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## **CONVERGENCE OF EXTENDED BERNSTEIN POLYNOMIALS IN THE COMPLEX PLANE**

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# CONVERGENCE OF EXTENDED BERNSTEIN POLYNOMIALS IN THE COMPLEX PLANE

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**1. Introduction.** Let  $f(x)$  be defined on  $[0, 1]$ . The following two theorems on the Bernstein polynomials corresponding to  $f$ ,

$$(1.1) \quad B_n(x; f) = \sum_{\lambda=0}^n f\left(\frac{\lambda}{n}\right) \binom{n}{\lambda} x^\lambda (1-x)^{n-\lambda}, \quad n = 1, 2, \dots,$$

are well known.

**THEOREM I.** *If  $f(x)$  is continuous on  $[0, 1]$ , then  $B_n(x; f) \rightarrow f(x)$  as  $n \rightarrow \infty$  uniformly on  $[0, 1]$ .*

**THEOREM II.** *If  $f(z)$ ,  $z = x + iy$ , is analytic in the interior  $E$  of the ellipse with foci at  $z = 0$  and  $z = 1$ , then  $B_n(z; f) \rightarrow f(z)$  as  $n \rightarrow \infty$  on  $E$ , this convergence being uniform on each closed subset of  $E$ .*

The first of these results is due to S. Bernstein [1], the second to L. V. Kantorovitch [6] (See also [4], [7]).

For  $f(x)$  defined on  $[0, \infty)$  the functions

$$(1.2) \quad P_k(x; f) = e^{-kx} \sum_{\lambda=0}^{\infty} \frac{(kx)^\lambda}{\lambda!} f\left(\frac{\lambda}{k}\right), \quad 0 < k,$$

form a natural extension of the Bernstein polynomials, the terms of (1.2) corresponding to a Poisson distribution in much the same manner as the terms of (1.1) correspond to a binomial distribution. The functions (1.2) have been considered by Favard [5], Szász [9], and Butzer [3] for the real case. The results of Favard and Szász include the following analogue of Theorem I.

**THEOREM III.** *If  $f(x)$  is continuous on  $[0, \infty)$ , and if  $f(x) = O(x^A)$  [Szász], or more generally, if  $f(x) = O(e^{Ax})$  [Favard] as  $x \rightarrow \infty$ , where  $A$  is a positive, real constant, then  $P_k(x; f) \rightarrow f(x)$  as  $k \rightarrow \infty$  for  $x$  on  $[0, \infty)$ , this convergence being uniform on each finite subinterval of  $[0, \infty)$ .*

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The order condition  $f(x) = O(x^4)$  can be replaced by  $O(e^{4x})$  in Szász' proof without difficulty through the application of the inequality

$$\begin{aligned} \sum_{|(\lambda/u)-x| \geq \delta} \frac{(tux)^\lambda}{\lambda!} &\leq \frac{1}{\delta^2 u^2} \sum_{\lambda=0}^{\infty} \frac{(\lambda - ux)^2 (tux)^\lambda}{\lambda!} \\ &= \frac{x}{\delta^2 u} [ux(t-1)^2 + t] e^{tux}, \end{aligned}$$

valid for  $0 < u, x, \delta, t$ , in Szász' treatment [9, p. 240] of  $S_1$ .

In this paper our objective is to obtain an analogue of Theorem II. Our principal results are stated in §2 below. In our analysis we depend heavily upon the work [10] of Szász and Yearley. Bohman [2] considers polynomials of the form  $e^{-Nz} \sum_{\lambda=0}^n ((Nz)^\lambda / \lambda!) f(\lambda/n)$ ,  $N = N(n)$ , in the complex plane, but there seems to be no existing treatment of the series (1.2) for the complex case.

**2. Principal results** Corresponding to the positive number  $d$ , let  $p(d)$  denote the parabolic set  $\{z \mid |z| < x + 2d^2\}$ . We will say that a function  $f(z)$  defined in  $p(d)$  has property  $B$  in  $p(d)$  if there corresponds to each  $b$ ,  $0 < b < d$ , a positive number  $B(b)$  such that for  $z \in p(b)$

$$(2.1) \quad |f(z)| \leq B(b) \exp \left\{ \frac{1}{2} x - |x|^{1/2} \left[ b^2 - \frac{1}{2} (|z| - x) \right]^{1/2} \right\}.$$

A collection of functions  $\{f_k(x)\}_{0 < k}$ , each defined in  $p(d)$ , will be said to have property  $B$  uniformly in  $p(d)$  if there corresponds to each  $b$ ,  $0 < b < d$ , a positive number  $B(b)$ , independent of  $k$ , such that (2.1) holds for each  $f_k$ . Our principal theorem is then

**THEOREM IV.** *Suppose that  $f(z)$  is analytic and has property  $B$  in  $p(d)$ , where  $d$  is a positive number. Then the functions*

$$(2.2) \quad P_k(z; f) = e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} f\left(\frac{\lambda}{k}\right), \quad 0 < k,$$

satisfy the following four conditions. (1)  $P_k(z; f)$  is an entire function of  $z$  for each  $k$ . (2)  $P_k(z; f) \rightarrow f(z)$  as  $k \rightarrow \infty$  in  $p(d)$ . (3) The convergence in (2) is uniform on each compact subset of  $p(d)$ . (4) The functions  $\{P_k(z/\chi_k; f)\}_{0 < k}$ , where  $\chi_k = \exp[1/(2k)]$ , have property  $B$  uniformly in  $p(d)$ .

We note the result of Pollard [8] and Szász and Yearley [10] that, in order that a function  $f(z)$  be analytic and have property  $B$  in  $p(d)$ ,  $0 < d$ , it is necessary and sufficient that  $f(z)$  possess a Laguerre series (of order 0),

$$f(z) \sim \sum_{n=0}^{\infty} a_n L_n(z), a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx,$$

which converges to it in  $p(d)$ . As a consequence of this result, the hypothesis in Theorem IV that  $f(z)$  be analytic and have property  $B$  in  $p(d)$  can be replaced by the hypothesis that  $f(z)$  possess a Laguerre series which converges to it in  $p(d)$ . The result of Szász and Yeadley [10] is valid as well for general Laguerre series.

3. *Lemmas for Theorem IV.* It is convenient to develop the proof of Theorem IV in lemmas. Unless the contrary is stated we assume  $z$  arbitrary and  $0 < k$ .

LEMMA 1. *If  $f(z)$  is a polynomial, then  $P_k(z; f)$  is a polynomial of the same degree as  $f$ .*

*Proof.* We can suppose  $f \equiv z^n$ , where  $n$  is a nonnegative integer. We have

$$e^{-z} \sum_{\lambda=0}^{\infty} \frac{z^\lambda}{\lambda!} \lambda^n = e^{-z} (zD_z)^n e^z = \sum_{j=0}^n c_j^{(n)} z^j,$$

where the  $c_j^{(n)}$  are constants. We obtain then

$$P_k(z; f) = e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \left(\frac{\lambda}{k}\right)^n = \frac{1}{k^n} \sum_{j=0}^n c_j^{(n)} (kz)^j$$

and the lemma follows.

We may observe that  $c_n^{(n)} = 1$ . It follows that  $P_k(z; f) \rightarrow z^n$  as  $k \rightarrow \infty$  for every  $z$ , the convergence being uniform on each compact set. The same result then holds for any polynomial.

LEMMA 2. *Denote by  $G_k^{(n)}(z)$  the polynomial*

$$G_k^{(n)}(z) = P_k(z; L_n), \quad n = 0, 1, 2, \dots,$$

where  $L_n$  is the  $n$ th Laguerre polynomial of order 0. Then

$$(3.1) \quad |G_k^{(n)}(z)| \leq \exp(-kx + k\chi_k |z|), \quad n = 1, 2, \dots,$$

and

$$(3.2) \quad \sum_{n=0}^{\infty} G_k^{(n)}(z) w^n = \frac{1}{1-w} \exp \left\{ -kz + kz \exp \left[ \frac{-w}{k(1-w)} \right] \right\}, \quad |w| < 1.$$

*Proof.* The inequality (3.1) follows from the fact that [11, p. 162]

$$(3.3) \quad |L_n(x)| \leq \exp(\tfrac{1}{2}x), \quad 0 \leq x, n = 1, 2, \dots$$

For the Laguerre polynomials  $L_n$  we have [11, p. 100]

$$\sum_{n=0}^{\infty} L_n(z)w^n = \frac{1}{1-w} \exp\left(\frac{-zw}{1-w}\right), \quad |w| < 1,$$

from which we obtain

$$\begin{aligned} e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \sum_{n=0}^{\infty} L_n\left(\frac{\lambda}{k}\right)w^n &= \frac{e^{-kz}}{1-w} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \exp\left[\frac{-\lambda w}{k(1-w)}\right] \\ &= \frac{1}{1-w} \exp\left\{-kz + kz \exp\left[\frac{-w}{k(1-w)}\right]\right\}. \end{aligned}$$

For  $z, k, w$ , fixed,  $|w| < 1$ , the double series on the left here is absolutely convergent. Interchanging the order of summation in this series we get (3.2).

LEMMA 3. *Let*

$$H_k(z, w) = \mathcal{R}\left\{-kz + kz \exp\left[\frac{-w}{k(1-w)}\right]\right\}.$$

Then

$$(3.4) \quad H_k(z, w) \leq \chi_k r(|z| - rx)/(1 - r^2), \quad |w| = r < 1.$$

This is a principal lemma for the proof of Theorem IV. We show that

$$(3.5) \quad H_k(z, w) \leq \alpha r(|z| - rx)/(1 - r^2), \quad |w| = r < 1,$$

where  $\alpha = \alpha(r, k) = \exp\{r/[k(1+r)]\}$ . This inequality is slightly stronger than (3.4). The proof is based on the representation (3.6), the use of which was suggested by the referee and results in a simpler proof than that originally submitted by the authors for (3.4).

*Proof.* The inequality (3.5) is trivial for  $z = 0$  or  $w = 0$ . We assume then  $|z|, |w|, k$  fixed with  $z \neq 0, 0 < r < 1$ . We write

$$\begin{aligned} z &= |z|e^{i\phi}, & \rho &= r/(1-r^2), & e^{i\theta} &= w(1-\bar{w})/[r(1-w)], \\ a &= 1/k, & \Phi &= \phi - a\rho \sin \theta. \end{aligned}$$

We have then

$$(3.6) \quad w/(1-w) = \rho(r + e^{i\theta}),$$

and we find that (3.5) holds provided

$$(3.7) \quad T(\theta, \phi) = (a\alpha r\rho - 1) \cos \phi + e^{-a\rho(r+\cos \theta)} \cos \Phi \leq a\alpha\rho$$

for  $|\theta|, |\phi| \leq \pi$ . Since  $T$  is symmetric in the origin in the  $(\theta, \phi) -$

plane, it is enough to show that (3.7) holds for  $(\theta, \phi)$  in the rectangle  $R: 0 \leq \theta \leq \pi, |\phi| \leq \pi$ .

Suppose first that  $1 \leq a\alpha r\rho$ . Since  $e^t \leq 1 + te^t, 0 \leq t$ , we then have

$$T \leq a\alpha r\rho - 1 + \alpha \leq a\alpha r\rho + a\alpha r/(1 + r) = a\alpha\rho,$$

which is (3.7) for this case.

Suppose then that  $a\alpha r\rho < 1$ . Let  $(\theta, \phi)$  denote a maximal point of  $T$  on  $R$ . We consider three possible cases

$$\theta = 0, \quad \theta = \pi, \quad 0 < \theta < \pi.$$

If  $\theta = 0$ , then

$$T = (a\alpha r\rho - 1 + e^{-a\alpha r/(1-r)}) \cos \phi.$$

If the coefficient of  $\cos \phi$  here is nonnegative, we have immediately

$$T \leq a\alpha r\rho \leq a\alpha\rho.$$

If this coefficient is negative, we have

$$\begin{aligned} T &\leq e^{a\alpha r/(1-r)}(e^{a\alpha r/(1-r)} - 1) - a\alpha r\rho \\ &\leq a\alpha r/(1 - r) - a\alpha r\rho \leq a\alpha\rho. \end{aligned}$$

If  $\theta = \pi$ , then

$$T = (a\alpha r\rho - 1 + \alpha) \cos \phi \leq a\alpha\rho.$$

Accordingly, to complete the proof it remains to consider the case  $0 < \theta < \pi$ .

At  $(\theta, \phi)$  both first partial derivatives of  $T$  vanish. Accordingly we obtain

$$(3.8) \quad \begin{aligned} \sin(\theta + \Phi) &= \sin \theta \cos \Phi + \cos \theta \sin \Phi = 0, \\ (a\alpha r\rho - 1) \sin \phi + e^{-a\rho(r+\cos \theta)} \sin \Phi &= 0. \end{aligned}$$

From these relations we then get

$$\begin{aligned} T \sin \theta &= (a\alpha r\rho - 1) \sin \theta \cos \phi + e^{-a\rho(r+\cos \theta)} \sin \theta \cos \Phi \\ &= (a\alpha r\rho - 1) \sin \theta \cos \phi - e^{-a\rho(r+\cos \theta)} \cos \theta \sin \Phi \\ &= (a\alpha r\rho - 1) \sin(\theta + \phi). \end{aligned}$$

Now from (3.8)  $\theta + \Phi = n\pi$ , where  $n = 0, \pm 1, \dots$ . Thus  $\theta + \phi = \theta + \Phi + a\rho \sin \theta = n\pi + a\rho \sin \theta$ , and

$$(3.9) \quad T \sin \theta = (a\alpha r\rho - 1) \sin(n\pi + a\rho \sin \theta).$$

From (3.9) we get, since  $a\alpha r\rho < 1$  and  $0 < \theta < \pi$ ,

$$(3.10) \quad T \sin \theta \leq (1 - \alpha \alpha r \rho) \alpha \rho \sin \theta \leq \alpha \rho \sin \theta .$$

The inequality (3.10) gives  $T \leq \alpha \alpha \rho$ , which completes the proof.

LEMMA 4. *Let  $\alpha, \beta, \gamma$  be positive constants such that  $\alpha \leq \beta$ . Put  $u(t) = 4\alpha^2/t + t\beta^2/(4 + t)$ . Then*

$$I(\alpha, \beta, \gamma) = \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left[ -u(t) - \frac{4\gamma^2}{t} \right] dt \leq M_1(\gamma) \exp(\alpha^2 - 2\alpha\beta) ,$$

where

$$M_1(\gamma) = e[2 + \sqrt{\pi}/(16\alpha^3)]/(e - 1) .$$

This lemma and the next two are closely related to results obtained by Szász and Yearley [10]. Our proofs are somewhat different from theirs. The precise bound  $M_3$  appearing in Lemma 6 does not occur in their article.

*Proof.* If  $\alpha = \beta$ , then  $u(t) = \alpha^2 + 16\alpha^2/[t(4 + t)] > \alpha^2 = 2\alpha\beta - \beta^2$  for  $0 < t$ . If  $\alpha < \beta$ , then  $u(t)$  has the minimum value  $2\alpha\beta - \alpha^2$  on this interval. Thus

$$I \leq \exp(\alpha^2 - 2\alpha\beta) \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left( -\frac{4\gamma^2}{t} \right) dt .$$

For  $0 < t \leq 1$  we have  $t(1 - 1/e) \leq 1 - e^{-t}$ , and for  $1 \leq t$  we have  $1 - 1/e \leq 1 - e^{-t}$ . This gives

$$I \leq [e/(e - 1)] \exp(\alpha^2 - 2\alpha\beta) \times \left[ \int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt + \int_1^\infty t^{-3/2} \exp(-4\alpha^2/t) dt \right] .$$

Now

$$\int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt \leq \int_0^\infty t^{-5/2} \exp(-4\alpha^2/t) dt = \sqrt{\pi}/(16\gamma^3) ,$$

$$\int_1^\infty t^{-3/2} \exp(-4\gamma^2/t) dt \leq \int_1^\infty t^{-3/2} dt = 2 ,$$

and the lemma follows.

LEMMA 5. *If  $0 < b < c$ , and*

$$J(b, c, z) = \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left[ -\frac{4c^2}{t} + \frac{2e^{-t/2}}{1 - e^{-t}} (|z| - xe^{t/2}) \right] dt ,$$

then

$$J(b, c, z) \leq M_2(b, c) \exp \left\{ x - 2|x|^{1/2} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}$$

for  $z \in p(b)$ , where

$$M_2(b, c) = e^{4b^2} M_1((c^2 - b^2)^{3/2}).$$

*Proof.* Suppose  $z \in p(b)$ , so that  $0 < b^2 + x$ . From the inequalities  $e^{-t/2}/(1 - e^{-t}) \leq 1/t$ ,  $e^{-t/2}(1 - e^{-t/2})/(1 - e^{-t}) \leq 2/(4 + t)$ , valid for  $0 < t$ , we then obtain for  $0 < t$

$$\begin{aligned} \frac{2e^{-t/2}}{1 - e^{-t}} (|z| - xe^{-t/2}) &= \frac{2e^{-t/2}}{1 - e^{-t}} [|z| - x + x(1 - e^{-t/2})] \\ &\leq \frac{2e^{-t/2}}{1 - e^{-t}} [|z| - x + (x + b^2)(1 - e^{-t/2})] \\ &\leq 2(|z| - x)/t + 4(x + b^2)/(4 + t) \\ &= 2(|z| - x)/t + x + b^2 - t(x + b^2)/(4 + t). \end{aligned}$$

Thus

$$J \leq e^{x+b^2} \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left\{ \frac{-4(c^2 - b^2)}{t} - \frac{4}{t} \left[ b^2 - \frac{1}{2}(|z| - x) \right] - \frac{t(x + b^2)}{4 + t} \right\} dt.$$

Since  $b^2 - \frac{1}{2}(|z| - x) \leq x + b^2$ , Lemma 4 is applicable. Applying this lemma we then get for  $z \in p(b)$

$$J \leq e^{x+b^2} M_1((c^2 - b^2)^{3/2}) \cdot \exp \left\{ b^2 - \frac{1}{2}(|z| - x) - 2(x + b^2)^{1/2} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}.$$

Now  $|x|^{1/2} - b \leq (x + b^2)^{1/2}$  for  $z \in p(b)$ , and the lemma follows readily.

LEMMA 6. Suppose  $0 < b < c$ . Then

$$\begin{aligned} \sum_{n=0}^\infty |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) \\ \leq M_3(b, c) \exp \left\{ x - 2|x|^{1/2} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\} \end{aligned}$$

for  $z \in p(b)$ , where

$$M_3(b, c) = (2c\sqrt{\pi}) M_2(b, c).$$

*Proof.* Let  $C_r$ ,  $0 < r < 1$ , denote the circle of radius  $r$  about the origin in the  $w$ -plane. Making use of Lemmas 2 and 3 and a classical



integral formula we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 r^{2n} &= \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \left| \exp \left\{ -kz + kz \exp \left[ \frac{-w}{k(1-w)} \right] \right\} \right|^2 |dw| \\ &= \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \exp [2H_k(z; w)] |dw| \\ &\leq \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \exp \{2\chi_k r(z - rx)/(1 - r^2)\} |dw| \\ &= \frac{1}{1 - r^2} \exp [2\chi_k r(|z| - rx)/(1 - r^2)]. \end{aligned}$$

Thus, if  $0 < t$ , then

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 e^{-nt} \\ \leq [1/(1 - e^{-t})] \exp \{2e^{-t/2}(|z| - xe^{-t/2})/(1 - e^{-t})\}. \end{aligned}$$

On the other hand,

$$\exp(-4c\sqrt{n}) = (2c/\sqrt{\pi}) \int_0^{\infty} t^{-3/2} \exp(-nt - 4c^2/t) dt.$$

Hence, applying Lemma 5, we get

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) \\ &= (2c/\sqrt{\pi}) \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \int_0^{\infty} t^{-3/2} \exp(-nt - 4c^2/t) dt \\ &= (2c/\sqrt{\pi}) \int_0^{\infty} t^{-3/2} \exp \left( - (4c^2/t) \left[ \sum_{n=0}^{\infty} |G_k(z/\chi_k)|^2 \exp(-nt) \right] dt \right) \\ &\leq (2c/\sqrt{\pi}) \int_0^{\infty} \frac{t^{-3/2}}{1 - e^{-t}} \exp \left[ \frac{-4c^2}{t} + \frac{2e^{-t/2}}{1 - e^{-t}} (|z| - xe^{-t/2}) \right] dt \\ &\leq (2c/\sqrt{\pi}) M_2(b, c) \exp \left\{ x - 2|x|^{1/2} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\} \end{aligned}$$

for  $z \in p(b)$ . This is the required inequality.

4. *Proof of Theorem IV.* Assume the hypotheses of Theorem IV hold. We note first that under these hypotheses  $f(x)$  satisfies

$$(4.1) \quad |f(x)| \leq Ae^{x/2}, \quad 0 \leq x,$$

for some positive constant  $A$ . It is seen then that the series in (2.2) converges for  $z, k$  arbitrary,  $0 < k$ . Thus conclusion (1) of Theorem IV holds.

Next, by the theorem of Pollard, and Szász and Yearley noted in § 2 above, the hypotheses of Theorem IV imply that  $f$  can be repre-

sented in  $p(d)$  by a convergent Laguerre series:

$$(4.2) \quad f(z) = \sum_{n=0}^{\infty} a_n L_n(z), z \in p(d); \quad a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx .$$

From the convergence in  $p(d)$  of the series (4.2) it follows that, if  $\varepsilon$  is an arbitrary positive number, then

$$(4.3) \quad |a_n| \leq A_\varepsilon \exp [2n(-d + \varepsilon)], \quad n = 1, 2, \dots,$$

for a suitably chosen positive constant  $A_\varepsilon$ . From (4.3) we obtain

$$(4.4) \quad \sum_{n=0}^{\infty} |a_n| < \infty, M(c; f) = \sum_{n=0}^{\infty} |a_n|^2 \exp (4c\sqrt{n}) < \infty$$

the latter provided  $0 < c < d$ .

Now consider  $P_k(z; f)$ . We have formally

$$(4.5) \quad \begin{aligned} P_k(z; f) &= e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \sum_{n=0}^{\infty} a_n L_n(\lambda/k) \\ &= \sum_{n=0}^{\infty} a_n \left[ e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} L_n\left(\frac{\lambda}{k}\right) \right] \\ &= \sum_{n=0}^{\infty} a_n G_k^{(n)}(z) . \end{aligned}$$

Making use of (3.3) and the first inequality in (4.4) we see that the series in the first line of (4.5) converges absolutely for  $z, k$  arbitrary,  $0 < k$ . This justifies the formal manipulation in (4.5) and we accordingly have

$$(4.6) \quad P_k(z; f) = \sum_{n=0}^{\infty} a_n G_k^{(n)}(z)$$

for  $z, k$  arbitrary,  $0 < k$ . From (4.6) we get

$$|P_k(z; f)|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \exp (4c\sqrt{n}) \sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 \exp (-4c\sqrt{n}) .$$

Thus, by Lemma 6, if  $0 < b < c < d$ , then

$$|P_k(z/\chi_k; f)|^2 \leq M(c; f) \cdot M_3(b, c) \cdot \exp \left\{ x - 2|x|^{1/2} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}$$

for  $z \in p(b)$ . For a fixed  $b, 0 < b < d$ , on taking  $c = \frac{1}{2}(b + d)$ , say, we find then that conclusion (4) holds with

$$B(b) = [M(c; f)M_3(b, c)]^{1/2}, \quad c = \frac{1}{2}(b + d) .$$

It remains to consider conclusions (2) and (3). It is enough to show

that, if  $S$  is a compact subset of  $p(d)$ , then  $P_k(z; f) \rightarrow f(z)$ ,  $k \rightarrow \infty$ , uniformly on  $S$ . For  $0 < b$ ,  $0 < x_0$  let

$$U(b, x_0) = \{z \mid |z| < x + 2b^2, x < x_0\}.$$

Choose  $b_1, b_2, b_3; x_1, x_2, x_3$  such that  $0 < b_1 < b_2 < b_3 < d$ ,  $0 < x_1 < x_2 < x_3$ , and  $S \subset U(b_1, x_1)$ . Making use of conclusion (4), we infer that there exists a constant  $M^*$  such that

$$|P_k(z/\chi_k; f)| \leq M^*, z \in U(b_3, x_3).$$

Choose  $k_0 = \max\{[4 \cdot \ln(b_3/b_2)]^{-1}, [2 \cdot \ln(x_3/x_2)]^{-1}\}$ . Then for  $k_0 < k$  and  $z \in U(b_2, x_2)$  we have  $z\chi_k \in U(b_3, x_3)$ . Thus

$$(4.7) \quad |P_k(z; f)| = |P_k(z\chi_k/\chi_k; f)| \leq M^*, k_0 < k, z \in U(b_2, x_2).$$

Recalling (4.1), we have also, by Theorem III,

$$P_k(x; f) \rightarrow f(x), k \rightarrow \infty, 0 < x < x_2.$$

By an application of Vitali's theorem,  $\{P_k(z; f)\}_{k_0 < k}$  converges uniformly on  $U(b_1, x_1)$  to a function  $F(z)$ , analytic on  $U(b_1, x_1)$ . Since  $f(z)$  is analytic on  $U(b_1, x_1)$  and  $F(x) = f(x)$ ,  $0 < x < x_1$ , it follows that  $F(z) = f(z)$  throughout  $U(b_1, x_1)$ , and the proof of complete.

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