak{V}} = b \sum \phi_i^\tau \pm \sum_{j=1}^{p-1} \eta_{0j} + \varDelta_{\mathfrak{V}} ,$$

where b is odd and $\varDelta_{\mathfrak{V}}$ is orthogonal to all ϕ_i^τ, η_{0j}. Therefore by (36.22)

$$0 = ((\tilde{1}_{\mathfrak{P}\mathfrak{Q}^*} - \mu_{01})^*, (\tilde{1}_{\mathfrak{V}^*} - \psi)^\tau) = 1 \pm 1 \pm b\frac{(u-1)}{3}.$$

Since $b \neq 0$ this implies that $|b|(u-1)/3 = 2$. Hence $u = 7$. Thus by Lemma 35.3 (i) $7 \geqq (p^2 + p + 1)/13$, hence $p < 10$. Hence $p = 5$ or $p = 7$. In either of these cases $u \mid (p^2 + p + 1)$ by Lemma 34.1 since $(u, 6) = 1$. Thus $7 \mid 31$ or $7 \mid 57$ which is not the case.

LEMMA 36.5. $|\mathfrak{P}| = p^q$.

Proof. If $|\mathfrak{P}| \neq p^q$ then $N(\mathfrak{U}) \subseteq \mathfrak{S}$ as $\mathfrak{P}$ is a T.I. set in $\mathfrak{G}$. This contradicts Lemma 36.4.

LEMMA 36.6. $\mathfrak{U}$ *is cyclic.*

Proof. By Lemma 34.1 if $\mathfrak{U}$ is not cyclic then $\mathfrak{U} = \mathfrak{U}_1 \times \mathfrak{U}_2$, where each $\mathfrak{U}_i$ is cyclic and $|\mathfrak{U}_i| \mid (p-1)/2$. Let $|\mathfrak{U}_i| = (p-1)/2y_i$ for $i = 1, 2$. If $y_1 y_2 \geqq 4$ then Lemma 35.3 (i) implies that

$$\frac{p^2}{13} < \frac{p^2+p+1}{13} < \frac{(p-1)^2}{4y_1y_2} \leqq \frac{(p-1)^2}{16} < \frac{p^2}{16}$$

which is not the case. Thus $y_1y_2 < 4$. If $y_1y_2 = 2$ then $p \equiv 1 \pmod 4$ and so $|\mathfrak{U}| = (p-1)^2/8$ is even. If $y_1y_2 = 3$ then $p \equiv 1 \pmod 3$ and so $3 \mid u$ which is not the case. Thus $y_1y_2 = 1$ and $u = [(p-1)/2]^2$. Therefore $((p-1)/2, 6) = 1$. Thus $p \geqq 11$. Furthermore $u \equiv 1/4 \pmod p$. Since $u^* \equiv 1 \pmod p$ by Lemma 36.4 we have that $u^* = ux$ and $x \equiv 4 \pmod p$. By Lemma 34.2 $v = (3^p - 1)/2$. Hence Lemma 36.3 and (36.10), (36.11) and (36.12) imply that

$$\frac{\frac{3^p-1}{2} - 1}{3p} \leqq 3p - 1 \quad \text{or} \quad \frac{u^* - 1}{3p} < 3p - 1\,. \tag{36.23}$$

The first possibility implies that $3^p - 3 \leqq 18p^2 - 6p$. Thus $3^{p-2} \leqq 2p^2$. Hence $p < 7$ by (5.4). The second possibility in (36.23) yields that

$$\frac{(p-1)^2}{4}x - 1 \leqq 9p^2 - 3p\,.$$

Therefore

$$(p-1)^2 x \leqq 36p^2 - 12p + 4 < 36p^2\,.$$

As $p \geqq 11$ this implies that

$$x < 36\left(\frac{p}{p-1}\right)^2 = 36\left(1 + \frac{1}{p-1}\right)^2 \leqq 36\left(\frac{121}{100}\right) < 45\,. \tag{36.24}$$

Let $x = 4 + zp$ for some integer z. Then since $p \geqq 11$ (36.24) yields that $z < 4$. Furthermore

$$p < 41\,; \quad \text{if} \quad z \geqq 2\,, \quad p < 20\,; \quad \text{if} \quad z = 3\,, \quad p < 14\,. \tag{36.25}$$

As $p < 41$ and $((p-1)/2, 6) = 1$, $p = 11$ or $p = 23$. If $p = 23$ then by (36.25) $x = 27$ which is impossible as $x \equiv 1 \pmod 3$. If $p = 11$, then $x = 15$, 26 or 37. As $x \equiv 1 \pmod 6$ this implies that $x = 37$. Then $u = 25$ and so $37 \equiv 1 \pmod{11}$ by Lemma 36.4 which is not the case.

LEMMA 36.7. *$u = p^2 + p + 1$ or $u = (p^2 + p + 1)/3$ or $u = (p^2 + p + 1)/7$.*

Proof. If $u \mid [(p-1)/2]^2$ then by Lemmas 34.1 and 36.6 $u \mid (p-1)/2$. Thus by Lemma 35.3 (i) $(p-1)/2 > (p^2+p+1)/13$. Hence $2p^2 - 11p + 15 < 0$ which implies that $p < 5$. Therefore by Lemma 34.1 $p^2 + p + 1 = uy$, y an integer. By Lemma 35.3 (i) $y < 13$. If r is a prime such that $p^2 + p + 1 \equiv 0 \pmod r$ then either $r = 3$ or

$r \equiv 1 \pmod 3$. Hence $y = 1, 3, 7$ or 9. If $y = 9$ then $p^2 + p + 1 \equiv 0 \pmod 9$. Hence $p \equiv 1 \pmod 3$. Thus $p \equiv 1, 4$ or $7 \pmod 9$. In none of these cases is $p^2 + p + 1 \equiv 0 \pmod 9$. Hence $y = 1, 3$ or 7.

LEMMA 36.8. $u = u^* = p^2 + p + 1$.

Proof. Let $u^* = ux$. Assume that $x \neq 1$. $u^* \equiv 1 \pmod{6p}$ by Lemma 36.4. If $u = p^2 + p + 1$, then $u \equiv u^* \equiv 1 \pmod{6p}$, thus $x \equiv 1 \pmod{6p}$ and so $x \geqq 1 + 6p$. If $u = (p^2 + p + 1)/3$, then $x \equiv 3 \pmod p$. Furthermore $x \equiv 1 \pmod 6$ since $u \equiv u^* \equiv 1 \pmod 6$ and $p \equiv 1 \pmod 6$ since $p^2 + p + 1 \equiv 0 \pmod 3$. Thus if $x = 3 + zp$ then $1 \equiv 3 + z \pmod 6$. Hence $x \geqq 3 + 4p$. If $u = (p^2 + p + 1)/7$ then $x \equiv 7 \pmod p$. If $x = 7$ then by Lemma 36.6 the S_7-subgroup of $\mathfrak{U}^*$ is generated by two elements. Hence $7^2 - 1 \equiv 0 \pmod p$ by Lemma 36.4. However $7^2 - 1 = 48$ and $(p, 48) = 1$. Thus $x \neq 7$. Let $x = 7 + zp$. Then $p^2 + p + 1 \equiv u \equiv 1 \pmod 6$. Hence $p \equiv 5 \pmod 6$. Thus $1 \equiv x \equiv 7 + 5z \pmod 6$, hence $z \equiv 0 \pmod 6$. Therefore $x \geqq 7 + 6p$. Thus in any case

$$u^* = ux, \qquad x \geqq 4p + 3. \tag{36.26}$$

Therefore $(u^* - 1)/3p > (u - 1)/3$. Hence by (36.13) and the coherence of $\mathscr{U}$

$$(\phi^\tau, \Gamma_{\mathfrak{P}}) = 0. \tag{36.27}$$

Assume first that $(\phi^\tau, \Gamma_{\mathfrak{P}}) \neq 0$, then by (36.12) and the coherence of $\mathscr{U}$

$$\frac{u^* - 1}{3p} \leqq 3p - 1. \tag{36.28}$$

Suppose now that $(\phi^\tau, \Gamma_{\mathfrak{P}}) = 0$. Then by (36.10) $(\psi^\tau, \Gamma_{\mathfrak{U}}) \neq 0$. Hence the coherence of $\mathscr{V}$ and (36.11) imply that

$$\frac{v^* - 1}{3p} \leqq 3p - 1. \tag{36.29}$$

By (36.27) and (36.5) $(\eta_{01}, \Gamma_{\mathfrak{U}}) \not\equiv 0 \pmod 2$. If also $(\eta_{10}, \Gamma_{\mathfrak{U}})$ were odd then by (36.8) $(\eta_{ij}, \Gamma_{\mathfrak{U}}) \neq 0$ for $1 \leqq i \leqq q - 1$, $1 \leqq j \leqq p - 1$. Thus by (36.11) $(\psi^\tau, \Gamma_{\mathfrak{U}}) = 0$ contrary to what has been proved. Therefore $(\eta_{10}, \Gamma_{\mathfrak{U}}) \equiv 0 \pmod 2$. Hence by (36.7) $(\Gamma_{\mathfrak{Q}}, \phi^\tau) \neq 0$. Thus by (36.14) and (36.29)

$$\frac{u^* - 1}{3p} \leqq \frac{v - 1}{p} \leqq \frac{v^* - 1}{p} < 9p - 3.$$

Now (36.28) implies that in any case

(36.30) $$\frac{u^* - 1}{9p} \leqq 3p - 1 .$$

For any prime r let $\mathfrak{U}_r$ be the S_r-subgroup of $\mathfrak{U}^*$.

Suppose first that $u = p^2 + p + 1$, then $x > 6p$. Hence (36.30) implies that

$$6(p^2 + p + 1) - 1 \leqq 27p - 9 .$$

Therefore $2p^2 - 7p + 4 \leqq 0$ which is impossible for $p \geqq 5$.

Suppose now that $u = (p^2 + p + 1)/3$ then $x \geqq 4p + 3$ by (36.26). Hence (36.30) implies that

$$4(p^2 + p + 1) < 81p .$$

Thus $4p < 81$ or $p < 22$. Since $p \equiv 1 \pmod 3$ this yields that $p = 7$, $p = 13$ or $p = 19$.

If $p = 7$ then $u = 19$. If $|\mathfrak{U}_{19}| = 19^n$ then $n \geqq 6$ as $|\mathfrak{U}_{19}| \equiv 1 \pmod 7$. Thus (36.30) implies that $19^6 \leqq 27.7^2 \leqq 19^4$. If $p = 13$ then $u = 61$. Let $|\mathfrak{U}_{61}| = 61^n$, then $n \geqq 3$ as $|\mathfrak{U}_{61}| \equiv 1 \pmod{13}$. Hence (36.30) implies that $61^3 \leqq 27.13^2 < 61^3$. If $p = 19$ then $u = 127$. Let $|\mathfrak{U}_{127}| = 127^n$, then $n \geqq 3$ as $|\mathfrak{U}_{127}| \equiv 1 \pmod{19}$. Hence (36.30) implies that $127^3 \leqq 27.19^2 < 127^3$.

Assume finally that $u = (p^2 + p + 1)/7$ then $x \geqq 6p + 1$. Thus (36.30) implies that

$$\frac{6(p^2 + p + 1)}{7} \leqq 27p .$$

Therefore $6p < 27.7$, so $p < 32$. Since $p^2 + p + 1 \equiv 0 \pmod 7$, $p \equiv 2 \pmod 7$ or $p \equiv 4 \pmod 7$. Thus $p = 11$ or $p = 23$.

If $p = 11$ then $u = 19$. Let $|\mathfrak{U}_{19}| = 19^n$; then $n \geqq 3$ as $|\mathfrak{U}_{19}| \equiv 1 \pmod{11}$. Hence (36.30) implies that $19^3 \leqq 27.11^2 = 287.11 < 19^3$. If $p = 23$ then $u = 79$. As $|\mathfrak{U}_{79}| \equiv 1 \pmod{23}$, $|\mathfrak{U}_{79}| \geqq 79^3$. Hence (36.30) implies that $79^3 \leqq 27.23^2 < 79^3$.

Therefore $u = u^*$ in all cases. Hence $u \equiv 1 \pmod p$ by Lemmas 36.4 and 36.5. Since $(p, 6) = 1$, $7 \not\equiv 1 \pmod p$ and $3 \not\equiv 1 \pmod p$. Hence by Lemma 36.7 $u = p^2 + p + 1$.

The proof of Theorem 27.1 under Hypothesis 36.1 is now immediate.

Let $q = 3$ and p have the same meaning as in the earlier part of this section. By Lemma 35.2 $|\mathfrak{Q}| = q^p$. By Lemma 36.5 $|\mathfrak{P}| = p^q$. The other properties of Condition (i) follow from the structure of $\mathfrak{S}$ and $\mathfrak{T}$ and Theorem 14.1. Thus Condition (i) is verified. By Lemma 35.6 $C(\mathfrak{Q}) \subseteq \mathfrak{Q}$. Hence $C(\mathfrak{Q}^*) = \mathfrak{P}^*\mathfrak{Q}$. By Lemma 36.3 $C(\mathfrak{P}) \subseteq \mathfrak{P}$, hence $C(\mathfrak{P}^*) = \mathfrak{P}\mathfrak{Q}^*$ by Lemma 36.5. The other properties of Condition (iii) follow from the structure of $\mathfrak{S}$ and $\mathfrak{T}$. Thus Condition (iii) is verified. Lemmas 36.6 and 36.8 imply that $\mathfrak{U} = \boldsymbol{C}(\mathfrak{U})$ is cyclic. By Lemmas 34.12 and 36.4 $\mathfrak{U} = \mathfrak{U}^*$ is a T.I. set in $\mathfrak{G}$. Hence Lemma 36.4 completes the verification of Condition (iv).

Lemmas 34.1, 36.3, 36.5 and 36.8 imply that $\mathfrak{P}\mathfrak{U}$ is a Frobenius group. Lemma 36.8 implies that $|\mathfrak{U}| = (p^q - 1)/(p - 1)$. Lemmas 36.4, 36.6 and 36.8 imply that if $u_0 \mid |\mathfrak{U}|$ then $u_0 \equiv 1 \pmod{pq}$. Thus $(|\mathfrak{U}|, p - 1) = 1$. The other statements in Condition (ii) follow from the structure of $\mathfrak{S}$ and $\mathfrak{T}$.

By Theorem 35.1 this completes the proof of Theorem 27.1 in all cases.

37. Statement of the Result Proved in Chapter VI

The purpose of this chapter is to prove the following result.

THEOREM 37.1. *There are no groups* $\mathfrak{G}$ *which satisfy conditions* (i)–(iv) *of Theorem* 27.1.

Once it is proved, Theorem 37.1 together with Theorem 27.1 will serve to complete the proof of the main theorem of this paper. In this chapter there is no reference to anything in Chapters II–V other than the statement of Theorem 27.1. The following notation is used throughout this chapter.

$\mathfrak{G}$ is a fixed group which satisfies conditions (i)–(iv) of Theorem 27.1.

$$|\mathfrak{U}| = u = \frac{p^q - 1}{p - 1}$$

$$\mathfrak{U}^* = C(\mathfrak{U}) \quad \text{and} \quad |\mathfrak{U}^*| = u^* .$$

$$\mathfrak{U}^* = \langle U_1 \rangle, \ U = U_1^{u^*/u} . \quad \text{Thus } \mathfrak{U} = \langle U \rangle$$

$$\mathfrak{Q}_0 = [\mathfrak{Q}, \mathfrak{P}^*] \quad \text{so that} \quad \mathfrak{Q} = \mathfrak{Q}^* \times \mathfrak{Q}_0 .$$

P and Q are fixed elements of $\mathfrak{P}^{*\#}$ and $\mathfrak{Q}^{*\#}$ respectively.

For any integer $n > 0$, $\mathscr{Z}_n$ is the ring of integers mod n. If n is a prime power then $\mathscr{F}_n$ is the field of n elements.

U acts as a linear transformation on $\mathfrak{P}$. Let $m(t)$ be the minimal polynomial of U on $\mathfrak{P}$. Then $m(t)$ is an irreducible polynomial of degree q over $\mathscr{F}_p$. Let ω be a fixed root of $m(t)$ in $\mathscr{F}_{p^q}$. Then ω is a primitive uth root of unity in $\mathscr{F}_{p^q}$ and $\omega, \omega^p, \cdots, \omega^{p^{q-1}}$ are all the characteristic roots of U on $\mathfrak{P}$.

38. The Sets $\mathscr{A}$ and $\mathscr{B}$

LEMMA 38.1. *There exists an element* $Y \in \mathfrak{Q}_0^\#$ *such that* $\mathfrak{P}^*$ *normalizes* $Y\mathfrak{U}^*Y^{-1}$

Proof. $\mathfrak{Q}^*$ normalizes $\mathfrak{U}^*$ and $\mathfrak{Q}^*$ is contained in a cyclic subgroup of $N(\mathfrak{U}^*)$ of order pq. Hence some element of order p in $C(\mathfrak{Q}^*)$ normalizes $\mathfrak{U}^*$. Since $C(\mathfrak{Q}^*) = \mathfrak{Q}\mathfrak{P}^*$ every subgroup of order p in $C(\mathfrak{Q}^*)$ is of the form $Y^{-1}\mathfrak{P}^*Y$ for some $Y \in \mathfrak{Q}_0$. Hence it is possible to choose $Y \in \mathfrak{Q}_0$ such that $Y^{-1}\mathfrak{P}^*Y$ normalizes $\mathfrak{U}^*$. Since $[\mathfrak{P}^*, \mathfrak{U}] \subseteq \mathfrak{P}$,

$\mathfrak{P}^*$ does not normalize $\mathfrak{U}^*$, hence $Y \in \mathfrak{Q}_0^\sharp$ and $\mathfrak{P}^*$ normalizes $Y\mathfrak{U}^*Y^{-1}$.

From now on let

$$(38.1) \qquad Z_1 = YU_1Y^{-1}, \quad Z = YUY^{-1} = Z_1^{u^*/u}$$

where Y satisfies Lemma 38.1. Notice that $\mathfrak{Q}^*$ normalizes $\langle Z_1 \rangle$, since $\mathfrak{Q}^*$ normalizes $\mathfrak{U}^*$ and Y centralizes $\mathfrak{Q}^*$. Define $v, w \in \mathscr{X}_{u^*}$ by

$$(38.2) \qquad P^{-1}Z_1P = Z_1^v, \quad Q^{-1}Z_1Q = Z_1^w$$

LEMMA 38.2. *If* $Z_0 \in \langle Z_1 \rangle$, $a \in \mathscr{X}_p$, $b \in \mathscr{X}_q$ *then* $\langle Z_0 \rangle = \langle Z_0^{v^a w^b - 1} \rangle$ *unless* $a = 0$ *and* $b = 0$.

Proof. $Z_0^{-1}P^{-a}Q^{-b}Z_0Q^bP^a = Z_0^{v^a w^b - 1}$. Hence P^aQ^b acts trivially on $\langle Z_0 \rangle / \langle Z_0^{v^a w^b - 1} \rangle$. However if $Z_0 \neq 1$ then $\mathfrak{P}^*\mathfrak{Q}^*\langle Z_0 \rangle$ is a Frobenius group with Frobenius kernel $\langle Z_0 \rangle$. Thus $\langle Z_0 \rangle = \langle Z_0^{v^a w^b - 1} \rangle$ as required.

LEMMA 38.3. *Every element of* $\mathfrak{P}\mathfrak{U}$ *has a unique representation in the form* $P^{m_1(U)}U^a$, *where* $a \in \mathscr{X}_u$ *and* $m_1(t)$ *is a polynomial of degree at most* $q - 1$ *over* $\mathscr{X}_p$.

Proof. There are up^q ordered pairs $(m_1(t), a)$ with $a \in \mathscr{X}_u$ and $m_1(t)$ of degree at most $q - 1$ over $\mathscr{X}_p$. Thus it is sufficient to show the uniqueness of $(m_1(t), a)$ in such a representation.

If $P^{m_1(U)}U^a = P^{m_1'(U)}U^{a'}$. Then reading mod $\mathfrak{P}$ yields that $a = a'$. Since $m(t)$ is irreducible we get that $m_1(t) \equiv m_1'(t) \pmod{m(t)}$. Thus $m_1(t) = m_1'(t)$ as required.

LEMMA 38.4. *Every element of* $\mathfrak{P}\mathfrak{U} - \mathfrak{U}$ *has a unique representation in the form* $U^xP^yU^z$, *where* $x, z \in \mathscr{X}_u$ *and* $y \in \mathscr{X}_p$, $y \neq 0$.

Proof. If $X \in \mathfrak{P}\mathfrak{U} - \mathfrak{U}$ and

$$X = U^xP^yU^z = U^{x_1}P^{y_1}U^{z_1}$$

then reading mod $\mathfrak{P}$ we get that $x + z \quad x_1 + z_1$. Hence

$$U^{x-x_1}P^yU^{-x+x_1} = P^{y_1}$$

Since $X \notin \mathfrak{U}$, $y \neq 0$. As $(u, p - 1) = 1$ we have that $x = x_1$, and so $y = y_1$, $z = z_1$. The representation is unique. There are $u^2(p - 1)$ ordered triples (x, y, z) with $x, z \in \mathscr{X}_u$ and $y \in \mathscr{X}_p$, $y \neq 0$. Each triple gives rise to an element of $\mathfrak{P}\mathfrak{U} - \mathfrak{U}$ and $|\mathfrak{P}\mathfrak{U} - \mathfrak{U}| = u^2(p - 1)$. The result now follows.

LEMMA 38.5. *Let* $x, z, g \in \mathscr{X}_p = \mathscr{F}_p$; $y, f, h \in \mathscr{X}_u$. *Then*

$$P^x U^y P^z U^f P^g U^h = 1$$

if and only if

(i) $y + f + h = 0$
(ii) $x\omega^y + z + g\omega^{y+h} = 0.$

Proof. Let $R = P^x U^y P^z U^f P^g U^h$. Then

$$R = P^{x+zU^{-y}} + gU^{-y-f} U^{y+h+f} .$$

Thus by Lemma 38.3 $R = 1$ if and only if

$$y + h + f = 0, \quad x + zt^{-y} + gt^{-y-f} \equiv 0 \,(\mathrm{mod}\, m(t)) .$$

The first equation allows us to rewrite the second as

$$xt^y + z + gt^{y+h} \equiv 0 \,(\mathrm{mod}\, m(t)) .$$

Thus the lemma is proved.

DEFINITION 38.1. The set $\mathscr{A}$ is defined to consist of all ordered triples (a_1, a_2, a_3) such that

(i) $a_i \in \mathscr{X}_u$, $a_i \neq 0$ for $i = 1, 2, 3$.
(ii) $a_1 + a_2 + a_3 = 0$.
(iii) $PU^{a_1}P^{-2}U^{a_2}PU^{a_3} = 1$.

DEFINITION 38.2. $\mathscr{B}$ is the set of all elements $a_1 \in \mathscr{X}_u$ such that $(a_1, a_2, a_3) \in \mathscr{A}$ for suitable a_2, a_3.

LEMMA 38.6. $|\mathscr{A}| = |\mathscr{B}|$.

Proof. If $(a_1, a_2, a_3) \in \mathscr{A}$ then by Lemma 38.4 a_2 and a_3 are determined by a_1.

LEMMA 38.7. $(a_1, a_2, a_3) \in \mathscr{A}$ *if and only if*

(i) $a_i \in \mathscr{X}_u$, $a_i \neq 0$ *for* $i = 1, 2, 3$
(ii) $a_1 + a_2 + a_3 = 0$
(iii) $\omega^{a_1} + \omega^{a_1+a_3} - 2 = 0$.

Proof. By Lemma 38.5,

$$PU^{a_1}P^{-2}U^{a_2}PU^{a_3} = 1$$

if and only if $a_1 + a_2 + a_3 = 0$ and $\omega^{a_1} - 2 + \omega^{a_1+a_3} = 0$. This implies the result.

LEMMA 38.8. *If* $(a_1, a_2, a_3) \in \mathscr{A}$, *then* $(-a_2, -a_1, -a_3) \in \mathscr{A}$.

Proof. If $(a_1, a_2, a_3) \in \mathscr{A}$ then by Lemma 38.7 $\omega^{-a_2} - 2 + \omega^{a_1} = 0$. As $a_1 = -a_2 - a_3$ this yields that

$$\omega^{-a_2} - 2 + \omega^{-a_2-a_3} = 0 .$$

As $-a_2 - a_1 - a_3 = 0$ the result follows from Lemma 38.7.

LEMMA 38.9. *For $0 \leqq i \leqq p-1$ let $\mathfrak{C}_i$ be the conjugate class of $\mathfrak{P}\mathfrak{U}$ which contains P^i and let $\mathfrak{K}_i$ be the sum of the elements in $\mathfrak{C}_i$ in the group ring of $\mathfrak{P}\mathfrak{U}$ over the integers. Let*

$$\mathfrak{K}_1^2 = \sum_{i=0}^{p-1} c_i \mathfrak{K}_i .$$

If $q > 3$, then $c_2 \geqq 2$.

Proof. Let $\mu_0, \mu_1, \cdots$ be all the irreducible characters of $\mathfrak{P}\mathfrak{U}/\mathfrak{P}$ and let $\chi_1, \chi_2, \cdots$ be all the other irreducible characters of $\mathfrak{P}\mathfrak{U}$. It is a well known consequence of the orthogonality relations ([4] p. 316) that

$$c_2 = \frac{up^q}{p^{2q}}\left\{\sum_i \frac{\mu_i(P)^2\overline{\mu_i(P^2)}}{\mu_i(1)} + \sum_j \frac{\chi_j(P)^2\overline{\chi_j(P^2)}}{\chi_j(1)}\right\} .$$

Since $\mathfrak{U}$ is cyclic, $\mu_i(P) = \mu_i(P^2) = \mu_i(1) = 1$ for all i. By 3.16 $\chi_j(1) = u$ for all j. Thus

$$c_2 = \frac{u}{p^q}\left\{u + \frac{1}{u}\sum_j \chi_j(P)^2\overline{\chi_j(P^2)}\right\} . \tag{38.3}$$

By the orthogonality relations

$$\sum_j |\chi_j(P^i)|^2 \leqq |C(P^i)| \leqq p^q \quad \text{for} \quad 1 \leqq i \leqq p-1 .$$

Therefore

$$|\sum_j \chi_j(P)^2\overline{\chi_j(P^2)}| \leqq (\max_j |\chi_j(P^2)|) \sum_j |\chi_j(P)|^2 \leqq p^{3q/2} . \tag{38.4}$$

By (38.3) and (38.4)

$$|p^q c_2 - u^2| \leqq p^{3q/2} .$$

Thus

$$p^q c_2 \geqq u^2 - p^{3q/2} . \tag{38.5}$$

Since $u = \dfrac{p^q - 1}{p - 1} > p^{q-1}$ (38.5) yields that

$$c_2 \geqq \frac{u^2}{p^q} - p^{q/2} > p^{q-2} - p^{q/2} = p^{q/2}(p^{q/2-2} - 1) .$$

As $q > 3$ and q is a prime we have $q \geqq 5$, and the lemma follows.

LEMMA 38.10. $|\mathscr{A}| = |\mathscr{B}| > 0$.

Proof. Assume first that $q = 3$. Consider the set of polynomials of the form $f_a(t) = t^3 + at^2 + (a + 6)t - 1$ with $a \in \mathscr{Z}_p$. There are p of these and none of them has 0 as a root. Thus if $f_a(t)$ were reducible for every value of a there would exist $a \neq b$ such that $f_a(t)$ and $f_b(t)$ have a common root $c \in \mathscr{F}_p$. Then

$$ac^2 + (a + 6)c = bc^2 + (b + 6)c .$$

Since $c \neq 0$ this yields that $a(c + 1) = b(c + 1)$, hence $c = -1$. However $f_a(-1) = -8 \neq 0$. Thus there exists some polynomial $f_a(t)$ which is irreducible over $\mathscr{F}_p$. Let α be a root of $f_a(t)$ in $\mathscr{F}_{p^3}$. Then

$$\alpha^{p^2+p+1} = -f_a(0) = 1, \quad (1 + \alpha)^{p^2+p+1} = -f_a(-1) = 8 .$$

Therefore $\alpha = \omega^{a_3}$ for some $a_3 \in \mathscr{Z}_u$, $a_3 \neq 0$, and $1 + \alpha = 2\omega^{-a_1}$ for some $a_1 \in \mathscr{Z}_u, a_1 \neq 0$. Furthermore $-\omega^{a_3} + 2\omega^{-a_1} = 1$. Thus $\omega^{a_1} + \omega^{a_1+a_3} - 2 = 0$. Since $\omega^{a_1} \neq 1$, $a_1 + a_3 \neq 0$. Hence by Lemmas 38.6 and 38.7 $|\mathscr{A}| = |\mathscr{B}| > 0$.

Assume now that $q > 3$. Then Lemma 38.9 implies the existence of $a, b \in \mathscr{Z}_u$, with $a \neq 0$ or $b \neq 0$ such that

$$U^{-a}PU^{a}U^{-b}PU^{b} = P^2 .$$

Therefore

$$PU^{b}P^{-2}U^{-a}PU^{a-b} = 1 . \tag{38.6}$$

Let $a_1 = b$, $a_2 = -a$, $a_3 = a - b$. Then $a_1 + a_2 + a_3 = 0$. If $b = 0$ then (38.6) becomes $P^{-1}U^{-a}PU^{a} = 1$; as $\mathfrak{P}\mathfrak{U}$ is a Frobenius group this implies $a = 0$ contrary to the choice of a and b. If $a = 0$ then (38.6) implies that $PU^{b}P^{-1}U^{-b} = 0$, hence $b = 0$. If $a - b = 0$ then (38.6) yields that $PU^{a}P^{-2}U^{-a}P = 1$ or U^{a} commutes with P^2. Thus $a = 0$, hence also $b = 0$. Therefore a_1, a_2, a_3 are all non zero and by Definition 38.1 and Lemma 38.6 $|\mathscr{A}| = |\mathscr{B}| > 0$.

The following result about finite fields is of importance for the proof of Theorem 37.1.

LEMMA 38.11. *For* $x \in \mathscr{F}_{p^q}$ *define* $N(x) = x^{1+p+\cdots+p^{q-1}}$ *and for* $x \neq 2$ *let* $x^{\sigma} = \dfrac{1}{2 - x}$. *If* $\alpha \in \mathscr{F}_{p^q} - \mathscr{F}_p$, *then for some* i, $N(\alpha^{\sigma^i}) \neq 1$.

Proof. Assume that the result is false and $N(\alpha^{\sigma^i}) = 1$ for all i. We will first prove by induction that

(38.7) $$\alpha^{\sigma^i} = \frac{-(i-1)\alpha + i}{-i\alpha + (i+1)} \qquad \text{for } i = 1, 2, \cdots$$

If $i = 1$ (38.7) follows from the definition of σ. Assume now that (38.7) holds for $i = k - 1$. Then

$$\begin{aligned}\alpha^{\sigma^k} &= \frac{1}{2 - \left\{\dfrac{-(k-2)\alpha + k - 1}{-(k-1)\alpha + k}\right\}} \\ &= \frac{-(k-1)\alpha + k}{-2(k-1)\alpha + 2k + (k-2)\alpha - (k-1)} \\ &= \frac{-(k-1)\alpha + k}{-k\alpha + (k+1)}.\end{aligned}$$

This establishes (38.7).

Now (38.7) implies that for $j \geqq 1$,

$$\prod_{i=1}^{j} \alpha^{\sigma^i} = \frac{\prod_{i=1}^{j}\{-(i-1)\alpha + i\}}{\prod_{i=1}^{j}\{-i\alpha + (i+1)\}} = \frac{1}{-j\alpha + (j+1)}.$$

Therefore

$$N(-j\alpha + j + 1) = \frac{1}{\prod_{i=1}^{j} N(\alpha^{\sigma^i})} = 1.$$

Thus

(38.8) $$N(-a\alpha + a + 1) = 1 \qquad \text{for } a \in \mathscr{F}_p.$$

Define $f(t)$ by

(38.9) $$f(t) = (t - \alpha)(t - \alpha^p) \cdots (t - \alpha^{p^{q-1}}).$$

Thus $f(t)$ has coefficients in $\mathscr{F}_p$ and (38.8) yields that

(38.10) $$a^q f\left(\frac{a+1}{a}\right) = a^q N\left(\frac{a+1}{a} - \alpha\right) = N(a + 1 - a\alpha) = 1$$
$$\text{for } a \in \mathscr{F}_p,\ a \neq 0.$$

Let $b = \dfrac{a+1}{a}$ for $a \neq 0$, then $a = \dfrac{1}{b-1}$. Hence (38.10) yields that

$$\frac{1}{(b-1)^q} f(b) = 1 \qquad \text{for } b \in \mathscr{F}_p,\ b \neq 1.$$

Therefore

(38.11) $$f(b) - (b-1)^q = 0 \qquad \text{for } b \in \mathscr{F}_p,\ b \neq 1.$$

$f(t) - (t-1)^q$ is a polynomial of degree at most q. By (38.11) $f(t) - (t-1)^q$ has at least $(p-1)$ roots. As $(p-1) > q$ we must have that $f(t) = (t-1)^q$. By (38.9) α is a root of $f(t)$, hence $\alpha = 1$ contrary to the choice of α. The proof is complete.

39. The Proof of Theorem 37.1

LEMMA 39.1. *There exist functions f, g, and h such that*
(i) *f and h map $\mathscr{X}_p \times \mathscr{X}_u \times \mathscr{X}_p$ into $\mathscr{X}_u$,*
(ii) *g maps $\mathscr{X}_p \times \mathscr{X}_u \times \mathscr{X}_p$ into $\mathscr{X}_p$,*
(iii) $P^x U^y P^z U^{f(x,y,z)} P^{g(x,y,z)} U^{h(x,y,z)} = 1$.
Furthermore for $x \neq 0$, $y \neq 0$, $z \neq 0$ (iii) determines $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ uniquely and $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ are all non-zero.

Proof. By Lemma 38.4 the functions exist and are uniquely defined by

$$P^x U^y P^z U^f P^g U^h = 1$$

provided that $P^x U^y P^z$ does not lie in $\mathfrak{U}$. It is easily seen that if $x \neq 0$, $y \neq 0$ and $z \neq 0$, $P^x U^y P^z$ does not lie in $\mathfrak{U}$.

Suppose that $f(x,y,z) = 0$. Then $P^x U^y P^{z+g} = U^{-h} \in \mathfrak{U}$. Then $y = -h$ and $U^y P^{z+g} U^{-y} = P^{-x} \in \mathfrak{P}^*$. Therefore either $y = 0$ or $x = 0$.

Suppose that $g(x,y,z) = 0$. Then $P^x U^y P^z = U^{-f-h}$. Thus $y = -f - h$ and $U^y P^z U^{-y} = P^{-x}$. Hence $x = 0$ or $y = 0$.

Suppose that $h(x,y,z) = 0$. Then $U^y P^z U^f P^{g+x} = 1$. Hence $y + f = 0$, then $U^y P^z U^{-y} = P^{-g-x}$. Thus $y = 0$ or $z = 0$. This completes the proof of the lemma.

Throughout the rest of this section f, g, h will denote the functions defined in Lemma 39.1. For $x \in \mathscr{X}_p$, Y as in Lemma 38.1, define

$$Y_x = Y^{-1} P^{-x} Y P^x .$$

LEMMA 39.2.
(i) $Y_x = Y^{-1} P^{-x} Y P^x = P^{-x} Y P^x Y^{-1}$
(ii) $Y P^z Y^{-1} = Y_{-z}^{-1} P^z$
(iii) $Y P^g Y^{-1} = P^g Y_g$,
for $x, z, g \in \mathscr{X}_p$.

Proof. Since $P \in \mathfrak{P}^* \subseteq N(\mathfrak{Q}_0)$ and $\mathfrak{Q}_0$ is abelian, (i) is immediate. (iii) is a direct consequence of (i). By definition $Y_{-z} = Y^{-1} P^z Y P^{-z}$. Thus $Y_{-z}^{-1} = P^z Y^{-1} P^{-z} Y = Y P^z Y^{-1} P^{-z}$ which implies (ii).

LEMMA 39.3. *For $x \in \mathscr{X}_p$, $P^{-x} U P^x = Y_x^{-1} U^{v^x} Y_x$.*

Proof. By (38.2) $P^x Z P^{-x} = Z^{v^{-x}}$. By (38.1) $Z = YUY^{-1}$. Hence

$$Y^{-1}P^x YUY^{-1}P^{-x}Y = U^{v^{-x}} .$$

Conjugating both sides by P^x, we get that

$$Y_x^{-1}UY_x = P^{-x}U^{v^{-x}}P^x .$$

If both sides are raised to the v^xth power, the lemma follows.

LEMMA 39.4.

$$Y_x Z^y Y_{-z}^{-1} = P^{-x}Z^{-h(x,y,z)}Y_{g(x,y,z)}^{-1}P^{-g(x,y,z)}Z^{-f(x,y,z)}P^{-z} .$$

Proof. Substitute (38.1) into (iii) of Lemma 39.1 to get

$$P^x Y^{-1}Z^y YP^z Y^{-1}Z^f YP^g Y^{-1}Z^h Y = 1 .$$

Conjugate by $Y^{-1}P^x$ to get

$$(P^{-x}YP^x Y^{-1})Z^y(YP^z Y^{-1})Z^f(YP^g Y^{-1})Z^h P^x = 1 .$$

Now use the results of Lemma 39.2 to derive that

$$Y_x Z^y Y_{-z}^{-1}P^z Z^f P^g Y_g Z^h P^x = 1$$

which implies the lemma.

LEMMA 39.5. *If* $(a_1, a_2, a_3) \in \mathscr{A}$, *then*

$$Y_2 Z^{a_1 v} Y_3^{-1} Y_1 Z^{a_2 v^3} Y_2^{-1} = Y_1 Z^{-a_3 v^2} Y_3^{-1} .$$

Proof. In the definition of $\mathscr{A}$ conjugate (iii) by P^2. Then

$$P^{-1}U^{a_1}P^{-2}U^{a_2}PU^{a_3}P^2 = 1 ,$$

or

$$(P^{-1}U^{a_1}P)(P^{-3}U^{a_2}P^3) = P^{-2}U^{-a_3}P^2 .$$

Hence Lemma 39.3 yields that

$$(Y_1^{-1}U^{a_1 v}Y_1)(Y_3^{-1}U^{a_2 v^3}Y_3) = Y_2^{-1}U^{-a_3 v^2}Y_2 .$$

Since $\mathfrak{Q}$ is abelian, this implies that

$$Y_2 U^{a_1 v}Y_3^{-1}Y_1 U^{a_2 v^3}Y_2^{-1} = Y_1 U^{-a_3 v^2}Y_3^{-1} .$$

Conjugating by Y^{-1} implies the result by (38.1) and the fact that $\mathfrak{Q}$ is abelian.

LEMMA 39.6. *For* $(a_1, a_2, a_3) \in \mathscr{A}$ *define*

$$\begin{aligned}
g_1 &= g(2,\, a_1 v,\, -3)\\
g_2 &= g(1,\, -a_3 v^2,\, -3)\\
g_3 &= g(1,\, a_2 v^3,\, -2)\\
k_1 &= h(2,\, a_1 v,\, -3) - h(1,\, -a_3 v^2,\, -3)v^{-1}\\
k_2 &= -f(2,\, a_1 v,\, -3) - h(1,\, a_2 v^3,\, -2)v^{-2}\\
k_3 &= -f(1,\, a_2 v^3,\, -2)v^{-1} + f(1,\, -a_3 v^2,\, -3)\\
k\ &= -g_3 - 1\,.
\end{aligned}$$

Then

$$Y_{g_1} Z^{k_1} P Y_{g_2}^{-1} = P^{-g_1} Z^{k_2} P^2 Y_{g_3}^{-1} P^k Z^{k_3} P^{g_2}\,. \tag{39.1}$$

Proof. Use Lemmas 39.4 and 39.5 to obtain

$$\begin{aligned}
&P^{-2} Z^{-h(2,a_1v,-3)} Y^{-1}_{g(2,a_1v,-3)} P^{-g(2,a_1v,-3)} Z^{-f(2,a_1v,-3)} P^3\,.\\
&P^{-1} Z^{-h(1,a_2v^3,-2)} Y^{-1}_{g(1\ a_2v^3,-2)} P^{-g(1,a_2v^3,-2)} Z^{-f(1,a_2v^3,-2)} P^2\\
&= Y_2 Z^{a_1 v} Y_3^{-1} Y_1 Z^{a_2 v^3} Y_2^{-1} = Y_1 Z^{-a_3 v^2} Y_3^{-1}\\
&= P^{-1} Z^{-h(1,-a_3v^2,-3)} Y^{-1}_{g(1\ -a_3v^2,-3)} P^{-g(1,-a_3v^2,-3)} Z^{-f(1,-a_3v^2,-3)} P^3\,.
\end{aligned}$$

Multiply on the left by $Y_{g(2,a_1v,-3)} Z^{h(2,a_1v,-3)} P^2$ and on the right by

$$P^{-3} Z^{f(1,-a_3v^2,-3)} P^{g(1,-a_3v^2,-3)}$$

to get

$$A Y^{-1}_{g(1\ a_2v^3,-2)} B = Y_{g(2,a_1v,-3)} C Y^{-1}_{g(1,-a_3v^2,-3)}$$

where

$$\begin{aligned}
A &= P^{-g(2,a_1v,-3)} Z^{-f(2,a_1v,-3)-h(1,a_2v^3,-2)v^{-2}} P^2\\
B &= P^{-g(1,a_2v^3,-2)-1} Z^{-f(1,a_2v^3,-2)v^{-1}+f(1,-a_3v^2,-3)} P^{g(1,-a_3v^2,-3)}\\
C &= Z^{h(2,a_1v,-3)-h(1,-a_3v^2,-3)v^{-1}} P\,,
\end{aligned}$$

or equivalently

$$A = P^{-g_1} Z^{k_2} P^2\,, \quad B = P^k Z^{k_3} P^{g_2}\,, \quad C = Z^{k_1} P\,.$$

The lemma follows.

LEMMA 39.7. *Let* $(a_1, a_2, a_3) \in \mathscr{A}$. *Use the notation of Lemma* 39.6. *If* $k_1 \neq 0$, *then there exist elements* $c_1, c_3 \in \mathscr{Z}_p$ *such that*

(i) $k_3 \neq 0$

(ii) $k_2 + k_3 v^{c_3} = k_1$

(iii) $Y^{-1} P Y P^{-g_2} = P^{-c_1} Y^{-1} P^{-c_3} Y$.

Proof. Conjugate (39.1) by Q. Since $\mathfrak{P}^* \mathfrak{Q} = C(Q)$, this yields that

$$Y_{g_1}Z^{wk_1}PY_{g_2}^{-1} = P^{-g_1}Z^{wk_2}P^2Y_{g_3}^{-1}P^kZ^{wk_3}P^{g_2} .$$

Taking inverses we get

$$Y_{g_2}P^{-1}Z^{-wk_1}Y_{g_1}^{-1} = P^{-g_2}Z^{-wk_3}P^{-k}Y_{g_3}P^{-2}Z^{-wk_2}P^{g_1} .$$

Multiplying this by (39.1) on the left yields

$$Y_{g_1}Z^{(1-w)k_1}Y_{g_1}^{-1} = P^{-g_1}Z^{k_2}P^2Y_{g_3}^{-1}P^kZ^{(1-w)k_3}P^{-k}Y_{g_3}P^{-2}Z^{-wk_2}P^{g_1} .$$

Conjugating by P^{-g_1} yields

$$P^{g_1}Y_{g_1}Z^{(1-w)k_1}Y_{g_1}^{-1}P^{-g_1} = Z^{k_2}P^2Y_{g_3}^{-1}P^kZ^{(1-w)k_3}P^{-k}Y_{g_3}P^{-2}Z^{-wk_2} .$$

Use Lemma 39.2 (iii) and (38.1) to get

$$\begin{aligned} &YP^{g_1}Y^{-1}YU^{(1-w)k_1}Y^{-1}YP^{-g_1}Y^{-1} \\ &\quad = YU^{k_2}Y^{-1}P^2Y_{g_3}^{-1}P^kYU^{(1-w)k_3}Y^{-1}P^{-k}Y_{g_3}P^{-2}YU^{-wk_2}Y^{-1} . \end{aligned}$$

Conjugate this by Y to obtain

$$P^{g_1}U^{(1-w)k_1}P^{-g_1} = U^{k_2}Y^{-1}P^2Y_{g_3}^{-1}P^kYU^{(1-w)k_3}Y^{-1}P^{-k}Y_{g_3}P^{-2}YU^{-wk_2} .$$

Multiply on the left by U^{-k_2} and on the right by U^{wk_2} to obtain

$$\begin{aligned} U^{-k_2}P^{g_1}U^{(1-w)k_1}P^{-g_1}U^{wk_2} &= W_1U^{k_3(1-w)}W_1^{-1} , \\ W_1 &= Y^{-1}P^2Y_{g_3}^{-1}P^kY . \end{aligned} \tag{39.2}$$

Suppose that $U^{k_3(1-w)} = 1$. Then (39.2) implies that

$$P^{g_1}U^{(1-w)k_1}P^{-g_1} = U^{(1-w)k_2} .$$

By Hypothesis $k_1 \neq 0$, hence by Lemma 38.2, $U^{(1-w)k_1} \neq 1$. By Lemma 39.1 $g_1 \neq 0$. Thus the above equality cannot hold in the Frobenius group $\mathfrak{P}\mathfrak{U}$. Hence $U^{k_3(1-w)} \neq 1$. This proves statement (i) of the lemma.

Let $U_0 = W_1U^{k_3(1-w)}W_1^{-1}$. By (39.2) U_0 is a conjugate of $U^{k_3(1-w)}$ which lies in $\mathfrak{P}\mathfrak{U}$. All conjugates of $U^{k_3(1-w)}$ which lie in $\mathfrak{U}$ are of the form

$$U^{k_3(1-w)v^{c_3}w^{c'}} ,$$

with $c_3 \in \mathscr{X}_p$, $c' \in \mathscr{X}_q$. Hence

$$U_0 = W_1U^{k_3(1-w)}W_1^{-1} = W_2^{-1}U^{k_3(1-w)v^{c_3}w^{c'}}W_2 \tag{39.3}$$

for some $W_2 \in \mathfrak{P}$. Thus $W_2W_1 \in N(\mathfrak{U})$. Since $Q \in N(\mathfrak{U})$, we get that $Q^{-1}W_2W_1Q \in N(\mathfrak{U})$. By (39.2) $W_1Q = QW_1$, thus $Q^{-1}W_2W_1Q = Q^{-1}W_2QW_1$. Hence

$$W_2Q^{-1}W_2^{-1}Q = W_2W_1(Q^{-1}W_1^{-1}W_2^{-1}Q) \in N(\mathfrak{U}) .$$

However $W_2 Q^{-1} W_2^{-1} Q \in \mathfrak{P}$. Since $\mathfrak{P} \cap N(\mathfrak{U}) = 1$, this yields that $Q \in C(W_2)$. Hence $W_2 \in \mathfrak{P} \cap C(Q) = \mathfrak{P}^*$. Thus

$$(39.4) \qquad W_2 = P^{c_2}$$

for some $c_2 \in \mathscr{X}_p$. Now (39.2) and (39.4) show that

$$W_2 W_1 \in \mathfrak{Q}_0 \mathfrak{P}^* \cap N(\mathfrak{U}) .$$

Since $P \in N(\langle Z \rangle)$, we have $Y^{-1}PY \in N(\mathfrak{U})$, thus $\mathfrak{Q}_0 \mathfrak{P}^* \cap N(\mathfrak{U}) = \langle Y^{-1}PY \rangle$. Therefore

$$(39.5) \qquad W_2 W_1 = Y^{-1} P^{c_0} Y$$

for some $c_0 \in \mathscr{X}_p$. Consequently

$$(W_2 W_1)^{-1} U^{k_3(1-w)v^{c_3} w^{c'}} W_2 W_1 = U^{k_3(1-w)v^{c_3+c_0} w^{c'}} .$$

If this is compared with (39.3) we see that

$$(39.6) \qquad c_0 + c_3 = 0, \quad c' = 0 .$$

Using (39.4) and (39.6) in (39.5) leads to

$$(39.7) \qquad W_1 = P^{-c_2} Y^{-1} P^{-c_3} Y .$$

Comparing (39.2) and (39.7), we get

$$P^{-c_2} Y^{-1} P^{-c_3} Y = Y^{-1} P^{2} Y_{g_3}^{-1} P^k Y .$$

Conjugating by Y^{-1} gives

$$(39.8) \qquad Y P^{-c_2} Y^{-1} P^{-c_3} = P^{2} Y_{g_3}^{-1} P^k .$$

If we substitute (39.7) into (39.2) we get

$$U^{-k_2} P^{g_1} U^{(1-w)k_1} P^{-g_1} U^{k_2 w} = P^{-c_2} U^{k_3(1-w)v^{c_3}} P^{c_2} .$$

Multiply on the left by $U^{-k_3 v^{c_3}} P^{c_2}$ and on the right by $U^{-k_2 w} P^{g_1} U^{k_1 w}$ to get

$$U^{-k_3 v^{c_3}} P^{c_2} U^{-k_2} P^{g_1} U^{k_1} = U^{-w k_3 v^{c_3}} P^{c_2} U^{-k_2 w} P^{g_1} U^{k_1 w} .$$

Since the right hand side is the left hand side conjugated by Q, we see that Q centralizes the left hand side. Hence

$$(39.9) \qquad U^{-k_3 v^{c_3}} P^{c_2} U^{-k_2} P^{g_1} U^{k_1} = P^{c_1}$$

for some $c_1 \in \mathscr{X}_p$. Reading (39.9) mod $\mathfrak{P}$ yields that

$$k_1 = k_2 + k_3 v^{c_3}$$

which proves (ii) of the lemma. Substituting (ii) of Lemma 39.2

into (39.8) we get that

$$P^{i}Y_{g_3}^{-1}P^{k} = Y_{c_2}^{-1}P^{-c_2-c_3} . \tag{39.10}$$

Substituting (39.10) into (39.1) leads to

$$Y_{g_1}Z^{k_1}PY_{g_2}^{-1} = P^{-g_1}Z^{k_2}Y_{c_2}^{-1}P^{-c_2-c_3}Z^{k_3}P^{g_2} .$$

Multiply on the left by P^{g_1} and on the right by P^{-g_2}. Then using Lemma 39.2 (ii) and (iii) this becomes

$$YP^{g_1}Y^{-1}Z^{k_1}PYP^{-g_2}Y^{-1} = Z^{k_2}Y_{c_2}^{-1}P^{-c_2-c_3}Z^{k_3} .$$

Use $Z = YUY^{-1}$ to get

$$YP^{g_1}U^{k_1}Y^{-1}PYP^{-g_2}Y^{-1} = YU^{k_2}Y^{-1}Y_{c_2}^{-1}P^{-c_2-c_3}YU^{k_3}Y^{-1} .$$

Conjugate by Y and multiply on the left by U^{-k_2} to get

$$U^{-k_2}P^{g_1}U^{k_1}Y^{-1}PYP^{-g_2} = Y^{-1}Y_{c_2}^{-1}P^{-c_2-c_3}YU^{k_3} . \tag{39.11}$$

Conjugate by Q and take inverses, then

$$P^{g_2}Y^{-1}P^{-1}YU^{-k_1w}P^{-g_1}U^{k_2w} = U^{-k_3w}Y^{-1}P^{c_2+c_3}Y_{c_2}Y .$$

Multiply by (39.11) on the right to get

$$P^{g_2}Y^{-1}P^{-1}YU^{-k_1w}P^{-g_1}U^{k_2(w-1)}P^{g_1}U^{k_1}Y^{-1}PYP^{-g_2} = U^{k_3(1-w)} .$$

Conjugate by W_1^{-1} to get

$$\begin{aligned} &W_1P^{g_2}Y^{-1}P^{-1}YU^{-k_1w}P^{-g_1}U^{k_2(w-1)}P^{g_1}U^{k_1}Y^{-1}PYP^{-g_2}W_1^{-1} \\ &\quad = W_1U^{k_3(1-w)}W_1^{-1} . \end{aligned}$$

Using (39.2) and (39.3), this yields

$$\begin{aligned} &W_1P^{g_2}Y^{-1}P^{-1}Y\{U^{-k_1w}P^{-g_1}U^{k_2(w-1)}P^{g_1}U^{k_1}\}Y^{-1}PYP^{-g_2}W_1^{-1} \\ &\quad = U_0 = U^{-k_2}P^{g_1}U^{(1-w)k_1}P^{-g_1}U^{wk_2} . \end{aligned} \tag{39.12}$$

Now by the second equation in (39.12)

$$U^{-k_1w}P^{-g_1}U^{k_2w}U^{-k_2}P^{g_1}U^{k_1} = U^{-k_1w}P^{-g_1}U^{k_2w}U_0U^{-k_2w}P^{g_1}U^{k_1w} .$$

Thus the first equation in (39.12) implies that

$$U^{-k_2w}P^{g_1}U^{k_1w}Y^{-1}PYP^{-g_2}W_1^{-1} \in C(U_0) .$$

By (39.3) and (39.4), $C(U_0) = P^{-c_2}\mathfrak{U}^*P^{c_2}$. Hence

$$U^{-k_2w}P^{g_1}U^{k_1w}Y^{-1}PYP^{-g_2}W_1^{-1} = P^{-c_2}U_2P^{c_2} \tag{39.13}$$

for some $U_2 \in \mathfrak{U}^*$. We wish to show that $U_2 \in \mathfrak{U}$. To do this conjugate (39.13) by Q to get

$$U^{-k_2w^2}P^{g_1}U^{k_1w^2}Y^{-1}PYP^{-g_2}W_1^{-1} = P^{-c_2}U_2^{w}P^{c_2} \tag{39.14}$$

by (39.7). Multiply (39.13) by the inverse of (39.14) on the right to get

$$U^{-k_2w}P^{g_1}U^{k_1w}U^{-k_1w^2}P^{-g_1}U^{k_2w^2} = P^{-c_2}U_2^{1-w}P^{c_2}\,. \tag{39.15}$$

By Lemma 38.2 U_2 and U_2^{1-w} have the same order. Since the left hand side of (39.15) is in $\mathfrak{P}\mathfrak{U}$, this implies that the order of U_2 divides u, thus $U_2 \in \mathfrak{U}$.

Multiply (39.13) on the left by $U_2^{-1}P^{c_2}$ and on the right by $W_1P^{g_2}Y^{-1}P^{-1}Y$ to get

$$U_2^{-1}P^{c_2}U^{-k_2w}P^{g_1}U^{k_1w} = P^{c_2}W_1P^{g_2}Y^{-1}P^{-1}Y\,. \tag{39.16}$$

By (39.7) the right hand side is in $C(Q)$, while the left hand side is in $\mathfrak{P}\mathfrak{U}$. Since $C(Q) \cap \mathfrak{P}\mathfrak{U} = \mathfrak{P}^*$, this yields that

$$U_2^{-1}P^{c_2}U^{-k_2w}P^{g_1}U^{k_1w} = P^{c''} \tag{39.17}$$

for some $c'' \in \mathscr{Z}_p$. Conjugate by Q^{-1} to get

$$U_2^{-w^{-1}}P^{c_2}U^{-k_2}P^{g_1}U^{k_1} = P^{c''}\,.$$

Comparing this with (39.9) yields that

$$U_2^{w^{-1}}P^{c''} = U^{k_3v^{c_3}}P^{c_1}\,,$$

so that

$$U_2^{w^{-1}} = U^{k_3v^{c_3}}\,, \quad c_1 = c''\,.$$

Using (39.16) and (39.17) this yields

$$P^{c_1} = P^{c_2}W_1P^{g_2}Y^{-1}P^{-1}Y$$

or

$$P^{c_1-c_2}Y^{-1}PYP^{-g_2} = W_1\,.$$

Hence by (39.7)

$$P^{c_1-c_2}Y^{-1}PYP^{-g_2} = P^{-c_2}Y^{-1}P^{-c_3}Y\,.$$

This immediately implies (iii) of the lemma and thus completes the proof.

LEMMA 39.8. *Let* $(a_1, a_2, a_3) \in \mathscr{A}$, *and let* k_1 *have the same meaning as in Lemma* 39.6. *Then* $k_1 = 0$.

Proof. Suppose that $k_1 \neq 0$, so that Lemma 39.7 may be applied. Let

$$h_1 = h(2, a_1v, -3)$$
$$h_2 = h(1, a_2v^3, -2)$$
$$h_3 = h(1, -a_3v^2, -3) .$$

By Lemma 38.5 (i)

$$f(2, a_1v, -3) = -a_1v - h_1$$
$$f(1, a_2v^3, -2) = -a_2v^3 - h_2$$
$$f(1, -a_3v^2, -3) = a_3v^2 - h_3 .$$

Hence in the notation of Lemma 39.6

$$k_1 = h_1 - h_3v^{-1}$$
$$k_2 = a_1v + h_1 - h_2v^{-2}$$
$$k_3 = a_2v^3 + h_2v^{-1} + a_3v^2 - h_3 .$$

Since $a_1 + a_2 + a_3 = 0$, this yields that

$$k_3 = -a_1v^2 + h_2v^{-1} - h_3$$
$$k_1 - k_2 = -a_1v + h_2v^{-2} - h_3v^{-1} .$$

Thus

$$(k_1 - k_2)v = k_3$$

or

$$k_2 + k_3v^{-1} = k_1 .$$

By Lemma 39.7 (ii) this implies that $k_3(v^{c_3} - v^{-1}) = 0$. If $c_3 \neq -1$, then by Lemma 38.2, $(v^{c_3} - v^{-1})$ has an inverse in $\mathscr{X}_u$. Thus $k_3 = 0$ contrary to Lemma 39.7 (i). Therefore $c_3 = -1$. Now Lemma 39.7 (iii) becomes

$$Y^{-1}PYP^{-g_2} = P^{-c_1}Y^{-1}PY . \qquad (39.18)$$

Reading (39.18) mod $\mathfrak{Q}$ implies that $g_2 = c_1$. Thus (39.18) yields that $Y^{-1}PY$ and P^{-g_2} commute. Since $g_2 \neq 0$ by Lemma 39.1, this implies that

$$P^{-1}Y^{-1}PY \in \mathfrak{Q}_0 \cap C(P) = \{1\} .$$

Thus $Y \in \mathfrak{Q}_0 \cap C(P) = \{1\}$ which is not the case. Therefore $k_1 = 0$ as required.

LEMMA 39.9 *Let* $(a_1, a_2, a_3) \in \mathscr{A}'$, *let* k_2 *and* k_3 *have the same meaning as in Lemma* 39.6. *Then* $k_2 = k_3 = 0$.

Proof. Since $k_1 = 0$ by Lemma 39.8, (39.1) becomes

(39.19) $$Y_{g_1}PY_{g_2}^{-1} = P^{-g_1}Z^{k_2}P^2Y_{g_3}^{-1}P^kZ^{k_3}P^{g_2}.$$

Conjugating by Q and using (38.2) we get that

(39.20) $$Y_{g_1}PY_{g_2}^{-1} = P^{-g_1}Z^{wk_2}P^2Y_{g_3}^{-1}P^kZ^{wk_3}P^{g_2}.$$

Now (39.19) and (39.20) imply that

$$Z^{k_2}P^2Y_{g_3}^{-1}P^kZ^{k_3} = Z^{wk_2}P^2Y_{g_3}^{-1}P^kZ^{wk_3}.$$

Therefore

(39.21) $$P^2Y_{g_3}^{-1}P^kZ^{k_3(1-w)}P^{-k}Y_{g_3}P^{-2} = Z^{k_2(w-1)}.$$

Suppose that $k_3 \neq 0$. Then by Lemma 38.2 $k_3(1-w) \neq 0$. As $\langle Z\rangle$ is a T.I. set in $\mathfrak{G}$, (39.21) now implies that $P^2Y_{g_3}^{-1}P^k \in N(\langle Z\rangle)$. As $P \in N(\langle Z\rangle)$ this implies that

$$Y^{-1}P^{-g_3}YP^{g_3} = Y_{g_3} \in N(\langle Z\rangle) \cap \mathfrak{Q}_0 = \langle 1\rangle.$$

Therefore P^{g_3} commutes with Y. Hence $g_3 = 0$. This is contrary to Lemma 39.1. Thus $k_3 = 0$.

Now (39.21) implies that $k_2(w-1) = 0$. Therefore by Lemma 38.2 $k_2 = 0$.

LEMMA 39.10. *Let* $(a_1, a_2, a_3) \in \mathscr{A}$ *and* g_3 *have the same meaning as in Lemma* 39.6. *Then* $g_3 = 1$.

Proof. In view of Lemmas 39.8 and 39.9 equation (39.1) becomes

(39.22) $$Y_{g_1}PY_{g_2}^{-1} = P^{-g_1}P^2Y_{g_3}^{-1}P^kP^{g_2}.$$

Reading (39.22) mod $\mathfrak{Q}_0$ implies that

$$1 = -g_1 + 2 + k + g_2$$

or using the definition of k

(39.23) $$-1 - g_3 = k = -1 + g_1 - g_2.$$

Hence $g_3 = g_2 - g_1$ and (39.22) becomes

(39.24) $$Y_{g_1}PY_{g_2}^{-1} = P^{2-g_1}Y_{g_2-g_1}^{-1}P^{g_1-1}.$$

P acts as a linear transformation on $\mathfrak{Q}_0$. It is convenient to use the exponential notation. Thus $Y^P = P^{-1}YP$, so that $Y_x = Y^{-1+P^x}$. (39.24) can be rewritten as

$$P^{-1}Y_{g_1}PY_{g_2}^{-1} = P^{-(g_1-1)}Y_{g_2-g_1}^{-1}P^{g_1-1}.$$

In exponential notation this becomes

(39.25) $$Y^{(-1+P^{g_1})P+(1-P^{g_2})} = Y^{(1-P^{g_2-g_1})P^{g_1-1}} .$$

Define

(39.26) $$\begin{aligned} A &= (-1+P^{g_1})P + (1-P^{g_2}) - (1-P^{g_2-g_1})P^{g_1-1} \\ &= (1-P) + P^{g_1-1}(P^2-1) - P^{g_2-1}(P-1) . \end{aligned}$$

Since $\mathfrak{P}^*\mathfrak{Q}_0$ is a Frobenius group with Frobenius kernel $\mathfrak{Q}_0$, $1-P$ is an invertible linear transformation on $\mathfrak{Q}_0$. By (39.25) A annihilates Y. Hence also $A(1-P)^{-1}$ annihilates Y. By (39.26)

$$\begin{aligned} A(1-P)^{-1} &= 1 - P^{g_1-1}(P+1) + P^{g_2-1} \\ &= 1 - P^{g_1} + 1 - P^{g_1-1} - 1 + P^{g_2-1} . \end{aligned}$$

Therefore

$$Y_{g_2-1}Y_{g_1-1}^{-1}Y_{g_1}^{-1} = Y^{(-1+P^{g_2-1})-(-1+P^{g_1-1})-(-1+P^{g_1})} = 1 .$$

Thus

(39.27) $$Y_{g_2-1} = Y_{g_1}Y_{g_1-1} .$$

By Lemma 39.3

$$Y_{g_2-1}^{-1}U^{v^{g_2-1}}Y_{g_2-1} = P^{-(g_2-1)}UP^{(g_2-1)} .$$

By (39.27) this yields that

(39.28) $$Y_{g_1-1}^{-1}Y_{g_1}^{-1}U^{v^{g_2-1}}Y_{g_1}Y_{g_1-1} = P^{-(g_2-1)}UP^{(g_2-1)} .$$

Lemma 39.2 also implies that

$$Y_{g_1}^{-1}U^{v^{g_1}}Y_{g_1} = P^{-g_1}UP^{g_1} .$$

Raising this to the $v^{g_2-g_1-1}$th power we get that

(39.29) $$Y_{g_1}^{-1}U^{v^{g_2-1}}Y_{g_1} = P^{-g_1}U^{v^{g_2-g_1-1}}P^{g_1} .$$

Now (39.28) and (39.29) yield that

(39.30) $$Y_{g_1-1}^{-1}P^{-g_1}U^{v^{g_2-g_1-1}}P^{g_1}Y_{g_1-1} = P^{-(g_2-1)}UP^{(g_2-1)} .$$

Another application of Lemma 39.3 gives

(39.31) $$Y_{g_1-1}^{-1}U^{v^{g_1-1}}Y_{g_1-1} = P^{-(g_1-1)}UP^{(g_1-1)} .$$

Thus (39.30) and (39.31) imply that

(39.32) $$\begin{aligned} &Y_{g_1-1}^{-1}[P^{-g_1}U^{v^{g_2-g_1-1}}P^{g_1}, U^{v^{g_1-1}}]Y_{g_1-1} \\ &= [P^{-(g_2-1)}UP^{(g_2-1)}, P^{-(g_1-1)}UP^{(g_1-1)}] . \end{aligned}$$

Since $g_1 \neq 0$, $P^{-g_1}U^{v^{g_2-g_1-1}}P^{g_1} \notin \mathfrak{U}$. Therefore

$$[P^{-g_1}U^{v^{g_2-g_1-1}}P^{g_1}, U^{v^{g_1-1}}] \in \mathfrak{P}^{\sharp} .$$

As $\mathfrak{P}$ is a T.I. set in $\mathfrak{G}$ (39.32) now implies that

$$Y_{g_1-1} \in N(\mathfrak{P}) \cap \mathfrak{Q}_0 = 1 .$$

Therefore P^{g_1-1} commutes with Y and so $g_1 = 1$. Now (39.27) yields that $Y_{g_2-1} = Y_1$, or

$$Y^{-1}P^{-(g_2-1)}YP^{(g_2-1)} = Y^{-1}P^{-1}YP .$$

Consequently $P^{-(g_2-2)}YP^{(g_2-2)} = Y$. Hence $g_2 = 2$. Now (39.23) implies that $g_3 = 1$ as required.

LEMMA 39.11. *Let $\mathscr{B}$ have the same meaning as in Definition 38.2. If $a \in \mathscr{B}$ then $-a \in \mathscr{B}$.*

Proof. Let $a = a_1 \in \mathscr{B}$ and suppose that $(a_1, a_2, a_3) \in \mathscr{A}$. By Lemma 38.8 $(-a_2, -a_1, -a_3) \in \mathscr{A}$. Let $(-a_2, -a_1, -a_3)$ play the role of (a_1, a_2, a_3). By Lemma 39.10 $g_3 = g(1, -a_1v^3, -2) = 1$. Thus Lemmas 38.5 and 39.1 imply that

$$-a_1v^3 + f(1, -a_1v^3, -2) + h(1, -a_1v^3, -2) = 0 \tag{39.33}$$

$$\omega^{-a_1v^3} - 2 + \omega^{-a_1v^3 + h(1, -a_1v^3, -2)} = 0 . \tag{39.34}$$

Let $b_1 = -a_1v^3$, $b_2 = f(1, -a_1v^3, -2)$ and $b_3 = h(1, -a_1v^3, -2)$. By Lemma 39.1 $b_i \neq 0$ for $i = 1, 2, 3$. By (39.33) $b_1 + b_2 + b_3 = 0$. Now it follows from (39.34) and Lemma 38.7 that $(b_1, b_2, b_3) \in \mathscr{A}$. Thus $-av^3 = -a_1v^3 = b_1 \in \mathscr{B}$.

Since a was an arbitrary element of $\mathscr{B}$ we get that for any integer n, $a(-v^3)^n \in \mathscr{B}$. Thus in particular, $a(-v^3)^p \in \mathscr{B}$. Hence by (38.2), $-a = -av^{3p} \in \mathscr{B}$ as was to be shown.

It is now very easy to complete the proof of Theorem 37.1.

Define the set $\mathscr{C}$ by

$$\mathscr{C} = \{\omega^a \mid a \in \mathscr{B}\} .$$

Since $|\mathscr{B}| = |\mathscr{C}|$, Lemma 38.10 yields that $\mathscr{C}$ is not empty. The definition of $\mathscr{B}$ and Lemma 38.7 yield that $1 \notin \mathscr{C}$ and $\alpha \in \mathscr{C}$ if and only if $2 - \alpha \in \mathscr{C}$. Lemma 39.11 implies that $\alpha \in \mathscr{C}$ if and only if $\alpha^{-1} \in \mathscr{C}$. Therefore if $\alpha \in \mathscr{C}$ then $\dfrac{1}{2-\alpha} \in \mathscr{C}$. Since $u = 1 + p + \cdots + p^{q-1}$, we have $N(\alpha) = \alpha^{1+p+\cdots+p^{q-1}} = 1$ for $\alpha \in \mathscr{C}$. Thus if σ has the same meaning as in Lemma 38.11 then there exists $\alpha \in \mathscr{F}_{p^q} - \mathscr{F}_p$ such that $N(\alpha^{\sigma^i}) = 1$ for all values of i. This contradicts Lemma 38.11, and completes the proof of the main theorem of this paper.

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