COMMON FIXED POINTS FOR COMMUTING CONTRACTION MAPPINGS

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Kakutani [1] and Markov [2] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous (not necessarily linear) transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this paper is the following.

**Theorem.** Let B be a Banach space and let X be a nonempty compact convex subset of B. If $\mathcal{F}$ is a nonempty commutative family of contraction mappings of X into itself, then the family $\mathcal{F}$ has a common fixed point in X.

**Note 1.** A mapping $f: X \to X$ is said to be a contraction mapping if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in X$.

**Note 2.** If the norm for B is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is nonempty, compact, and convex. In the general case, however, the fixed-point set of a contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let B be the space of all ordered pairs $(a, b)$ of real numbers, where if $x = (a, b)$, then $\|x\| = \max \{|a|, |b|\}$. Define $X = \{x: \|x\| \leq 1\}$ and $f: X \to X$ as follows: if $x = (a, b)$, then $f(x) = (|b|, b)$. It is easily shown that $f$ is a contraction mapping and that $x = (1, 1)$ and $y = (1, -1)$ are fixed points for $f$. However, $1/2(x + y) = (1, 0)$ is not a fixed point for $f$.

In the proof of the theorem we shall make use of the following two lemmas.

**Lemma 1.** Let B be a Banach space and let M be a nonempty compact subset of B and let $K$ be the closed convex hull of M. Let $\rho$ be the diameter of M. If $\rho > 0$, then there exists an element $u \in K$ such that

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Proof. Since \( M \) is nonempty and compact, we may find \( x_0, x_1 \in M \) such that \( \|x_0 - x_1\| = \rho \). Let \( M_0 \subset M \) be maximal so that \( M_0 \supset \{x_0, x_1\} \) and \( \|x - y\| = 0 \) or \( \rho \) for all \( x, y \in M_0 \). Since \( M \) is compact and we are assuming \( \rho > 0 \), \( M_0 \) must be finite. Let us assume \( M_0 = \{x_0, x_1, \ldots, x_n\} \). Now let us define

\[
u = \sum_{k=0}^{n} \frac{1}{n+1} x_k \in K.
\]

Since \( M \) is compact, we can find \( y_0 \in M \) such that \( \|y_0 - u\| = \sup \{\|x - u\|: x \in M\} \). Now

\[
\|y_0 - u\| \leq \sum_{k=0}^{n} \frac{1}{n+1} \|y_0 - x_k\| \leq \rho
\]

because \( \|y_0 - x_k\| \leq \rho \) for all \( k = 0, 1, \ldots, n \). Therefore, if \( \|y_0 - u\| = \rho \), then we must have \( \|y_0 - x_k\| = \rho > 0 \) for all \( k = 0, 1, \ldots, n \), which means that \( y_0 \in M_0 \) by definition of \( M_0 \). But then we must have \( y_0 = x_k \) for some \( k = 0, 1, \ldots, n \), which is a contradiction. Therefore, \( \|y_0 - u\| < \rho \).

**Lemma 2.** Let \( X_0 \) be a nonempty convex subset of a Banach space and let \( f \) be a contraction mapping of \( X_0 \) into itself. If there is a compact set \( M \subset X_0 \) such that \( M = \{f(x): x \in M\} \) and \( M \) has at least two points, then there exists a nonempty closed convex set \( K_1 \) such that \( f(x) \in K_1 \cap X_0 \) for all \( x \in K_1 \cap X_0 \) and \( M \cap K_1' \neq \phi \). (\( K_1' \) is the complement of \( K_1 \).)

**Proof.** If we take \( K \) as the closed convex hull of \( M \), then by Lemma 1 there exists an element \( u \in K \) such that

\[
\rho_1 = \sup \{\|x - u\|: x \in M\} < \rho,
\]

where \( \rho \) is the diameter of \( M \). Since \( M \) has at least two points, we have \( \rho > 0 \), so that our use of Lemma 1 is valid.

For each \( x \in M \) let us define \( U(x) = \{y: \|y - x\| \leq \rho_1\} \). Since \( u \in U(x) \) for each \( x \in M \), we have \( K_1 = \bigcap_{x \in M} U(x) \neq \phi \). It is clear that \( K_1 \) is closed and convex. For any \( x \in K_1 \cap X_0 \) and any \( z \in M \) we have \( x \in U(z) \), i.e., \( \|x - z\| \leq \rho_1 \). Since \( M = \{f(y): y \in M\} \), there must exist \( y \in M \) such that \( z = f(y) \). Since \( f \) is a contraction mapping, we have

\[
\|f(x) - z\| = \|f(x) - f(y)\| \leq \|x - y\| \leq \rho_1;
\]

i.e., \( f(x) \in U(z) \). Since this is true for any \( z \in M \), we have \( f(x) \in K_1 \cap X_0 \). We have shown that \( f(x) \in K_1 \cap X_0 \) for all \( x \in K_1 \cap X_0 \).
Since $M$ is compact, there exist $x_0, x_1 \in M$ such that $\|x_0 - x_1\| = \rho > \rho_1$. Thus, we see that $x_1$ does not belong to $U(x_0) \supseteq K_1$, i.e., $x_1 \in M \cap K'_1 \neq \emptyset$.

**Proof of the theorem.** One may show by using Zorn's lemma that there exists a minimal nonempty compact convex set $X_0 \subseteq X$ such that $X_0$ is invariant under each $f \in \mathcal{F}$. If $X_0$ consists of a single point, then the theorem is proved. We shall now show that if $X_0$ consists of more than one point, then we obtain a contradiction.

We may use Zorn's lemma again to show that there exists a minimal nonempty compact (but not necessarily convex) set $M \subseteq X_0$ such that $M$ is invariant under each $f \in \mathcal{F}$. We will now show that $M = \{f(x) : x \in M\}$ for each $f \in \mathcal{F}$. Since each $f \in \mathcal{F}$ is continuous and $M$ is compact, $f(M)$ must also be compact. For all $f \in \mathcal{F}$ we have $f(M) \subseteq M$. Let us assume that for some $g \in \mathcal{F}$ we have $g(M) = N \neq M$. Now for any $x \in N$ there exists $y \in M$ such that $x = g(y)$. Since all functions in $\mathcal{F}$ commute, we have for all $f \in \mathcal{F}$ $f(x) = f(g(y)) = g(f(y)) \in N$ because $f(y) \in M$. Thus, we have $f(N) \subseteq N \subseteq M$ for all $f \in \mathcal{F}$. But since $N$ is a nonempty compact subset of $X_0$ which is invariant under each $f \in \mathcal{F}$ and since $N \subseteq M$ and $N \neq M$, we have contradicted the minimality of $M$. Consequently, our assumption that $M \neq N$ is false. We may assume that $M$ has at least two points; otherwise, the theorem is proved.

We may now apply Lemma 2 to each $f \in \mathcal{F}$. Referring to the notation of Lemma 2, we see that the set $K_1 \cap X_0$ is invariant under each $f \in \mathcal{F}$. Since $K_1$ is closed, we see that $K_1 \cap X_0$ is a nonempty compact convex subset of $X_0$. Since $X_0 \cap K'_1 \supseteq M \cap K'_1 \neq \emptyset$, we see that $K_1 \cap X_0 \neq X_0$. Thus, we see that if $X_0$ has more than one point, then we obtain a contradiction to the minimality of $X_0$.

**References**

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