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Introduction. Let $\Phi(\omega)$ be a random variable which takes only a finite number of values

$$x_1, x_2 \cdots, x_p ,$$

with probabilities

$$\pi_1, \pi_2, \cdots, \pi_p .$$

Let $A(x)$ be the distribution function of $\Phi(\omega)$.

We shall be concerned here with infinite convolutions of the type

$$F(x, r) = A\left(\frac{x}{r_1}\right) * A\left(\frac{x}{r_2}\right) * \cdots * A\left(\frac{x}{r_n}\right) * \cdots \tag{I.1}$$

where $r = (r_1, r_2, \cdots, r_n, \cdots)$ is a given sequence of non-vanishing real numbers. From standard theorems of Probability theory it follows that the convolution product in (I.1) converges (if we exclude the trivial case $p = 1$, $x_1 = 0$) if and only if $\sum r_n^2 < \infty$ and either

$$E(\Phi) = 0 \tag{I.2}$$

or

$$E(\Phi) \neq 0 \text{ but } \sum r_n \text{ is convergent.} \tag{I.3}$$

In either case, the limit distribution $F(x, r)$ is continuous and pure.[1] A proof of this result in the case that $\Phi(\omega)$ takes only the two values ± 1 with equal probabilities can be found in [4].

We can and shall restrict our study to the case $E(\Phi) = 0$. Our main result here concerns the distributions $F(x, r)$ generated by sequence $\{r_n\}$ such that, for some $0<\beta<1$,

$$r_n = 0[\beta^n] . \tag{I.4}$$

Under this hypothesis it is easy to see that for a given $A(x)$, when β is sufficiently small, $F(x, r)$ is necessarily singular. This result follows from the simple fact that the set of points of increase of $F(x, r)$, for all sufficiently small β, has zero measure. On the other hand, as β increases towards one $F(x, r)$ in general will become

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[1] That is either absolutely continuous or purely singular.

absolutely continuous. We are interested in finding a lower bound for the β's for which this may happen. Considerations involving the set of points of increase of $F(x, r)$ stop short of being helpful for, as we shall see, in general we can be sure that $F(x, r)$ will remain singular beyond the first β for which this set acquires positive measure.

Our discovery consists in the fact that, without further information on the relations between the values $x_1, x_2, \cdots, x_p$ and the sequence $\{r_n\}$, the best possible dividing line between singularity and absolute continuity of $F(x, r)$ is given by the entropy [8] of the distribution $\Phi(\omega)$.

More specifically we shall show that

THEOREM I.1 *If* $\{r_n\}$ *satisfies* I.4 *and*

$$-\sum_{i=1}^{p} \pi_i \log \pi_i < \log 1/\beta ,$$

then the function $F(x, r)$ *is necessarily singular.*

Our methods bring also to light some peculiarities of the Pisot-Vijayaraghavan numbers. The latter are algebraic integers whose conjugates are all in absolute value less than one [6]. Let α be in the interval (1, 2) and set $\beta = 1/\alpha$. Let $H_p(\alpha)$ denote the entropy of the distribution of the random variable

$$y_p = \varepsilon_1\beta + \varepsilon_2\beta^2 + \cdots + \varepsilon_p\beta^p , \tag{I.5}$$

where the ε_i are independent random variables taking the values ± 1 with equal probabilities. We can show the following result:

THEOREM I.2. *For every* α *in* (1, 2) *the ratio* $H_p(\alpha)/p$ *is convergent, and if* α *is a P.V. number*

$$\lim_{p\to\infty} H_p(\alpha)/p < \log \alpha . \tag{I.6}$$

We note that Theorems I.1 and I.2 combined provide an explanation for the singularity [2] of the distribution function of the random variable

$$y = \sum_{n=1}^{\infty} \varepsilon_n\beta^n ,$$

when β is the reciprocal of a P. V. number[2]. This is obtained by letting $A(x)$ be the distribution of the variable y_p and setting

[2] Cfr. [3] for further references and a history of this question.

$r_n = \beta^{n_p}$, then taking p sufficiently large.

The inequality in (I.6) contains the fact that the numbers

$$\pm \beta \pm \beta^2 \pm \cdots \pm \beta^p ,$$

for sufficiently large p, cannot be all distinct. This implies that each P. V. number in (1, 2) satisfies a polynomial equation with coefficients ± 1 or 0. Two different proofs of this results can be found in [7] and [3]. Here we shall show that even more is true, namely

THEOREM I.3 *Let $1/\beta$ be a P. V. number in the interval* (1, 2), *For any increasing sequence of integers $\{n_k\}$ such that*

$$(1/\beta)n_k = 0(2^k)$$

the corresponding powers of β

$$\beta^{n_1}, \beta^{n_2}, \cdots, \beta^{n_k}, \cdots$$

cannot be independent over the coefficients $\pm 1,0$.

A result of the same type as Theorem I.1 holds for convolutions of the form

$$F(x) = A_1(x) * A_2(x) * \cdots * A_n(x) * \cdots , \tag{I.7}$$

where for each n, $A_n(x)$ is the distribution of a random variable which takes only a finite number of values. Such a function when it is defined, is either totally discontinuous, continuous but purely singular or absolutely continuous. We shall be concerned with the cases in which $F(x)$ is continuous.

We can visualize (I.7) as being the distribution of a sum of independent random variables

$$y = x_1 + x_2 + \cdots + x_n + \cdots$$

where x_n has distribution $A_n(x)$. The continuity of $F(x)$ is assured as soon as y is not probabilistically equivalent to a series of constants [5].

In the case that

$$E(x_n) = 0, \quad \sum E(x_n^2) < \infty \tag{I.8}$$

the result corresponding to Theorem I.1 reads as follows. Let H_n denote the entropy of the distribution $A_n(x)$.

THEOREM I.4. *If $\{R_n\}$ is a sequence of positive numbers tending to zero for which*

$$\liminf R_n^2/(E(x_{n+1}^2) + E(x_{n+2}^2) + \cdots) > 0$$

then an estimate such as

$$\liminf (H_1 + H_2 + \cdots + H_n)/\log 1/R_n < 1 \tag{I.9}$$

implies the singularity of $F(x)$.

It can be shown that, although the condition (I.9) is best possible, no estimates on the entropy of the partial sums

$$y_n = x_1 + x_2 + \cdots x_n$$

are necessary for the singularity of $F(x)$. However, it is worthwhile to note that the situation is quite different if we look for necessary and sufficient conditions for F to be absolutely continuous with a derivative in the $L \log L$ class. To this effect we have the following theorem. Let $y = y_n + z_n$ with y_n and z_n independent and bounded and

$$E(z_n^2) \to 0 \quad \text{as} \quad n \to \infty .$$

Let $F(x)$ be the distribution of y and $F_n(x)$ be that of y_n. Suppose that $F(x)$ is continuous.

THEOREM I.5. *If* A_n *is a sequence of positive numbers tending to zero and such that*

$$\liminf A_n^2/E(z_n^2) > 0$$

then a necessary and sufficient condition for F *to be absolutely continuous and* F' *to be in the* $L \log L$ *class is that as* $n \to \infty$ *we have*

$$\log 1/A_n + \sum [F_n(kA_n + A_n) - F_n(kA_n)] \cdot \log [F_n(kA_n + A_n) - F_n(kA_n)] = 0(1) .$$

1. Auxiliary lemmas and definitions.

1.1 It will be useful to consider sequences $\{\mathfrak{P}_n\}$ of ordered probability distributions

$$\mathfrak{P}_n = (p_1(n), p_2(n), \cdots, p_{N(n)}(n)) .^3 \tag{1.11}$$

To be specific, for each n we shall suppose that

$$\begin{aligned} &\text{(a)} \quad p_1(n) \geqq p_2(n) \geqq \cdots \geqq p_{N(n)}(n) \geqq 0 \\ &\text{(b)} \quad p_1(n) + p_2(n) + \cdots + p_{N(n)}(n) = 1 . \end{aligned} \tag{1.12}$$

For a given probability distribution $\mathfrak{P} = (p_1, p_2, \cdots, p_N)$ the distribu-

[3] We shall assume that $N(n) \to \infty$ as $n \to \infty$.

tion function $F(x)$ defined by the condition that

$$F'(x) = \begin{cases} 0 \text{ for } x < 0 \text{ or } x \geqq 1 \\ Np_i \text{ for } (i-1)/N \leqq x < i/N, \end{cases}$$

will be called the "associated" distribution function.

Given a sequence $\{\mathfrak{P}_n\}$ such that (1.11) and (1.12) hold we shall have the *sequence* $\{F_n\}$ of associated distribution functions and the family $\overline{\{F_n\}}$ of the distributions that can be obtained as limits of the F_n's.

1.2. LEMMA 1.21. *For a given $\{\mathfrak{P}_n\}$ the functions of $\overline{\{F_n\}}$, except for a possible jump at the origin, are absolutely continuous.*

Proof. Because of (a) we have that for each $i = 1, 2, \cdots, N$

$$1 \geqq p_1 + p_2 + \cdots p_i \geqq ip_i ;$$

this gives that for $(i-1)/N \leqq x < i/N$.

$$F'(x) \leqq N/i < 1/x .$$

Thus the associated distribution functions are uniformly absolutely continuous on the right of one, hence the same will hold for their limits.

LEMMA 1.22. *The functions F_n are uniformly absolutely continuous if and only if the functions of $\overline{\{F_n\}}$ are continuous at the origin.*

Proof. First of all it is clear that the functions F_n are uniformly absolutely continuous if and only if they are equicontinuous. But if the F_n's are equicontinuous, their limits are continuous. Vice versa if their limits are continuous at the origin, a standard argument shows that the F_n's must be equicontinuous.

DEFINITION. If in the family $\overline{\{F_n\}}$ there are discontinuous functions, we shall say that $\{\mathfrak{P}_n\}$ is a "singular" sequence. In this case the maximum of the jumps of the functions of $\overline{\{F_n\}}$ will be called the "deficiency" of the family $\{\mathfrak{P}_n\}$.

Clearly $\{\mathfrak{P}_n\}$ has deficiency $\geqq \gamma > 0$ if and only if there exists a subsequence $\{n_k\}$ such that for any $\varepsilon > 0$ we have

$$\lim_{k \to \infty} [p_{[\varepsilon N]}(n_k) + \cdots + p_N(n_k)] \leqq 1 - \gamma .^4$$

[4] $[\varepsilon N]$ is to mean "integral part of εn".

1.3. We shall now give a sufficient condition for a $\{\mathfrak{P}_n\}$ to be a singular sequence.

LEMMA 1.3. *The sequence* $\{\mathfrak{P}_n\}$ *fails to be singular only if*

$$\lim_{n\to\infty} -(\sum p_i(n) \log p_i(n))/\log N(n) = 1 . \tag{1.31}$$

As a matter of fact, if (1.31) *does not hold, the deficiency* γ *of* $\{\mathfrak{P}_n\}$ *satisfies the inequality*

$$1 - \gamma \leqq \liminf_{n\to\infty} -(\sum p_i(n) \log p_i(n))/\log N(n) . \tag{1.32}$$

Proof. Since a sequence $\{\mathfrak{P}_n\}$ is singular with deficience $\geqq\gamma$ if a a subsequence $\{\mathfrak{P}_{n_k}\}$ is such, we can assume that

$$\lim_{n\to\infty} -(\sum p_i(n) \log p_i(n))/\log N(n) = 1 - \gamma_0 < 1 . \tag{1.33}$$

We define the quantities $a_k(=a_k(n))$ for $k = 1, 2, \cdots, N(n)$ by setting

$$a_k = k[p_k - p_{k+1}] \qquad (\text{set } p_{N+1} = 0) .$$

Note that from our assumption 1.12 (a) it follows that $a_k \geqq 0$. We also have that

$$p_k = a_k/k + a_{k+1}/(k+1) + \cdots + a_N/N \qquad (k = 1, 2, \cdots, N) .$$

From 1.12 (b) we have

$$a_1 + a_2 + \cdots + a_N = p_1 + p_2 + \cdots + p_N = 1$$

From the concavity of $Y(x) = -x \log x$, for any nonnegative constants $(b_1, b_2, \cdots, b_N)$ we have

$$\sum a_k Y(b_k) \leqq Y(\sum a_k b_k) .$$

Using this inequality with $(b_1, b_2, \cdots, b_N)$ replaced by

$$\left(0, 0, \cdots, 0, \frac{1}{i}, \frac{1}{i+1}, \cdots, \frac{1}{N}\right)$$

we get

$$-\sum_{k=i}^{N} a_k/k \log 1/k \leqq -\left(\sum_{k=i}^{N} a_k/k\right) \log \left(\sum_{k=i}^{N} a_k/k\right) = -p_i \log p_i .$$

Summing with respect to i we obtain

$$-\sum_{i=1}^{N} p_i \log p_i \geqq -\sum_{i=1}^{N} \sum_{k=i}^{N} a_k/k \log 1/k = \sum_{k=1}^{N} a_k \log k .$$

Making the substitution $a_k = \theta_k - \theta_{k-1}$ (setting $\theta_0 = 0$, $\theta_k = a_1 + a_2 + \cdots + a_k$), we get

$$(1.34) \quad -\sum_{i=1}^{N} p_i \log p_i \geqq \log N + \sum_{k=1}^{N-1} \theta_k \log k/(1+k) \geqq -\sum_{k=1}^{N} \theta_k/k + \log N .$$

Now let $\nu(n)$ be a sequence of integers taken with the sole restriction that $\nu(n) \sim \varepsilon N(n)$ for some $\varepsilon > 0$. Then from (1.34) we deduce that

$$\log N + \sum_{i=1}^{N} p_i \log p_i \leqq \theta_{\nu(n)} \log N + \log N/\nu(n) .$$

Dividing by $\log N$ and passing to the limit we get

$$\gamma_0 \leqq \liminf_{n\to\infty} \theta_{\nu(n)} .$$

In view of the fact that $p_1 + p_2 + \cdots + p_\nu \geqq a_1 + a_2 + \cdots + a_\nu$ and the definition of $\nu(n)$ we deduce that the functions of $\{\overline{F_n}\}$ must all have a jump at the origin at least as big as γ_0. This establishes the inequality in (1.32).

Suppose we are given a sequence of probability distributions

$$Q_n = \{q_1(n), q_2(n), \cdots, q_{N(n)}(n)\} .$$

In case the $q_i(n)$ are not ordered we shall say that the sequence $\{Q_n\}$ is singular if and only if the ordered sequence $\{\mathfrak{P}_n\}$ that we obtain by rearranging the probabilities of the Q_n's is a singular sequence. Similarly the deficiency of Q_n will be the deficiency of P_n. Lemma 1.3 remains valid for unordered sequences of probability distributions.

2. Proofs of the results.

2.1. Theorem I.1 can be readily obtained from Theorem I.4. We shall thus concentrate in proving the latter. To this end we need the following result. Let y and y_n denote random variables with distributions $F(x)$ and $F_n(x)$ respectively with $F(x)$ continuous. Assume in addition that the random variable $z_n = y - y_n$ is independent of y_n and that $E(z_n^2) \to 0$.

LEMMA 2.1. *If R_n is a sequence tending to zero in such a way that*

$$\liminf R_n^2/E(z_n^2) > 0 ,$$

then a necessary and sufficient condition for the distribution $F(x)$ to have a singular part is that for an M so large that the quantity

$$\lambda_n = \sum_{|kR_n| \leqq M} [F_n(kR_n + R_n) - F_n(kR_n)] \tag{2.11}$$

is bounded away from zero, the probability distributions

$$Q_n = [(F_n(kR_n + R_n) - F_n(kR_n))/\lambda_n;\ \forall \text{ integers } k \ni |kR_n| \leqq M]$$

form a singular sequence.

This result follows from Lemma 2.5 of [3].

2.2. Assume then, with the notation and the hypotheses of Theorem I.4, that

$$\liminf_{n\to\infty} (H_1 + H_2 + \cdots + H_n)/\log 1/R_n < 1 . \tag{2.21}$$

Let us set

$$y_n = x_1 + x_2 + \cdots + x_n .$$

We shall visualize the probability space Ω where y_n and y are defined as the product of the probability spaces $\Omega_1, \Omega_2, \cdots, \Omega_n$ where the variables $x_1, x_2, \cdots, x_n$ are defined. Then the equivalence relation

$$\omega' \sim \omega'' \text{ if and only if } x_n(\omega') = x_i(\omega'') \text{ for } i = 1, 2, \cdots, n$$

generates a partition of Ω which is finer than the partition generated by the relation

$$\omega' \approx \omega'' \text{ if and only if } y_n(\omega') = y_n(\omega'') . \tag{2.22}$$

Thus denoting by D_n the entropy of the distribution of y_n, in view of well known properties of the entropy function, we shall have

$$D_n \leqq H_1 + \cdots, + H_n . \tag{2.23}$$

Suppose now that y_n takes the values

$$y_{1\,n}, y_{2\,n}, \cdots, y_{N(n)\,n}$$

with respective probabilities

$$q_1(n), q_2(n), \cdots, q_{N(n)}(n) .$$

We shall consider, for a given M, a partition of the indices $1, 2, \cdots, N(n)$ into two sets S' and S'' defined as follows. S' is the set of all i such that $|y_{i\,n}| \leqq M$ and S'' is the complement. Let

$$Q'_n = \sum' q_i , \qquad D'_n = -\sum' q_i/Q'_n \log q_i/Q'_n$$
$$Q''_n = \sum'' q_i , \qquad D''_n = -\sum'' q_i/Q''_n \log q_i/Q''_n ,$$

where the summations $\sum'$ and $\sum''$ are carried out over S' and S'' respectively. We have then that

$$(2.24) \qquad Q'_n D'_n + Q''_n D''_n = D_n + Q'_n \log Q'_n + Q''_n \log Q''_n \leqq D_n .$$

By choosing M sufficiently large, by the hypothesis I.8, we can guarantee that Q'_n after a while remains as close to one as we wish. In particular, since $D''_n \geqq 0$, by 2.23, 2.24 and the hypothesis 2.21 we can make sure that we have also

$$(2.25) \qquad \liminf_{n \to \infty} D'_n / \log 1/R_n < 1 .$$

In the subset of Ω where $|y_n| \leqq M$ we introduce a partition by means of the equivalence $\omega' \cong \omega''$ if and only if $y_n(\omega')$ and $y_n(\omega'')$ belong to the same interval $[kR_n + R_n, kR_n)$. Since this partition is even coarser than the one induced by the equivalence in 2.22, for the entropy E'_n associated with this partition we must have

$$(2.26) \qquad E'_n \leqq D'_n .$$

On the other hand if we let λ_n be as in 2.11 and set

$$E_n = -\sum_{|kR_n| \leqq M} [F_n(kR_n + R_n) - F_n(kR_n)]/\lambda_n \log [F_n(kR_n + R_n) - F_n(kR_n)]/\lambda_n ,$$

we must have

$$E'_n \sim E_n .$$

Combining this relation with (2.26) and (2.25) we deduce that

$$\liminf_{n \to \infty} E_n / \log 1/R_n < 1 .$$

Using this inequality in conjunction with Lemmas 1.3 and 2.1 we obtain the singularity of the distribution $F(x)$. This completes the proof of Theorem I.4

REMARK. It should be pointed out that Theorem I.4 gives a simpler condition and is more general than Theorem 2.6 of [3].

2.3. *Proof of Theorem* I.2. For a given n and p we can write the variable y_{np} defined in I.5 in the form

$$y_{np}(\omega) = y_p(\omega_1) + \beta^p y_p(\omega_2) + \cdots + \beta^{p(n-1)} y_p(\omega_n) ,$$

where $y_p(\omega_1), y_p(\omega_2), \cdots, y_p(\omega_n)$ are supposed independent and equally distributed. If $m = np + r$ where $0 \leqq r < p$ we have that

$$y_m(\omega) = y_{np}(\omega) + \beta^{np} y_r(\omega_{n+1}) .$$

By an argument similar to that used in § 2.2 we deduce that the entropy of the variable $y_m(\omega)$ is less than or equal to the sum of the entropies of the summands. In other words, with the notation of the introduction, we must have

$$H_m(\alpha) \leqq nH_p(\alpha) + H_r(\alpha) .$$

Dividing by m and passing to the limit as $m \to \infty$ we obtain

$$\limsup_{m\to\infty} H_m(\alpha)/m \leqq H_p(\alpha)/p .$$

Since p was arbitrarily chosen we deduce that

$$\lim_{m\to\infty} H_m(\alpha)/m = \inf H_p(\alpha)/p . \tag{2.31}$$

Clearly the limit in (2.31) may differ from $\log 2$ only when α satisfies polynomial equations with coefficients ± 1 or 0.

If α is a P. V. number it can be shown (for instance by means of Lemma 2.5 of [3]) that $H_p(\alpha)/p$ eventually takes values below $\log \alpha$. This accounts for the estimate in I.6. It would be of some interest to know whether or not I.6 holds for other than P. V. numbers.

2.4. *Proof of Theorem* I.3. For a P. V. number α we have the following estimates. If each a_i $(i = 1, 2, \cdots, n)$ takes only the values ± 1 or 0 then either

$$a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$$

or

$$| a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n | \geqq c \tag{2.41}$$

where c is a constant depending only on α. This result can be easily deduced from the definition of P. V. numbers. (See for instance Lemma 1.51 of [3]).

Let then $\{n_k\}$ be a sequence of constants satisfying the condition of Theorem I.3. If the numbers

$$\pm\beta^{n_1} \pm \beta^{n_2} \pm \cdots \pm \beta^{n_k}$$

(with $\beta = 1/\alpha$) were all distinct, the minimum distance between any two of them, in view of (2.41) would be greater than a fixed constant divided by 2^k. Theorem 1.2 of [3] would then apply, and we would deduce that the distribution of the random variable

$$y = \sum_{k=1}^{\infty} \pm \beta^{n_k}$$

is absolutely continuous with a bounded derivative. This is of course absurd since it is known [2] that the distribution of

$$\sum_{n=1}^{\infty} \pm \beta^n$$

is singular.

2.5. We shall omit the proof of Theorem I.5, for it can be carried out step by step as it was done for Theorem 1.10 of [3]. The difference here is that Orlicz spaces methods would have to replace the L_p spaces methods used there. There is one point of the proof that is worth noting. Namely, it is known [1] that a bounded functional on an Orlicz space does not necessarily have an integral representation. However, at a point of our proof of Theorem 1.10 in [3] we use the Riesz representation theorem. Nevertheless, in carrying out the proof of Theorem I.5 even this point need not be modified. In fact, for functionals of the type

$$L(y) = \int_{-\infty}^{-\infty} y(x)dF(x)$$

such a representation holds in an Orlicz space just as well as in an L_p space.

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