A TOPOLOGICAL MEASURE CONSTRUCTION

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1. Introduction. The purpose of this paper is to give two results in topological measure theory that generalize two well known results for metric spaces.

The principal one of these, which is given in § 3, concerns the construction of a measure from a nonnegative set function. Carathéodory [1] has done this in a natural way in defining Carathéodory linear measure in a finite dimensional Euclidean space. It is well known that this Carathéodory construction can be applied in metric spaces to produce measures for which the open sets are measurable.\(^1\) Our treatment produces a measure for which the open \(F_\sigma\) sets, in a regular topological space, are measurable, and is identical with the Carathéodory measure in case the topology is metrizable. Since each open set in a metric space is an \(F_\sigma\) this provides a generalization of the metric result.

Our other result, which is given in § 2, concerns a necessary and sufficient condition for the measurability of open sets. A well known condition for this in metric spaces is that the measure be additive on any two sets which are a positive distance apart. When this condition is changed to require the additivity on two sets whose closures do not intersect, it becomes a necessary and sufficient condition for the measurability of the open \(F_\sigma\) sets, for a normal topological space. We show that the condition of normality can be weakened to one of \(\phi\) Normal” (see definition 2.4.5 below). Since a metric space is normal and therefore \(\phi\) Normal, and since each open set of a metric space is an \(F_\sigma\), this provides a clear generalization of the metric result.

At first glance the weakened normality condition of 2.4.5 appears to add little to topological measure theory. However, this is just the condition that results from our construction in § 3 (even though the topology is not necessarily normal) and hence the results of § 2 help us to obtain the results of § 3.

Nowhere in this paper is an assumption of local compactness made.

2. Conditions for measurability.

2.1. Definitions.

\(^1\) See, for example, Method II, page 105, of [3].
.1 \sim B = the complement of \( B \).

.2 \sigma F = the union of \( F = \bigcup \beta \in F \beta \)  
= the set of points \( x \) for which \( x \in \beta \) for some \( \beta \in F \).

.3 \( x \in \text{domain } f \) if and only if \( (x, y) \in f \) for some \( y \).

.4 \omega = the set of nonnegative integers.

2.2. DEFINITIONS.

.1 \( \phi \) measures \( \mathcal{S} \) if and only if \( \phi \) is such a function of the subsets of \( \mathcal{S} \) that:

\[
0 \leq \phi(A) \quad \text{whenever} \quad A \subset \mathcal{S};
\]

and

\[
\phi(A) \leq \sum \beta \in F \phi(\beta)
\]

whenever \( F \) is a countable family for which

\[
A \subset \sigma F \subset \mathcal{S}.
\]

.2 \( A \) is \( \phi \) measurable if and only if \( A \in \text{domain } \phi \) and

\[
\phi(T) = \phi(TA) + \phi(T \sim A)
\]

for each \( T \in \text{domain } \phi \).

.3 measurable \( \phi = EA \) (\( A \) is \( \phi \) measurable)

.4 section \( \phi_T = \) the function \( \psi \) on domain \( \phi \) such that

\[
\psi(A) = \phi(TA)
\]

for each \( A \in \text{domain } \phi \).

.5 \( \psi \) is a submeasure of \( \phi \) if and only if \( \psi = \text{section } \phi_T \) for some \( T \) for which \( \phi(T) < \infty \).

2.3. DEFINITIONS.

.1 \( \mathcal{X} \) is a topology if and only if \( \mathcal{X} \) is such a family of sets that

\[
\sigma F \in \mathcal{X} \quad \text{whenever} \quad F \subset \mathcal{X},
\]

and
Thus a topology $\mathcal{I}$ is closed to finite intersections and unrestricted unions. For example, the family of all open sets of a metric space is a topology.

.2 $\mathcal{I}$ topologizes $\mathcal{S}$ if and only if $\mathcal{I}$ is a topology and $\mathcal{S} = \sigma\mathcal{I}$.

.3 $C$ is $\mathcal{I}$ closed if and only if $C = \sigma\mathcal{I} \sim A$ for some $A \in \mathcal{I}$.

.4 Closure $\mathcal{I}A = \text{the intersection of all } \mathcal{I} \text{ closed sets which contain } A$.

.5 $\mathcal{I}$ is regular if and only if corresponding to each $A \in \mathcal{I}$ and each $x \in A$ there is a $B \in \mathcal{I}$ such that $x \in B$ and Closure $\mathcal{I}B \subset A$.

.6 $F_{\sigma}$ $\mathcal{I} = \bigcap B \ (B \in \mathcal{I} \text{ and } B = \sigma F \text{ for some countable family } F \text{ of } \mathcal{I} \text{ closed sets})$.

.7 $G_{\delta}$ $\mathcal{I} = \bigcap C \ (C \text{ is } \mathcal{I} \text{ closed and } C \text{ is the intersection of a countable subfamily of } \mathcal{I})$.

Thus $F_{\sigma}$ $\mathcal{I}$ and $G_{\delta}$ $\mathcal{I}$ are the familiar open $F_{\sigma}$'s and closed $G_{\delta}$'s.

This paper deals with a fixed topology, $\mathcal{I}$, which topologizes the space, $\mathcal{S}$. It is assumed that the hypothesis

$\mathcal{I}$ topologizes $\mathcal{S}$

is added to every theorem. Also the "$\mathcal{I}$" will be dropped from such expressions as "$C$ is $\mathcal{I}$ closed" whenever no confusion will result. Thus we write "$F_{\sigma}$" for "$F_{\sigma}$ $\mathcal{I}$," "Closure $A$" for "Closure $\mathcal{I}A$," etc.

In definitions 2.4 below the well known topological concepts of compactness and normality, are followed by generalizations involving both topology and measure.

2.4. Definitions.

.1 $A$ is compact if and only if $A$ is closed and for each $F \subseteq \mathcal{I}$ for which $A \subseteq \sigma F$ there is a finite subfamily $H$ of $F$ for which $A \subseteq \sigma H$.

.2 $A$ is $\phi$ compact if and only if $A$ is closed and for each $F \subseteq \mathcal{I}$ for which $A \subseteq \sigma F$, for each submeasure $\psi$ for $\phi$, for each $\varepsilon > 0$, there is a finite subfamily $H$ of $F$ for which $\psi(A) \leq \psi(A \sigma H) + \varepsilon$. 
A is normal if and only if A is closed and for each C and B for which C is closed, $C \subset A$, $C \subset B \in \mathcal{X}$, there exists $D \in \mathcal{X}$ for which $C \subset D$ and Closure $D \subset B$.

A is $\phi$ normal if and only if $A$ is closed and for each $C$ and $B$ for which $C$ is closed, $C \subset A$, $C \subset B \in \mathcal{X}$, and for each submeasure $\psi$ for $\phi$, for each $\varepsilon > 0$, there exists $D \in \mathcal{X}$ and a closed set $C'$ for which $C' \subset C$, $C' \subset D$, Closure $D \subset B$, and

(1) $\psi(C) \leq \psi(C') + \varepsilon$.

We define the slightly less general notion of $\phi$ Normality by changing the condition (1) to condition (2) below.

A $\phi$ Normal if and only if $A$ is closed and for each $C$ and $B$ for which $C$ is closed, $C \subset A$, $C \subset B \in \mathcal{X}$, and for each submeasure $\psi$ of $\phi$, for each $\varepsilon > 0$, there exists $D \in \mathcal{X}$ and a closed set $C'$ for which $C' \subset C$, $C' \subset D$, Closure $D \subset B$, and

(2) $\psi(C \sim C') \leq \varepsilon$.

$\phi$ is $\mathcal{X}$ additive if and only if $\phi(A \cup B) = \phi(A) + \phi(B)$ whenever Closure $A \cap$ Closure $B = 0$.

$\phi$ is a $\rho$ metric measure if and only if $\rho$ metrizes $\mathcal{S}$, $\phi$ measures $\mathcal{S}$, and $\phi(A \cup B) = \phi(A) + \phi(B)$ whenever $A$ and $B$ are subsets of $\mathcal{S}$ which are a positive $\rho$ distance apart.

2.5. THEOREM. If $\phi$ measures $\mathcal{S}$ then:

.1 if $A$ is compact then $A$ is $\phi$ compact;

.2 if $A$ is normal then $A$ is $\phi$ Normal;

.3 if $A$ is $\phi$ Normal then $A$ is $\phi$ normal.

2.6. THEOREM. If $\rho$ metrizes $\mathcal{S}$, $\mathcal{X}$ is the family of $\rho$-open sets, and $\phi$ measures $\mathcal{S}$ then:

.1 $\phi$ is $\mathcal{X}$ additive if and only if $\phi$ is a $\rho$ metric measure;

.2 $\mathcal{X}$ is a regular topology;

.3 $\mathcal{S}$ is normal.

A well known theorem is the following:
2.7. **Theorem.** If \( \rho \) metrizes \( \mathcal{S} \) and \( \phi \) measures \( \mathcal{S} \) then \( \phi \) is a \( \rho \) metric measure if and only if the \( \rho \)-open sets are \( \phi \) measurable.

The primary aim of this section is to generalize Theorem 2.7 to the case where the metric space is replaced by a regular, \( \phi \) Normal, topology. This is done in Theorem 2.19.

2.8. **Theorem.** If \( \phi \) measures \( \mathcal{S} \), \( C \) is closed and \( C \subset A \) then:

1. if \( A \) is compact then \( C \) is compact;
2. if \( A \) is \( \phi \) compact then \( C \) is \( \phi \) compact.

**Proof.** .1 is well known.

Suppose \( F' \subset \mathcal{F} \), \( C \subset \sigma F' \), \( \psi \) is a submeasure of \( \phi \), \( \psi' = \text{section } \psi C \), \( 0 < \varepsilon < \infty \), \( F' = \text{EB} (B \in F \text{ or } B = \mathcal{S} \sim C) \). Since \( A \subset \sigma F' \), \( F' \subset \mathcal{F} \), \( \psi' \) is a submeasure of \( \phi \), and \( A \) is \( \phi \) compact, we can and do choose such a finite subfamily \( H' \) of \( F' \) that

\[
\psi'(A) \leq \psi'(A \sigma H') + \varepsilon .
\]

Letting \( H = \text{EB} (B \in H' \text{ and } B \neq \mathcal{S} \sim C) \), it follows that \( H \) is a finite subfamily of \( F' \), and

\[
\psi(C) = \psi'(A) \leq \psi'(A \sigma H') + \varepsilon = \psi'(A \sigma H) + \varepsilon = \psi(C \sigma H) + \varepsilon .
\]

Consequently \( C \) is \( \phi \) compact, and .2 is proved.

2.9. **Theorem.** If \( \mathcal{X} \) is regular, and \( \phi \) measures \( \mathcal{S} \) then

1. if \( A \) is compact then \( A \) is normal,
2. if \( A \) is \( \phi \) compact then \( A \) is \( \phi \) normal.

**Proof.** .1 is well known.

Suppose \( A \) is \( \phi \) compact, \( C \subset A \), \( C \) is closed, \( C \subset B \in \mathcal{X} \), \( \psi \) is a submeasure of \( \phi \), and \( \varepsilon > 0 \). First use the fact that \( \mathcal{X} \) is regular to secure such a function \( f \) on \( C \) that \( f(x) \in \mathcal{X}, x \in f(x) \), Closure \( f(x) \subset B \) for each \( x \in C \). Now again use the regularity of \( \mathcal{X} \) to secure such a function \( g \) on \( C \) that \( g(x) \in \mathcal{X}, x \in g(x) \), Closure \( g(x) \subset f(x) \) for each \( x \in C \). Now use the (2.8) facts that \( C \) is \( \phi \) compact and \( C \subset \bigcup x \in C g(x) \) to secure such a finite subset \( Q \) of \( C \) that

\[
\psi(C) \leq \psi(C \cap \bigcup x \in Q g(x)) + \varepsilon .
\]
Let
\[ D = \bigcup x \in Qf(x), \quad C' = \bigcup x \in Q(C \cap \text{Closure } g(x)) \]
and observe that \( D \in \mathcal{X} \), \( C' \) is closed, \( C' \subset C \),
\[ C' = \bigcup x \in Q(C \cap \text{Closure } g(x)) \subset \bigcup x \in Qf(x) = D, \]
Closure \( D = \text{Closure} \bigcup x \in Qf(x) = \bigcup x \in Q \text{ Closure } f(x) \subset B \), and
\[ \psi(C) \leq \psi(C \cap \bigcup x \in Qg(x)) + \varepsilon \]
\[ \leq \psi(C \cap \bigcup x \in Q \text{ Closure } g(x)) + \varepsilon \]
\[ = \psi(C') + \varepsilon. \]
Consequently \( A \) is \( \phi \) normal and the proof is complete.

The following theorem is given in Chapter 5 of [3].

2.10. THEOREM. If \( \phi \) measures \( \mathcal{S} \) and if for each \( n \in \omega \) and each submeasure \( \psi \) of \( \phi \),
\[ A_n \subset A_{n+1} \subset \mathcal{S}, \]
and
\[ \psi(A_n \cup (\mathcal{S} \sim A_{n+1})) = \psi(A_n) + \psi(\mathcal{S} \sim A_{n+1}) \]
then
.1 \( \bigcup n \in \omega A_n \) is \( \phi \) measurable, and
.2 \( \theta(\bigcup n \in \omega A_n) = \lim_{n \to \infty} \theta(A_n), \) whenever \( \theta \) is a submeasure of \( \phi \).

By considering complements one easily established the following corollary of 2.10.

2.11. THEOREM. If \( \phi \) measures \( \mathcal{S} \) and if for each \( n \in \omega \) and each submeasure \( \psi \) of \( \phi \),
\[ A_{n+1} \subset A_n \subset \mathcal{S}, \]
and
\[ \psi(A_{n+1} \cup (\mathcal{S} \sim A_n)) = \psi(A_{n+1}) + \psi(\mathcal{S} \sim A_n) \]
then
.1 \( \bigcap n \in \omega A_n \) is \( \phi \) measurable, and
.2 \( \theta(\bigcap n \in \omega A_n) = \lim_{n \to \infty} \theta(A_n), \) whenever \( \theta \) is a submeasure of \( \phi \).

The following lemma is easily verified.
2.12. **Lemma.** If \( \phi \) measures \( \mathcal{S} \) then \( \phi \) is \( X \) additive if and only if \( \psi \) is \( X \) additive for each submeasure \( \psi \) of \( \phi \).

2.13. **Theorem.** If \( \phi \) measures \( \mathcal{S} \), \( \phi \) is \( X \) additive, \( A_n \subset \mathcal{S} \), and \( \text{Closure} \ A_{n+1} \cap \text{Closure} \ (\mathcal{S} \sim A_n) = 0 \) for each \( n \in \omega \), then

1. \( \bigcap n \in \omega A_n \) is \( \phi \) measurable, and
2. \( \theta(\bigcap n \in \omega A_n) = \lim \theta(A_n) \), whenever \( \theta \) is a submeasure of \( \phi \).

**Proof.** Observe that \( A_{n+1} \subset A_n \subset \mathcal{S} \), for \( n \in \omega \). Let \( \psi \) be a submeasure of \( \phi \). Since, by 2.12, \( \psi \) is \( X \) additive, it follows that

\[
\psi(A_{n+1} \cup (\mathcal{S} \sim A_n)) = \psi(A_{n+1}) + \psi(\mathcal{S} \sim A_n)
\]

for each \( n \in \omega \). Application of 2.11 completes the proof.

2.14. **Theorem.** If \( \phi \) measures \( \mathcal{S} \), \( \phi(\mathcal{S}) < \infty \), \( \phi \) is \( X \) additive, \( C \) is \( \phi \) normal, \( C \subset B \subset \mathcal{I} \), and \( 0 < \varepsilon < \infty \), then there exist sets \( C' \) and \( D \) such that \( C' \) is \( \phi \) measurable, \( C' \in \text{Gdelta} \), \( C' \subset C \), \( D \in \mathcal{I} \), \( C' \subset D \), \( \text{Closure} \ D \subset B \), and \( \phi(C \sim C') \leq \varepsilon \).

**Proof.** Since \( \phi(\mathcal{S}) < \infty \) it follows that \( \phi \) is a submeasure of \( \phi \). We use the facts that \( C \) is \( \phi \) normal and \( C \subset B \) to inductively obtain such sequences \( c' \) and \( d \) that \( c'_0 = C \), \( d_0 = B \), \( c'_n \) is closed, \( d_n \in \mathcal{I} \), \( c'_{n+1} \subset c'_n \subset d_n \), \( \text{Closure} \ d_{n+1} \subset d_n \), and \( \phi(c'_n) \leq \phi(c'_{n+1}) + \varepsilon/2^{n+1} \), for each \( n \in \omega \).

Let \( C' = \bigcap n \in \omega d_n \), \( D = d_1 \). Since

\[
\text{Closure} \ d_{n+1} \cap \text{Closure} \ (\mathcal{S} \sim d_n) = 0
\]

for each \( n \in \omega \), it follows from 2.13.1 that \( C' \) is \( \phi \) measurable. Also

\[
C' = \bigcap n \in \omega d_n = \bigcap n \in \omega \text{Closure} \ d_{n+1}
\]

is closed, \( C' \in \text{Gdelta} \), \( D \in \mathcal{I} \), \( C' \subset C \), \( C' \subset d_1 = D \) and \( \text{Closure} \ D \subset d_1 \subset d_0 = B \).

We now use induction to deduce that, for any \( m \in \omega \),

\[
\phi(C) = \phi(c'_m) \leq \phi(c'_{m+1}) + \sum n \in \omega (\varepsilon/2^{n+1})
\]

\[
\leq \phi(c'_m) + \varepsilon.
\]

Since \( c'_m \subset d_m \), \( c'_m \subset C \), and \( c'_m \subset C \cap d_m \), for each \( m \in \omega \), it follows from (1) that

\[
\phi(C) \leq \phi(c'_m) + \varepsilon \leq \phi(C \cap d_m) + \varepsilon.
\]
for each $m \in \omega$. Let $\theta = \text{section } \phi C$ and, with the help of 2.13.2., observe that

$$
\phi(CC') = \theta(C')
= \lim_{m \to \infty} \theta(d_m)
= \lim_{m \to \infty} \phi(Cd_m)
\geq \lim_{m \to \infty} \phi(C) - \epsilon
= \phi(C) - \epsilon,
$$

and, since $C'$ is $\phi$ measurable,

$$
\phi(C \sim C') = \phi(C) - \phi(CC') \leq \epsilon.
$$

2.15. THEOREM. If $\phi$ measures $\mathcal{S}$, $\phi(\mathcal{S}) < \infty$, $\phi$ is $\mathcal{X}$ additive, $C$ is $\phi$ normal, and $C \subset B \in \mathcal{X}$, then for some $\phi$ measurable set $K$, $C \subset K \subset B$.

Proof. Repeatedly use 2.14 to secure such a sequence, $k$, of $\phi$ measurable sets that $k_n \subset B$, and $\phi(C \sim k_n) \leq (1/2^n)$ for each $n \in \omega$. Let

$$
K' = \bigcup n \in \omega k_n, \quad K = K' \cup C.
$$

Thus $K'$ is $\phi$ measurable, $K \subset B$,

$$
0 \leq \phi(C \sim K') \leq \phi(C \sim k_n) \leq (1/2^n)
$$

for each $n \in \omega$, $\phi(C \sim K') = 0$, $C \sim K$ is $\phi$ measurable, $K = K' \cup (C \sim K')$ is $\phi$ measurable and $C \subset K \subset B$.

The following lemma is well known.

2.16. LEMMA. If $\phi$ measures $\mathcal{S}$ and $A \subset \mathcal{S}$ then $A$ is $\phi$ measurable if and only if $A$ is $\psi$ measurable for each submeasure $\psi$ of $\phi$.

2.17. THEOREM. If $\phi$ measures $\mathcal{S}$, $\phi$ is $\mathcal{X}$ additive and $\mathcal{S}$ is $\phi$ normal then $\mathcal{F}_{\text{sigma}} \subset \text{measurable } \phi$.

Proof. Let $B \in \mathcal{F}_{\text{sigma}}$, and let $\psi$ be a submeasure of $\phi$. Choose such a countable subfamily $F$ of closed sets that $B = \sigma F$. Check that $\psi(\mathcal{S}) < \infty$, $\psi$ is $\mathcal{X}$ additive, $C$ is $\phi$ normal, and $C$ is $\psi$ normal, for each $C \in F$.

Thus we can and do use 2.15 to secure such a function $K$ on $F$ that $K(C)$ is $\psi$ measurable and $C \subset K(C) \subset B$ for each $C \in F$. It follows that
B = \sigma F = \bigcup C \in FC
\subset \bigcup C \in FK(C)
\subset B ,
B = \bigcup C \in FK(C) ,

B is \psi measurable.
Thus B is \psi measurable for each submeasure \psi of \phi, and, by 2.16, B is \phi measurable.
The desired conclusion is at hand.

2.18. THEOREM. If \phi measures \mathcal{S} , \phi(\mathcal{S}) < \infty , C is \phi Normal C \subset B \in \mathcal{I} , and 0 < \varepsilon < \infty , then there exists such a member C' of Gdelta that C' \subset B and \phi(C \sim C') \leq \varepsilon .

Proof. Repeatedly use the fact that C is \phi Normal to obtain such sequences c' and d that c'_0 = C , d'_0 = B , c'_n is closed, d_n \in \mathcal{I} , c'_{n+1} \subset c'_n \subset d_n , Closure d_{n+1} \subset d_n , and

\phi(c'_n \sim c'_{n+1}) \leq (\varepsilon/2^{n+1}) ,

for each n \in \omega .

Let C' = \bigcap n \in \omega d_n and observe that C' \in Gdelta , C' \subset B ,

\phi(C \sim C') = \phi(C \sim \bigcap n \in \omega d_n)
= \phi(\bigcup n \in \omega(C \sim d_n))
\leq \phi(\bigcup n \in \omega(C \sim c'_n))
= \phi(\bigcup n \in \omega(c'_0 \sim c'_n))
= \phi(\bigcup n \in \omega(c'_n \sim c'_{n+1}))
\leq \sum n \in \omega\phi(c'_n \sim c'_{n+1})
\leq \sum n \in \omega(\varepsilon/2^{n+1}) = \varepsilon .

Thus \phi(C \sim C') \leq \varepsilon , and the proof is complete.

2.19. THEOREM. If \phi measures \mathcal{S} , and \mathcal{S} is \phi Normal then Fsigma \subset measurable \phi

if and only if \phi is \mathcal{I} additive.

Proof. If \phi is \mathcal{I} additive, it follows from 2.17 that

Fsigma \subset measurable \phi ,
since (2.5.3) a set which is \phi Normal is also \phi normal.
Now suppose that Fsigma \subset measurable \phi . Let \psi be a sub-
measure of $\phi$, $0 < \epsilon < \infty$, $A \subset \mathcal{G}$, $B \subset \mathcal{G}$, $\bar{A} =$ Closure $A$, $\bar{B} =$ Closure $B$, $\alpha = \mathcal{G} \sim \bar{B}$, and suppose that $\bar{A} \bar{B} = 0$.

Since $\bar{A} \subset \alpha$ we may use 2.18 to secure such a member $C'$ of Gdelta that $C' \subset \alpha$ and $\psi(\bar{A} \sim C') \leq \epsilon$. Thus $\psi(A \sim C') \leq \epsilon$, $(A \cup B)C' = AC'$, $(A \cup B) \sim C' = (A \sim C') \cup B$, and, since $C' \in F_{\sigma \iota}$ is measurable $\Phi$,

$$
\psi(A \cup B) = \psi((A \cup B)C') + \psi((A \cup B) \sim C')
$$

$$
\geq \psi(AC') + \psi((A \sim C') \cup B)
$$

$$
\geq \psi(A) - \psi(A \sim C') + \psi(B)
$$

$$
\geq \psi(A) + \psi(B) - \epsilon .
$$

Thus $\psi(A \cup B) \geq \psi(A) + \psi(B)$, and since $\psi(A \cup B) \leq \psi(A) + \psi(B)$, it follows that

$$
\psi(A \cup B) = \psi(A) + \psi(B) .
$$

Therefore $\phi$ is $\mathcal{G}$ additive, and the proof is complete.

2.20. REMARK. Since any metric space is normal and therefore (2.52) $\phi$ Normal and since every open set of a metric space is an $F_{\sigma}$, it follows that 2.19 is a generalization of the following well known theorem.

THEOREM. If $\rho$ metrizes $\mathcal{G}$, $\phi$ measures $\mathcal{G}$, and $\mathcal{X}$ is the family of $\rho$-open sets, then $\mathcal{X}$ is measurable $\phi$ if and only if $\phi(A \cup B) = \phi(A) + \phi(A)$ whenever $A$ and $B$ are a positive $\rho$-distance apart.

We shall use 2.17 in the next section where the topological space is not known to be normal but is $\phi$ normal for the measure $\phi$ that is constructed there.

3. Measure construction.

It is well known that the set function, $\theta = \text{msm} g\rho H$, defined in 3.2 below, is a Borel measure (i.e., the $\rho$-open sets are $\theta$ measurable) in case $\rho$ metrizes $\mathcal{G}$ and $g$ is a nonnegative function of $H$. It is the purpose of this section to generalize this result to the topological case by defining a measure, $\phi = \text{mst} g\mathcal{X} H$, for which the open $F_{\sigma}$'s are $\phi$-measurable whenever $\mathcal{X}$ is regular, and which is equal to $\theta$ in case $\mathcal{X}$ is metrizable.

3.0. DEFINITIONS.

.1 mss $g\mathcal{G} H = \text{the function } \psi$, on the subsets of $\mathcal{G}$, such
that if $A \subseteq \mathcal{S}$ then $\psi(A)$ is the infimum of numbers of the form
\[
\sum B \in F g(B)
\]
where $F$ is such a countable subfamily of $H$ that $A \subseteq \sigma F$.

In connection with 3.1, we would like to remind the reader that an empty infimum is $\infty$.

.2 msr $g\rho H = \theta$, on the subsets of $\mathcal{S}$, such that, if $A \subseteq \mathcal{S}$ then
\[
\theta(A) = \lim_{n \to \infty} \text{msr } g\mathcal{S} H_n(A),
\]
where $H_n$ consists of those members of $H$ whose $\rho$-diameter is less than $1/2^n$, for each $n$.

.3 msf $gFH = \text{msr } g\mathcal{S} H_F$, where
\[
H_F = H \cap \text{ED}(D \subseteq B \text{ for some } B \in F).
\]

.4 Cover $\mathcal{X} = E F (F \subseteq \mathcal{X} \text{ and } \sigma F = \sigma \mathcal{X})$.

Thus, $E \in \text{Cover } \mathcal{X}$ if and only if $F \subseteq \mathcal{X}$ and $F$ covers the space covered by $\mathcal{X}$.

.5 mst $g\mathcal{X} H = \phi$, on the subsets of $\mathcal{S}$ such that, if $A \subseteq \mathcal{S}$ then
\[
\phi(A) = \sup \{F \in \text{Cover } \mathcal{X} \mid \phi_F(A)\}
\]
where $\phi_F = \text{msf } gFH$, for each $F \in \text{Cover } \mathcal{X}$.

.6 $F$ is a refinement of $F'$ if and only if each member of $F$ is a subset of some member of $F'$.

.7 $F' \cap \cap F'' = E B (B = \alpha \beta$ for some $\alpha \in F$ and some $\beta \in F''$).

3.1. LEMMA. If $F \in \text{Cover } \mathcal{X}$, $F' \in \text{Cover } \mathcal{X}$, and $F'' = F \cap \cap F'$, then $F'' \in \text{Cover } \mathcal{X}$ and $F''$ is a refinement of $F$.

The following theorem is well known.

3.2. THEOREM. If $g$ is a nonnegative function on $H$, and $\psi = \text{msr } g\mathcal{S} H$ then:

.1 $\psi$ measures $\mathcal{S}$;

.2 if $H \subseteq H'$ and $\psi' = \text{msr } g\mathcal{S} H'$, then $\psi'(A) \leq \psi(A)$ for
if $\psi(A) < \infty$ and $0 < \varepsilon < \infty$ then there exists such a countable subfamily $G$ of $H$ that $A \subset \sigma G$ and $\sum D \in G g(D) \leq \psi(A) + \varepsilon$.

3.3. Theorem. If $g$ is a nonnegative function of $H$, $F$ is a refinement of $F'$, $\phi_F = \text{msf} \ gFH$, and $\phi_{F'} = \text{msf} \ gF'H$, then

$$\phi_{F'}(A) \leq \phi_F(A)$$

for each $A \subset S$.

Proof. Let

$$H_F = H \cap \text{EB}(B \subset D \text{ for some } D \in F),$$

$$H_{F'} = H \cap \text{EB}(B \subset D \text{ for some } D \in F''),$$

and note that $\phi_F = \text{mss} \ g\mathcal{H}_F$. Application of 3.2.2 completes the proof.

The following theorem is well known.

3.4. Theorem. If $F$ is nonempty, $\psi$ measures $\mathcal{F}$ for each $\psi \in F$, and

$$\phi(D) = \sup \psi \in F' \psi(D)$$

whenever $D \subset \mathcal{F}$, then:

.1 $\phi$ measures $\mathcal{F}$;

.2 if for each $\psi' \in F$ and each $\psi'' \in F$, there exists a $\psi \in F$ for which

$$\psi'(D) \leq \psi(D) \text{ and } \psi''(D) \leq \psi(D)$$

whenever $D \subset \mathcal{F}$, then

$$\bigcap \psi \in F \text{ measurable } \psi \subset \text{measurable } \phi.$$

3.5. Theorem. If $\mathcal{F}$ is a regular topology, $g$ is a nonnegative function on $H$, and $\phi = \text{mst} \ g\mathcal{F}H$ then $F\sigma \subset \text{measurable } \phi$.

Proof. Let $\phi_F = \text{msf} \ gFH$ for each $F \in \text{Cover } \mathcal{F}$. Thus

$$\phi(A) = \sup F \in \text{Cover } \mathcal{F} \phi_F(A)$$

whenever $A \subset \mathcal{F}$. The proof is completed in six parts.

Part I. $\phi$ measures $\mathcal{F}$. 
Proof. Since, by 3.2.1, \( \phi_F \) measures \( \mathcal{S} \), for each \( F \in \text{Cover } \mathcal{X} \), and since \( \text{Cover } \mathcal{X} \) is not empty, it follows from 3.4.1 that \( \phi \) measures \( \mathcal{S} \).

**PART II.** If \( F \in \text{Cover } \mathcal{X}, \phi(A) < \infty \), and \( 0 < \varepsilon < \infty \) then there exists such a refinement \( F' \) of \( F \) that \( F' \in \text{Cover } \mathcal{X} \), and \( \phi(A) \leq \phi_F(A) + \varepsilon \).

Proof. Choose such a member \( F'' \) of \( \text{Cover } \mathcal{X} \) that \( \phi(A) \leq \phi_{F''}(A) + \varepsilon \), and let \( F'' = F \cap \cap F'' \) (see Definition 3.0.7). Thus \( S \subseteq \sigma F'' \), \( F'' \subset \mathcal{X} \), \( F'' \in \text{Cover } \mathcal{X} \), and using 3.1 and 3.3, we infer that \( F'' \) is a refinement of \( F \), \( F'' \) is a refinement of \( F'' \), and

\[
\phi(A) \leq \phi_{F''}(A) + \varepsilon \leq \phi_F(A) + \varepsilon.
\]

**PART III.** \( \mathcal{S} \) is \( \phi \) compact.

Proof. Recall definition 2.4.2. Let \( F \in \text{Cover } \mathcal{X}, \phi(T) < \infty, \psi = \text{section } \phi_T, \) and \( 0 < \varepsilon < \infty \). Use Part II to choose such a refinement \( F' \) of \( F \) that \( F' \in \text{Cover } \mathcal{X} \) and

\[
\phi(T) \leq \phi_{F'}(T) + \varepsilon/2 < \infty.
\]

Now use 3.2.3 and definition 3.0.3 to secure such a countable subfamily \( H' \) of \( H \) that \( T \subseteq \sigma H' \), \( H' \) is a refinement of \( F' \), and \( \sum D \in H' g(D) < \infty \); and let \( H'' \) be such a finite subfamily of \( H' \) that

\[
\sum D \in (H' \sim H'') g(D) \leq \varepsilon/2.
\]

Since \( H'' \) is a refinement of \( F \), choose such a finite subfamily \( G \) of \( F \) that \( H'' \) is a refinement of \( G \). Thus

\[
\phi_F(T) \leq \phi_{F'}(T \sigma H'') + \phi_{F'}(T \sigma (H' \sim H'')) \\
\leq \phi(T \sigma H'') + \phi_{F'}((H' \sim H'')) \\
\leq \phi(T \sigma G) + \sum D \in (H' \sim H'') g(D) \\
\leq \psi(T) + \varepsilon/2, \\
\psi(\mathcal{S}) = \phi(T) \leq \phi_F(T) + \varepsilon/2 \\
\leq \psi(T) + \varepsilon.
\]

Thus \( \mathcal{S} \) is \( \phi \) compact.

**PART IV.** \( \phi \) is \( \mathcal{X} \) additive.

Proof. Recall definition 2.4.6. Let \( \bar{A} = \text{Closure } A, \bar{B} = \text{Closure } B, \bar{A}ar{B} = 0 \). If \( \phi(A \cup B) = \infty \) then \( \phi(A \cup B) = \phi(A) + \phi(B) \).
Now assume \( \phi(A \cup B) < \infty \), \( 0 < \varepsilon < \infty \), and select such members \( G' \) and \( G'' \) of Cover \( \mathcal{X} \) that

\[
\phi(A) \leq \phi_\alpha(A) + \varepsilon/3, \quad \phi(B) \leq \phi_\beta(B) + \varepsilon/3,
\]

and let

\[
F' = \bigcup \alpha (\alpha = B \cap (\mathcal{S} \sim B) \text{ for some } \beta \in G'), \\
F'' = \bigcup \alpha (\alpha = \beta \cap (\mathcal{S} \sim A) \text{ for some } \beta \in G''), \\
F = F' \cup F''.
\]

Thus:

\[
F \subset \mathcal{X}; \quad \mathcal{S} \subset \sigma F; \quad F \in \text{Cover } \mathcal{X};
\]

\[
\begin{align*}
\text{if } D \in F \text{ and } DA \neq 0 & \quad \text{then } D \in F' ; \\
\text{if } D \in F \text{ and } DB \neq 0 & \quad \text{then } D \in F'' .
\end{align*}
\]

Now use 3.2.3 to secure such a countable subfamily \( H''' \) of \( H \) that \( H''' \) is a refinement of \( F \), \( A \cup B \subset H''' \), and

\[
\sum D \in H''' g(D) \leq \phi_\alpha (A \cup B) + \varepsilon/3 ;
\]

infer from (2) that

\[
\begin{align*}
\text{if } D \in H''' \text{ and } DA \neq 0 & \quad \text{then } DB = 0 ; \\
\text{if } D \in H''' \text{ and } DB \neq 0 & \quad \text{then } DA = 0 ;
\end{align*}
\]

let \( H' = H''' \cap ED(DA \neq 0), H'' = H''' \cap ED(DB \neq 0) .
\]

Thus \( H' \cup H'' \subset H''' \subset H, H' \cup H'' \) is countable, \( A \subset \sigma H', B \subset \sigma H'' \), \( H'H'' = 0, H' \) is a refinement of \( G', H'' \) is a refinement of \( G'' \), and

\[
\phi_\alpha(A) \leq \sum D \in H' g(D) , \quad \phi_\beta(B) \leq \sum D \in H'' g(D) .
\]

Consequently, with the help of (1), (4) and (3), we deduce that

\[
\phi(A) + \phi(B) \leq \phi_\alpha(A) + \phi_\beta(B) + 2\varepsilon/3
\]

\[
\leq \sum D \in H' g(D) + \sum D \in H'' g(D) + 2\varepsilon/3
\]

\[
\leq \sum D \in H''' g(D) + 2\varepsilon/3
\]

\[
\leq \phi_\alpha(A \cup B) + \varepsilon/3 + 2\varepsilon/3
\]

\[
\leq \phi(A \cup B) + \varepsilon .
\]

Thus, if \( \phi(A \cup B) < \infty \),

\[
\phi(A) + \phi(B) \leq \phi(A \cup B) .
\]

Therefore

\[
\phi(A \cup B) = \phi(A) + \phi(B) ,
\]

whenever \( A \subset \mathcal{S}, B \subset \mathcal{S} \), and \( \phi \) is \( \mathcal{X} \) additive.
PART V. $\mathcal{S}$ is $\phi$ normal.

Proof. Recall definition 2.4.4, use part IV and 2.9.2.

PART VI. $\mathcal{F}_\sigma \subseteq$ measurable $\phi$.

Proof. By parts IV and V, $\phi$ is $\mathcal{X}$ additive and $\mathcal{S}$ is $\phi$ normal. Application of 2.17 completes the proof.

The reader will observe that the regularity of $\mathcal{X}$ was not used in the proofs of Parts I—III.

3.6. Lemma. If $\rho$ metrizes $\mathcal{S}$, $\mathcal{X}$ is the family of $\rho$-open sets, $F \subseteq \mathcal{X}$, $n \in \omega$, and $G_n = ED$ (the $\rho$-diameter of $D < 1/2^n$), then:

1. $\mathcal{X} \cap G_n \in \text{Cover } \mathcal{X}$;
2. if $H' \subset G_n$, if $DA \neq 0$ whenever $D \in H'$, and if for each $x \in A$ there exists a $B \in F$ such that
   
   \[(\text{the } \rho\text{-distance from } x \text{ to } \mathcal{S} \sim B) > 1/2^n,\]

   then $H'$ is a refinement of $F$;
3. $D \in G_n$ if, and only if, $D \subset B \in G_n$ for some $B \in \mathcal{X}$.

Proof. Suppose $D \in G_n$, and let $d =$ the $\rho$-diameter of $D$, $\delta = 1/2^n - d$, and

\[B = \text{Ex}(\rho(x, y) < \delta/3 \text{ for some } y \in D).\]

Thus, we have immediately that $B \in \mathcal{X}$, $D \subset B$, and one can show that $B \in G_n$.

Therefore if $D \in G_n$ then $D \subset B \in G_n$ for some $B \in \mathcal{X}$.

The remainder of the proof is straightforward.

3.7. Theorem. If $\rho$ metrizes $\mathcal{S}$, $\mathcal{X}$ is the family of all $\rho$-open sets, $F \subseteq \mathcal{X}$, $g$ is a nonnegative function on $H$, $G_n = ED$ (the $\rho$-diameter of $D < 1/2^n$) for each $n \in \omega$, and $\theta =$ $\text{msm } g\rho H$, then:

0. $\mathcal{X} \subseteq \text{measurable } \phi$;
1. if $\theta(A) < \infty$, $n \in \omega$, and $0 < \varepsilon < \infty$, then there exists a countable subfamily $H'$ of $H$ for which $H' \subset G_n$, $A \subset H'$, and

\[\sum D \in H' g(D) \leq \theta(A) + \varepsilon;\]
.2 if \( \theta(A) < \infty, n \in \omega, 0 < \varepsilon < \infty \), and if for each \( x \in A \) there exists a \( B \in F \) such that 

\[
(\text{the } \rho\text{-distance from } x \text{ to } \mathcal{S} \sim B) \geq 1/2^n,
\]

then there exists a countable subfamily \( H' \) of \( H \) for which \( H' \subset G_n, A \subset \sigma H', \) \( H' \) is a refinement of \( F \), and \( \sum D \in H'g(D) \leq \theta(A) + \varepsilon. \)

Proof. .0 is well known.

Proof .1 Let \( \theta_n = \text{mss } g\mathcal{S}(HG_n) \). Thus \( \theta_n(A) \leq \theta(A) < \infty \), and we can use 3.2.3 to secure such a countable subfamily \( H' \) of \( (H \cap G_n) \) that \( A \subset \sigma H' \) and \( \sum D \in H'g(D) \leq \theta_n(A) + \varepsilon. \) Since \( \theta_n(A) \leq \theta(A) \) the proof is complete.

Proof .2 Use .1 to obtain such a countable subfamily \( H'' \) of \( (HG_n) \) that \( A \subset \sigma H'' \) and \( \sum D \in H''g(D) \leq \theta(A) + \varepsilon, \) and let \( H' = H'' \cap \text{ED}(DA \neq 0). \)

Thus \( A \subset \sigma H', \) and \( \sum D \in H'g(D) \leq \sum D \in H''g(D) \leq \theta(A) + \varepsilon. \)

Also \( DA \neq 0 \) for each \( D \in H' \), so by 3.6.2 \( H' \) is a refinement of \( F \).

3.8. Theorem. If \( \rho \) metrizes \( \mathcal{S}, \mathcal{X} \) is the family of all \( \rho \)-open sets, \( g \) is a nonnegative function on \( H, \phi = \text{mst } g\mathcal{X}H, \) and \( \theta = \text{msm } g\rho H \) then \( \phi = \theta. \)

Proof. Let \( \phi_F = \text{msf } gFH \) for each \( F \in \text{Cover } \mathcal{X} \), and complete the proof in Parts I and II by showing that \( \phi(A) \geq \theta(A) \), and \( \phi(A) \leq \theta(A) \) whenever \( A \subset \mathcal{S}. \)

PART I. If \( A \subset \mathcal{S} \) then \( \phi(A) \geq \theta(A). \)

Proof. Let 

\[
G_n = \text{ED}((\text{the } \rho\text{-diameter of } D) < 1/2^n),
\]

\[
F_n = \mathcal{X} \cap G_n, \quad H_n = H \cap G_n, \quad \text{and } \theta_n = \text{mss } g\mathcal{S}H_n, \text{ for each } n \in \omega.
\]

Thus, for each \( n \in \omega \), \( F_n \in \text{Cover } \mathcal{X} \), and by definition 3.0.3, and 3.6.3,

\[
\phi_{F_n} = \text{msf } gF_nH
= \text{mss } g\mathcal{S}(H \cap \text{ED}(D \subset B \text{ for some } B \in F_n))
= \text{mss } g\mathcal{S}H_n
= \theta_n.
\]

Consequently,
A TOPOLOGICAL MEASURE CONSTRUCTION 1083

\[ \phi(A) = \sup F \in \text{Cover} \; \mathcal{X} \phi(F)(A) \]

\[ \geq \sup n \in \omega \phi_{F_n}(A) \]

\[ = \sup n \in \omega \theta_n(A) \]

\[ = \theta(A). \]

PART II. If \( A \subset \mathcal{S} \) then \( \phi(A) \leq \theta(A) \).

Proof. The result being obvious if \( \theta(A) = \infty \), we assume that \( \theta(A) < \infty \). Let \( 0 < \varepsilon < \infty \), \( F \in \text{Cover} \; \mathcal{X} \), and let \( f' \) and \( f \) be such sequences that

\[ f'_n = \bigcup B \in F \; \text{Ex} \left( \text{(the } \rho\text{-distance from } x \text{ to } \mathcal{S} \sim B \text{) > } 1/2^n \right), \]

\[ f_0 = f'_0, \]

\[ f_{n+1} = f'_{n+1} \sim f'_n, \]

whenever \( n \in \omega \). We infer: \( f_n \) is a \( \rho \)-Borel set and, by 3.7.0, \( f_n \) is \( \phi \) measurable for each \( n \in \omega \); \( A = \bigcup n \in \omega (Af_n); \; \theta(A) = \sum n \in \omega \theta(Af_n); \) for each \( n \in \omega \), for each \( x \in (Af_n), \) there is a \( B \in F \) for which the distance from \( x \) to \( \mathcal{S} \sim B \) is greater than \( 1/2^n \). Consequently, we can use 3.7.2 to secure such a sequence, \( h \), that \( h_n \) is a countable subfamily of \( H, \; Af_n \subset \sigma h_n, \; h_n \) is a refinement of \( F \), and

\[ \sum D \in h_n g(D) \leq \theta(Af_n) + \varepsilon/2^{n+1}, \]

whenever \( n \in \omega \).

Let \( H' = \bigcup n \in \omega h_n \) and note that \( H' \) is a countable subfamily of \( H, \; A = \bigcup n \in \omega (Af_n) \subset \bigcup n \in \omega \sigma h_n = \sigma H', \; H' \) is a refinement of \( F \), and

\[ \phi_\rho(A) \leq \sum D \in H' g(D) \leq \sum n \in \omega \sum D \in h_n g(D) \]

\[ \leq \sum n \in \omega (\theta(Af_n) + \varepsilon/2^{n+1}) = \sum n \in \omega \theta(Af_n) + \varepsilon = \theta(A) + \varepsilon. \]

Thus \( \phi_\rho(A) \leq \theta(A) \) for each \( F \in \text{Cover} \; \mathcal{X} \), and

\[ \phi(A) = \sup F \in \text{Cover} \; \mathcal{X} \phi_\rho(F) \]

\[ \leq \theta(A). \]

3.9. Remark. Thus, because of Theorems 3.5.3 and 3.8, the topological measure \( \phi = \text{mst} \; g\mathcal{X}H \) is a generalization of the metric measure \( \theta = \text{msm} \; g\rho H \).

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Dallas O. Banks, *Bounds for eigenvalues and generalized convexity* .................................. 1031
Woodrow Wilson Bledsoe and A. P. Morse, *A topological measure construction* ........ 1067
George Clements, *Entropies of several sets of real valued functions* .................... 1085
Sandra Barkdull Cleveland, *Homomorphisms of non-commutative \(*\)-algebras* ...... 1097
William John Andrew Culmer and William Ashton Harris, *Convergent solutions of ordinary linear homogeneous difference equations* ..................................... 1111
Ralph DeMarr, *Common fixed points for commuting contraction mappings* .......... 1139
James Robert Dorroh, *Integral equations in normed abelian groups* ................. 1143
Adriano Mario Garsia, *Entropy and singularity of infinite convolutions* ................ 1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., *Convergence of extended Bernstein polynomials in the complex plane* ..................................... 1171
Irving Leonard Glicksberg, *A remark on analyticity of function algebras* .......... 1181
Charles John August Halberg, Jr., *Semigroups of matrices defining linked operators with different spectra* ................................................................. 1187
Philip Hartman and Nelson Onuchic, *On the asymptotic integration of ordinary differential equations* ................................................................. 1193
Isidore Heller, *On a class of equivalent systems of linear inequalities* ............... 1209
Joseph Hersch, *The method of interior parallels applied to polygonal or multiply connected membranes* .......................................................... 1229
Hans F. Weinberger, *An effectless cutting of a vibrating membrane* ................. 1239
Melvin F. Janowitz, *Quantifiers and orthomodular lattices* .............................. 1241
Tilla Weinstein, *Another conformal structure on immersed surfaces of negative curvature* .................................................................................. 1281
Gregers Louis Krabbe, *Spectral permanence of scalar operators* ................. 1289
Shige Toshi Kuroda, *Finite-dimensional perturbation and a representation of scattering operator* .................................................. 1305
Marvin David Marcus and Afton Herbert Cayford, *Equality in certain inequalities* .......................................................... 1319
Joseph Martin, *A note on uncountably many disks* ............................................ 1331
John W. Moon, *An extension of Landau’s theorem on tournaments* ............ 1343
Louis Joel Mordell, *On the integer solutions of \(y(y + 1) = x(x + 1)(x + 2)\)* 1347
Kenneth Roy Mount, *Some remarks on Fitting’s invariants* .......................... 1353
Miroslav Novotný, *Über Abbildungen von Mengen* ........................................ 1359
Robert Dean Ryan, *Conjugate functions in Orlicz spaces* .............................. 1371
John Vincent Ryff, *On the representation of doubly stochastic operators* .......... 1379
Donald Ray Sherbert, *Banach algebras of Lipschitz functions* ....................... 1387
James McLean Sloss, *Reflection of biharmonic functions across analytic boundary conditions with examples* ........................................ 1401
L. Bruce Treybig, *Concerning homogeneity in totally ordered, connected topological space* ................................................................. 1417
John Wermer, *The space of real parts of a function algebra* .......................... 1423
James Juei-Chin Yeh, *Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables* ............. 1427
William P. Ziemer, *On the compactness of integral classes* .......................... 1437