

# Pacific Journal of Mathematics

## **COMMON FIXED POINTS FOR COMMUTING CONTRACTION MAPPINGS**

RALPH DEMARR

# COMMON FIXED POINTS FOR COMMUTING CONTRACTION MAPPINGS

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Kakutani [1] and Markov [2] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous (not necessarily linear) transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this paper is the following.

**THEOREM.** *Let  $B$  be a Banach space and let  $X$  be a nonempty compact convex subset of  $B$ . If  $\mathcal{F}$  is a nonempty commutative family of contraction mappings of  $X$  into itself, then the family  $\mathcal{F}$  has a common fixed point in  $X$ .*

*Note 1.* A mapping  $f: X \rightarrow X$  is said to be a contraction mapping if  $\|f(x) - f(y)\| \leq \|x - y\|$  for all  $x, y \in X$ .

*Note 2.* If the norm for  $B$  is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is nonempty, compact, and convex. In the general case, however, the fixed-point set of a contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let  $B$  be the space of all ordered pairs  $(a, b)$  of real numbers, where if  $x = (a, b)$ , then  $\|x\| = \max\{|a|, |b|\}$ . Define  $X = \{x: \|x\| \leq 1\}$  and  $f: X \rightarrow X$  as follows: if  $x = (a, b)$ , then  $f(x) = (|b|, b)$ . It is easily shown that  $f$  is a contraction mapping and that  $x = (1, 1)$  and  $y = (1, -1)$  are fixed points for  $f$ . However,  $1/2(x + y) = (1, 0)$  is not a fixed point for  $f$ .

In the proof of the theorem we shall make use of the following two lemmas.

**LEMMA 1.** *Let  $B$  be a Banach space and let  $M$  be a nonempty compact subset of  $B$  and let  $K$  be the closed convex hull of  $M$ . Let  $\rho$  be the diameter of  $M$ . If  $\rho > 0$ , then there exists an element  $u \in K$  such that*

$$\sup \{ \|x - u\| : x \in M \} < \rho .$$

*Proof.* Since  $M$  is nonempty and compact, we may find  $x_0, x_1 \in M$  such that  $\|x_0 - x_1\| = \rho$ . Let  $M_0 \subset M$  be maximal so that  $M_0 \supset \{x_0, x_1\}$  and  $\|x - y\| = 0$  or  $\rho$  for all  $x, y \in M_0$ . Since  $M$  is compact and we are assuming  $\rho > 0$ ,  $M_0$  must be finite. Let us assume  $M_0 = \{x_0, x_1, \dots, x_n\}$ . Now let us define

$$u = \sum_{k=0}^n \frac{1}{n+1} x_k \in K .$$

Since  $M$  is compact, we can find  $y_0 \in M$  such that  $\|y_0 - u\| = \sup \{ \|x - u\| : x \in M \}$ . Now

$$\|y_0 - u\| \leq \sum_{k=0}^n \frac{1}{n+1} \|y_0 - x_k\| \leq \rho$$

because  $\|y_0 - x_k\| \leq \rho$  for all  $k = 0, 1, \dots, n$ . Therefore, if  $\|y_0 - u\| = \rho$ , then we must have  $\|y_0 - x_k\| = \rho > 0$  for all  $k = 0, 1, \dots, n$ , which means that  $y_0 \in M_0$  by definition of  $M_0$ . But then we must have  $y_0 = x_k$  for some  $k = 0, 1, \dots, n$ , which is a contradiction. Therefore,  $\|y_0 - u\| < \rho$ .

**LEMMA 2.** *Let  $X_0$  be a nonempty convex subset of a Banach space and let  $f$  be a contraction mapping of  $X_0$  into itself. If there is a compact set  $M \subset X_0$  such that  $M = \{f(x) : x \in M\}$  and  $M$  has at least two points, then there exists a nonempty closed convex set  $K_1$  such that  $f(x) \in K_1 \cap X_0$  for all  $x \in K_1 \cap X_0$  and  $M \cap K_1' \neq \phi$ . ( $K_1'$  is the complement of  $K_1$ .)*

*Proof.* If we take  $K$  as the closed convex hull of  $M$ , then by Lemma 1 there exists an element  $u \in K$  such that

$$\rho_1 = \sup \{ \|x - u\| : x \in M \} < \rho ,$$

where  $\rho$  is the diameter of  $M$ . Since  $M$  has at least two points, we have  $\rho > 0$ , so that our use of Lemma 1 is valid.

For each  $x \in M$  let us define  $U(x) = \{y : \|y - x\| \leq \rho_1\}$ . Since  $u \in U(x)$  for each  $x \in M$ , we have  $K_1 = \bigcap_{x \in M} U(x) \neq \phi$ . It is clear that  $K_1$  is closed and convex. For any  $x \in K_1 \cap X_0$  and any  $z \in M$  we have  $x \in U(z)$ , i.e.,  $\|x - z\| \leq \rho_1$ . Since  $M = \{f(y) : y \in M\}$ , there must exist  $y \in M$  such that  $z = f(y)$ . Since  $f$  is a contraction mapping, we have

$$\|f(x) - z\| = \|f(x) - f(y)\| \leq \|x - y\| \leq \rho_1 ;$$

i.e.,  $f(x) \in U(z)$ . Since this is true for any  $z \in M$ , we have  $f(x) \in K_1 \cap X_0$ . We have shown that  $f(x) \in K_1 \cap X_0$  for all  $x \in K_1 \cap X_0$ .

Since  $M$  is compact, there exist  $x_0, x_1 \in M$  such that  $\|x_0 - x_1\| = \rho > \rho_1$ . Thus, we see that  $x_1$  does not belong to  $U(x_0) \supset K_1$ , i.e.,  $x_1 \in M \cap K_1' \neq \phi$ .

*Proof of the theorem.* One may show by using Zorn's lemma that there exists a minimal nonempty compact convex set  $X_0 \subset X$  such that  $X_0$  is invariant under each  $f \in \mathcal{F}$ . If  $X_0$  consists of a single point, then the theorem is proved. We shall now show that if  $X_0$  consists of more than one point, then we obtain a contradiction.

We may use Zorn's lemma again to show that there exists a minimal nonempty compact (but not necessarily convex) set  $M \subset X_0$  such that  $M$  is invariant under each  $f \in \mathcal{F}$ . We will now show that  $M = \{f(x) : x \in M\}$  for each  $f \in \mathcal{F}$ . Since each  $f \in \mathcal{F}$  is continuous and  $M$  is compact,  $f(M)$  must also be compact. For all  $f \in \mathcal{F}$  we have  $f(M) \subset M$ . Let us assume that for some  $g \in \mathcal{F}$  we have  $g(M) = N \neq M$ . Now for any  $x \in N$  there exists  $y \in M$  such that  $x = g(y)$ . Since all functions in  $\mathcal{F}$  commute, we have for all  $f \in \mathcal{F}$   $f(x) = f(g(y)) = g(f(y)) \in N$  because  $f(y) \in M$ . Thus, we have  $f(N) \subset N \subset M$  for all  $f \in \mathcal{F}$ . But since  $N$  is a nonempty compact subset of  $X_0$  which is invariant under each  $f \in \mathcal{F}$  and since  $N \subset M$  and  $N \neq M$ , we have contradicted the minimality of  $M$ . Consequently, our assumption that  $M \neq N$  is false. We may assume that  $M$  has at least two points; otherwise, the theorem is proved.

We may now apply Lemma 2 to each  $f \in \mathcal{F}$ . Referring to the notation of Lemma 2, we see that the set  $K_1 \cap X_0$  is invariant under each  $f \in \mathcal{F}$ . Since  $K_1$  is closed, we see that  $K_1 \cap X_0$  is a nonempty compact convex subset of  $X_0$ . Since  $X_0 \cap K_1' \supset M \cap K_1' \neq \phi$ , we see that  $K_1 \cap X_0 \neq X_0$ . Thus, we see that if  $X_0$  has more than one point, then we obtain a contradiction to the minimality of  $X_0$ .

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