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CONVERGENCE OF EXTENDED BERNSTEIN POLYNOMIALS IN THE COMPLEX PLANE

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1. Introduction. Let $f(x)$ be defined on $[0, 1]$. The following two theorems on the Bernstein polynomials corresponding to f ,

$$(1.1) \quad B_n(x; f) = \sum_{\lambda=0}^n f\left(\frac{\lambda}{n}\right) \binom{n}{\lambda} x^\lambda (1-x)^{n-\lambda}, \quad n = 1, 2, \dots,$$

are well known.

THEOREM I. *If $f(x)$ is continuous on $[0, 1]$, then $B_n(x; f) \rightarrow f(x)$ as $n \rightarrow \infty$ uniformly on $[0, 1]$.*

THEOREM II. *If $f(z)$, $z = x + iy$, is analytic in the interior E of the ellipse with foci at $z = 0$ and $z = 1$, then $B_n(z; f) \rightarrow f(z)$ as $n \rightarrow \infty$ on E , this convergence being uniform on each closed subset of E .*

The first of these results is due to S. Bernstein [1], the second to L. V. Kantorovitch [6] (See also [4], [7]).

For $f(x)$ defined on $[0, \infty)$ the functions

$$(1.2) \quad P_k(x; f) = e^{-kx} \sum_{\lambda=0}^{\infty} \frac{(kx)^\lambda}{\lambda!} f\left(\frac{\lambda}{k}\right), \quad 0 < k,$$

form a natural extension of the Bernstein polynomials, the terms of (1.2) corresponding to a Poisson distribution in much the same manner as the terms of (1.1) correspond to a binomial distribution. The functions (1.2) have been considered by Favard [5], Szász [9], and Butzer [3] for the real case. The results of Favard and Szász include the following analogue of Theorem I.

THEOREM III. *If $f(x)$ is continuous on $[0, \infty)$, and if $f(x) = O(x^A)$ [Szász], or more generally, if $f(x) = O(e^{Ax})$ [Favard] as $x \rightarrow \infty$, where A is a positive, real constant, then $P_k(x; f) \rightarrow f(x)$ as $k \rightarrow \infty$ for x on $[0, \infty)$, this convergence being uniform on each finite subinterval of $[0, \infty)$.*

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The order condition $f(x) = O(x^A)$ can be replaced by $O(e^{Ax})$ in Szász' proof without difficulty through the application of the inequality

$$\sum_{|(\lambda/w) - x| \geq \delta} \frac{(tux)^\lambda}{\lambda!} \leq \frac{1}{\delta^2 u^2} \sum_{\lambda=0}^{\infty} \frac{(\lambda - ux)^2 (tux)^\lambda}{\lambda!} \\ = \frac{x}{\delta^2 u} [ux(t-1)^2 + t] e^{tux},$$

valid for $0 < u, x, \delta, t$, in Szász' treatment [9, p. 240] of S_4 .

In this paper our objective is to obtain an analogue of Theorem II. Our principal results are stated in §2 below. In our analysis we depend heavily upon the work [10] of Szász and Yeadley. Bohman [2] considers polynomials of the form $e^{-Nz} \sum_{\lambda=0}^n ((Nz)^\lambda / \lambda!) f(\lambda/n)$, $N = N(n)$, in the complex plane, but there seems to be no existing treatment of the series (1.2) for the complex case.

2. Principal results Corresponding to the positive number d , let $p(d)$ denote the parabolic set $\{z \mid |z| < x + 2d^2\}$. We will say that a function $f(z)$ defined in $p(d)$ has property B in $p(d)$ if there corresponds to each b , $0 < b < d$, a positive number $B(b)$ such that for $z \in p(b)$

$$(2.1) \quad |f(z)| \leq B(b) \exp \left\{ \frac{1}{2} x - |x|^{1/2} \left[b^2 - \frac{1}{2} (|z| - x) \right]^{1/2} \right\}.$$

A collection of functions $\{f_k(x)\}_{0 < k}$, each defined in $p(d)$, will be said to have property B uniformly in $p(d)$ if there corresponds to each b , $0 < b < d$, a positive number $B(b)$, independent of k , such that (2.1) holds for each f_k . Our principal theorem is then

THEOREM IV. *Suppose that $f(z)$ is analytic and has property B in $p(d)$, where d is a positive number. Then the functions*

$$(2.2) \quad P_k(z; f) = e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} f\left(\frac{\lambda}{k}\right), \quad 0 < k,$$

satisfy the following four conditions. (1) $P_k(z; f)$ is an entire function of z for each k . (2) $P_k(z; f) \rightarrow f(z)$ as $k \rightarrow \infty$ in $p(d)$. (3) The convergence in (2) is uniform on each compact subset of $p(d)$. (4) The functions $\{P_k(z; \chi_k; f)\}_{0 < k}$, where $\chi_k = \exp[1/(2k)]$, have property B uniformly in $p(d)$.

We note the result of Pollard [8] and Szász and Yeadley [10] that, in order that a function $f(z)$ be analytic and have property B in $p(d)$, $0 < d$, it is necessary and sufficient that $f(z)$ possess a Laguerre series (of order 0),

$$f(z) \sim \sum_{n=0}^{\infty} a_n L_n(z), \quad a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx,$$

which converges to it in $p(d)$. As a consequence of this result, the hypothesis in Theorem IV that $f(z)$ be analytic and have property B in $p(d)$ can be replaced by the hypothesis that $f(z)$ possess a Laguerre series which converges to it in $p(d)$. The result of Szász and Yearley [10] is valid as well for general Laguerre series.

3. *Lemmas for Theorem IV.* It is convenient to develop the proof of Theorem IV in lemmas. Unless the contrary is stated we assume z arbitrary and $0 < k$.

LEMMA 1. *If $f(z)$ is a polynomial, then $P_k(z; f)$ is a polynomial of the same degree as f .*

Proof. We can suppose $f \equiv z^n$, where n is a nonnegative integer. We have

$$e^{-z} \sum_{\lambda=0}^{\infty} \frac{z^\lambda}{\lambda!} \lambda^n = e^{-z} (zD_z)^n e^z = \sum_{j=0}^n c_j^{(n)} z^j,$$

where the $c_j^{(n)}$ are constants. We obtain then

$$P_k(z; f) = e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \left(\frac{\lambda}{k}\right)^n = \frac{1}{k^n} \sum_{j=0}^n c_j^{(n)} (kz)^j$$

and the lemma follows.

We may observe that $c_n^{(n)} = 1$. It follows that $P_k(z; f) \rightarrow z^n$ as $k \rightarrow \infty$ for every z , the convergence being uniform on each compact set. The same result then holds for any polynomial.

LEMMA 2. *Denote by $G_k^{(n)}(z)$ the polynomial*

$$G_k^{(n)}(z) = P_k(z; L_n), \quad n = 0, 1, 2, \dots,$$

where L_n is the n th Laguerre polynomial of order 0. Then

$$(3.1) \quad |G_k^{(n)}(z)| \leq \exp(-kx + k\chi_k |z|), \quad n = 1, 2, \dots,$$

and

$$(3.2) \quad \sum_{n=0}^{\infty} G_k^{(n)}(z) w^n = \frac{1}{1-w} \exp \left\{ -kz + kz \exp \left[\frac{-w}{k(1-w)} \right] \right\}, \quad |w| < 1.$$

Proof. The inequality (3.1) follows from the fact that [11, p. 162]

$$(3.3) \quad |L_n(x)| \leq \exp\left(\frac{1}{2}x\right), \quad 0 \leq x, n = 1, 2, \dots$$

For the Laguerre polynomials L_n we have [11, p. 100]

$$\sum_{n=0}^{\infty} L_n(z)w^n = \frac{1}{1-w} \exp\left(\frac{-zw}{1-w}\right), \quad |w| < 1,$$

from which we obtain

$$\begin{aligned} e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \sum_{n=0}^{\infty} L_n\left(\frac{\lambda}{k}\right)w^n &= \frac{e^{-kz}}{1-w} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \exp\left[\frac{-\lambda w}{k(1-w)}\right] \\ &= \frac{1}{1-w} \exp\left\{-kz + kz \exp\left[\frac{-w}{k(1-w)}\right]\right\}. \end{aligned}$$

For z, k, w , fixed, $|w| < 1$, the double series on the left here is absolutely convergent. Interchanging the order of summation in this series we get (3.2).

LEMMA 3. *Let*

$$H_k(z, w) = \mathcal{R}\left\{-kz + kz \exp\left[\frac{-w}{k(1-w)}\right]\right\}.$$

Then

$$(3.4) \quad H_k(z, w) \leq \chi_k r(|z| - rx)/(1 - r^2), \quad |w| = r < 1.$$

This is a principal lemma for the proof of Theorem IV. We show that

$$(3.5) \quad H_k(z, w) \leq \alpha r(|z| - rx)/(1 - r^2), \quad |w| = r < 1,$$

where $\alpha = \alpha(r, k) = \exp\{r/[k(1+r)]\}$. This inequality is slightly stronger than (3.4). The proof is based on the representation (3.6), the use of which was suggested by the referee and results in a simpler proof than that originally submitted by the authors for (3.4).

Proof. The inequality (3.5) is trivial for $z = 0$ or $w = 0$. We assume then $|z|, |w|, k$ fixed with $z \neq 0, 0 < r < 1$. We write

$$\begin{aligned} z &= |z| e^{i\phi}, & \rho &= r/(1 - r^2), & e^{i\theta} &= w(1 - \bar{w})/[r(1 - w)], \\ a &= 1/k, & \Phi &= \phi - \alpha\rho \sin \theta. \end{aligned}$$

We have then

$$(3.6) \quad w/(1 - w) = \rho(r + e^{i\theta}),$$

and we find that (3.5) holds provided

$$(3.7) \quad T(\theta, \phi) = (\alpha\alpha r\rho - 1) \cos \phi + e^{-\alpha\rho(r + \cos \theta)} \cos \Phi \leq \alpha\alpha\rho$$

for $|\theta|, |\phi| \leq \pi$. Since T is symmetric in the origin in the $(\theta, \phi) -$

plane, it is enough to show that (3.7) holds for (θ, ϕ) in the rectangle $R: 0 \leq \theta \leq \pi, |\phi| \leq \pi$.

Suppose first that $1 \leq a\alpha r\rho$. Since $e^t \leq 1 + te^t, 0 \leq t$, we then have

$$T \leq a\alpha r\rho - 1 + \alpha \leq a\alpha r\rho + a\alpha r/(1 + r) = a\alpha\rho,$$

which is (3.7) for this case.

Suppose then that $a\alpha r\rho < 1$. Let (θ, ϕ) denote a maximal point of T on R . We consider three possible cases

$$\theta = 0, \quad \theta = \pi, \quad 0 < \theta < \pi.$$

If $\theta = 0$, then

$$T = (a\alpha r\rho - 1 + e^{-a\alpha r/(1-r)}) \cos \phi.$$

If the coefficient of $\cos \phi$ here is nonnegative, we have immediately

$$T \leq a\alpha r\rho \leq a\alpha\rho.$$

If this coefficient is negative, we have

$$\begin{aligned} T &\leq e^{a\alpha r/(1-r)}(e^{a\alpha r/(1-r)} - 1) - a\alpha r\rho \\ &\leq a\alpha r/(1-r) - a\alpha r\rho \leq a\alpha\rho. \end{aligned}$$

If $\theta = \pi$, then

$$T = (a\alpha r\rho - 1 + \alpha) \cos \phi \leq a\alpha\rho.$$

Accordingly, to complete the proof it remains to consider the case $0 < \theta < \pi$.

At (θ, ϕ) both first partial derivatives of T vanish. Accordingly we obtain

$$(3.8) \quad \begin{aligned} \sin(\theta + \Phi) &= \sin \theta \cos \Phi + \cos \theta \sin \Phi = 0, \\ (a\alpha r\rho - 1) \sin \phi + e^{-a\alpha\rho(r+\cos \theta)} \sin \Phi &= 0. \end{aligned}$$

From these relations we then get

$$\begin{aligned} T \sin \theta &= (a\alpha r\rho - 1) \sin \theta \cos \phi + e^{-a\alpha\rho(r+\cos \theta)} \sin \theta \cos \Phi \\ &= (a\alpha r\rho - 1) \sin \theta \cos \phi - e^{-a\alpha\rho(r+\cos \theta)} \cos \theta \sin \Phi \\ &= (a\alpha r\rho - 1) \sin(\theta + \phi). \end{aligned}$$

Now from (3.8) $\theta + \Phi = n\pi$, where $n = 0, \pm 1, \dots$. Thus $\theta + \phi = \theta + \Phi + \alpha\rho \sin \theta = n\pi + \alpha\rho \sin \theta$, and

$$(3.9) \quad T \sin \theta = (a\alpha r\rho - 1) \sin(n\pi + \alpha\rho \sin \theta).$$

From (3.9) we get, since $a\alpha r\rho < 1$ and $0 < \theta < \pi$,

$$(3.10) \quad T \sin \theta \leq (1 - \alpha \alpha r \rho) a \rho \sin \theta \leq a \rho \sin \theta .$$

The inequality (3.10) gives $T \leq \alpha a \rho$, which completes the proof.

LEMMA 4. *Let α, β, γ be positive constants such that $\alpha \leq \beta$. Put $u(t) = 4\alpha^2/t + t\beta^2/(4 + t)$. Then*

$$I(\alpha, \beta, \gamma) = \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left[-u(t) - \frac{4\gamma^2}{t} \right] dt \leq M_1(\gamma) \exp(\alpha^2 - 2\alpha\beta) ,$$

where

$$M_1(\gamma) = e[2 + \sqrt{\pi}/(16\alpha^3)]/(e - 1) .$$

This lemma and the next two are closely related to results obtained by Szász and Yeadley [10]. Our proofs are somewhat different from theirs. The precise bound M_3 appearing in Lemma 6 does not occur in their article.

Proof. If $\alpha = \beta$, then $u(t) = \alpha^2 + 16\alpha^2/[t(4 + t)] > \alpha^2 = 2\alpha\beta - \beta^2$ for $0 < t$. If $\alpha < \beta$, then $u(t)$ has the minimum value $2\alpha\beta - \alpha^2$ on this interval. Thus

$$I \leq \exp(\alpha^2 - 2\alpha\beta) \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left(-\frac{4\gamma^2}{t} \right) dt .$$

For $0 < t \leq 1$ we have $t(1 - 1/e) \leq 1 - e^{-t}$, and for $1 \leq t$ we have $1 - 1/e \leq 1 - e^{-t}$. This gives

$$I \leq [e/(e - 1)] \exp(\alpha^2 - 2\alpha\beta) \times \left[\int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt + \int_1^\infty t^{-3/2} \exp(-4\alpha^2/t) dt \right] .$$

Now

$$\int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt \leq \int_0^\infty t^{-5/2} \exp(-4\alpha^2/t) dt = \sqrt{\pi}/(16\alpha^3) ,$$

$$\int_1^\infty t^{-3/2} \exp(-4\gamma^2/t) dt \leq \int_1^\infty t^{-3/2} dt = 2 ,$$

and the lemma follows.

LEMMA 5. *If $0 < b < c$, and*

$$J(b, c, z) = \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left[-\frac{4c^2}{t} + \frac{2e^{-t/2}}{1 - e^{-t}} (|z| - xe^{t/2}) \right] dt ,$$

then

$$J(b, c, z) \leq M_2(b, c) \exp \left\{ x - 2|x|^{1/2} \left[b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}$$

for $z \in p(b)$, where

$$M_2(b, c) = e^{4b^2} M_1((c^2 - b^2)^{3/2}).$$

Proof. Suppose $z \in p(b)$, so that $0 < b^2 + x$. From the inequalities $e^{-t/2}/(1 - e^{-t}) \leq 1/t$, $e^{-t/2}(1 - e^{-t/2})/(1 - e^{-t}) \leq 2/(4 + t)$, valid for $0 < t$, we then obtain for $0 < t$

$$\begin{aligned} \frac{2e^{-t/2}}{1 - e^{-t}} (|z| - xe^{-t/2}) &= \frac{2e^{-t/2}}{1 - e^{-t}} [|z| - x + x(1 - e^{-t/2})] \\ &\leq \frac{2e^{-t/2}}{1 - e^{-t}} [|z| - x + (x + b^2)(1 - e^{-t/2})] \\ &\leq 2(|z| - x)/t + 4(x + b^2)/(4 + t) \\ &= 2(|z| - x)/t + x + b^2 - t(x + b^2)/(4 + t). \end{aligned}$$

Thus

$$J \leq e^{x+b^2} \int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp \left\{ \frac{-4(c^2 - b^2)}{t} - \frac{4}{t} \left[b^2 - \frac{1}{2}(|z| - x) \right] - \frac{t(x + b^2)}{4 + t} \right\} dt.$$

Since $b^2 - \frac{1}{2}(|z| - x) \leq x + b^2$, Lemma 4 is applicable. Applying this lemma we then get for $z \in p(b)$

$$J \leq e^{x+b^2} M_1((c^2 - b^2)^{3/2}) \cdot \exp \left\{ b^2 - \frac{1}{2}(|z| - x) - 2(x + b^2)^{1/2} \left[b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}.$$

Now $|x|^{1/2} - b \leq (x + b^2)^{1/2}$ for $z \in p(b)$, and the lemma follows readily.

LEMMA 6. Suppose $0 < b < c$. Then

$$\begin{aligned} \sum_{n=0}^\infty |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) \\ \leq M_3(b, c) \exp \left\{ x - 2|x|^{1/2} \left[b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\} \end{aligned}$$

for $z \in p(b)$, where

$$M_3(b, c) = (2c\sqrt{\pi})M_2(b, c).$$

Proof. Let C_r , $0 < r < 1$, denote the circle of radius r about the origin in the w -plane. Making use of Lemmas 2 and 3 and a classical

integral formula we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 r^{2n} &= \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \left| \exp \left\{ -kz + kz \exp \left[\frac{-w}{k(1-w)} \right] \right\} \right|^2 |dw| \\ &= \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \exp [2H_k(z; w)] |dw| \\ &\leq \frac{1}{2\pi r} \int_{\sigma_r} \frac{1}{|1-w|^2} \exp \{2\chi_k r(z-rx)/(1-r^2)\} |dw| \\ &= \frac{1}{1-r^2} \exp [2\chi_k r(|z|-rx)/(1-r^2)]. \end{aligned}$$

Thus, if $0 < t$, then

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 e^{-nt} &\leq [1/(1-e^{-t})] \exp \{2e^{-t/2}(|z|-xe^{-t/2})/(1-e^{-t})\}. \end{aligned}$$

On the other hand,

$$\exp(-4c\sqrt{n}) = (2c/\sqrt{\pi}) \int_0^{\infty} t^{-3/2} \exp(-nt - 4c^2/t) dt ..$$

Hence, applying Lemma 5, we get

$$\begin{aligned} \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) &= (2c/\sqrt{\pi}) \sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \int_0^{\infty} t^{-3/2} \exp(-nt - 4c^2/t) dt \\ &= (2c/\sqrt{\pi}) \int_0^{\infty} t^{-3/2} \exp \left(-(4c^2/t) \left[\sum_{n=0}^{\infty} |G_k(z/\chi_k)|^2 \exp(-nt) \right] dt \right) \\ &\leq (2c/\sqrt{\pi}) \int_0^{\infty} \frac{t^{-3/2}}{1-e^{-t}} \exp \left[\frac{-4c^2}{t} + \frac{2e^{-t/2}}{1-e^{-t}} (|z|-xe^{-t/2}) \right] dt \\ &\leq (2c/\sqrt{\pi}) M_2(b, c) \exp \left\{ x - 2|x|^{1/2} \left[b^2 - \frac{1}{2}(|z|-x) \right]^{1/2} \right\} \end{aligned}$$

for $z \in p(b)$. This is the required inequality.

4. *Proof of Theorem IV.* Assume the hypotheses of Theorem IV hold. We note first that under these hypotheses $f(x)$ satisfies

$$(4.1) \quad |f(x)| \leq Ae^{x/2}, \quad 0 \leq x,$$

for some positive constant A . It is seen then that the series in (2.2) converges for z, k arbitrary, $0 < k$. Thus conclusion (1) of Theorem IV holds.

Next, by the theorem of Pollard, and Szász and Yearley noted in § 2 above, the hypotheses of Theorem IV imply that f can be repre-

sented in $p(d)$ by a convergent Laguerre series:

$$(4.2) \quad f(z) = \sum_{n=0}^{\infty} a_n L_n(z), \quad z \in p(\bar{d}); \quad a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx.$$

From the convergence in $p(d)$ of the series (4.2) it follows that, if ε is an arbitrary positive number, then

$$(4.3) \quad |a_n| \leq A_\varepsilon \exp [2n(-d + \varepsilon)], \quad n = 1, 2, \dots,$$

for a suitably chosen positive constant A_ε . From (4.3) we obtain

$$(4.4) \quad \sum_{n=0}^{\infty} |a_n| < \infty, \quad M(c; f) = \sum_{n=0}^{\infty} |a_n|^2 \exp(4c\sqrt{n}) < \infty$$

the latter provided $0 < c < d$.

Now consider $P_k(z; f)$. We have formally

$$(4.5) \quad \begin{aligned} P_k(z; f) &= e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \sum_{n=0}^{\infty} a_n L_n(\lambda/k) \\ &= \sum_{n=0}^{\infty} a_n \left[e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} L_n\left(\frac{\lambda}{k}\right) \right] \\ &= \sum_{n=0}^{\infty} a_n G_k^{(n)}(z). \end{aligned}$$

Making use of (3.3) and the first inequality in (4.4) we see that the series in the first line of (4.5) converges absolutely for z, k arbitrary, $0 < k$. This justifies the formal manipulation in (4.5) and we accordingly have

$$(4.6) \quad P_k(z; f) = \sum_{n=0}^{\infty} a_n G_k^{(n)}(z)$$

for z, k arbitrary, $0 < k$. From (4.6) we get

$$|P_k(z; f)|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \exp(4c\sqrt{n}) \sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 \exp(-4c\sqrt{n}).$$

Thus, by Lemma 6, if $0 < b < c < d$, then

$$|P_k(z/\chi_k; f)|^2 \leq M(c; f) \cdot M_3(b, c) \cdot \exp \left\{ x - 2|x|^{1/2} \left[b^2 - \frac{1}{2}(|z| - x) \right]^{1/2} \right\}$$

for $z \in p(b)$. For a fixed b , $0 < b < d$, on taking $c = \frac{1}{2}(b + d)$, say, we find then that conclusion (4) holds with

$$B(b) = [M(c; f)M_3(b, c)]^{1/2}, \quad c = \frac{1}{2}(b + d).$$

It remains to consider conclusions (2) and (3). It is enough to show

that, if S is a compact subset of $p(d)$, then $P_k(z; f) \rightarrow f(z)$, $k \rightarrow \infty$, uniformly on S . For $0 < b$, $0 < x_0$ let

$$U(b, x_0) = \{z \mid |z| < x + 2b^2, x < x_0\}.$$

Choose $b_1, b_2, b_3; x_1, x_2, x_3$ such that $0 < b_1 < b_2 < b_3 < d$, $0 < x_1 < x_2 < x_3$, and $S \subset U(b_1, x_1)$. Making use of conclusion (4), we infer that there exists a constant M^* such that

$$|P_k(z/\chi_k; f)| \leq M^*, z \in U(b_3, x_3).$$

Choose $k_0 = \max\{[4 \cdot \ln(b_3/b_2)]^{-1}, [2 \cdot \ln(x_3/x_2)]^{-1}\}$. Then for $k_0 < k$ and $z \in U(b_2, x_2)$ we have $z\chi_k \in U(b_3, x_3)$. Thus

$$(4.7) \quad |P_k(z; f)| = |P_k(z\chi_k/\chi_k; f)| \leq M^*, k_0 < k, z \in U(b_2, x_2).$$

Recalling (4.1), we have also, by Theorem III,

$$P_k(x; f) \rightarrow f(x), k \rightarrow \infty, 0 < x < x_2.$$

By an application of Vitali's theorem, $\{P_k(z; f)\}_{k_0 < k}$ converges uniformly on $U(b_1, x_1)$ to a function $F(z)$, analytic on $U(b_1, x_1)$. Since $f(z)$ is analytic on $U(b_1, x_1)$ and $F(x) = f(x)$, $0 < x < x_1$, it follows that $F(z) = f(z)$ throughout $U(b_1, x_1)$, and the proof of complete.

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Dallas O. Banks, <i>Bounds for eigenvalues and generalized convexity</i>	1031
Jerrold William Bebernes, <i>A subfunction approach to a boundary value problem for ordinary differential equations</i>	1053
Woodrow Wilson Bledsoe and A. P. Morse, <i>A topological measure construction</i>	1067
George Clements, <i>Entropies of several sets of real valued functions</i>	1085
Sandra Barkdull Cleveland, <i>Homomorphisms of non-commutative *-algebras</i>	1097
William John Andrew Culmer and William Ashton Harris, <i>Convergent solutions of ordinary linear homogeneous difference equations</i>	1111
Ralph DeMarr, <i>Common fixed points for commuting contraction mappings</i>	1139
James Robert Dorroh, <i>Integral equations in normed abelian groups</i>	1143
Adriano Mario Garsia, <i>Entropy and singularity of infinite convolutions</i>	1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., <i>Convergence of extended Bernstein polynomials in the complex plane</i>	1171
Irving Leonard Glicksberg, <i>A remark on analyticity of function algebras</i>	1181
Charles John August Halberg, Jr., <i>Semigroups of matrices defining linked operators with different spectra</i>	1187
Philip Hartman and Nelson Onuchic, <i>On the asymptotic integration of ordinary differential equations</i>	1193
Isidore Heller, <i>On a class of equivalent systems of linear inequalities</i>	1209
Joseph Hersch, <i>The method of interior parallels applied to polygonal or multiply connected membranes</i>	1229
Hans F. Weinberger, <i>An effectless cutting of a vibrating membrane</i>	1239
Melvin F. Janowitz, <i>Quantifiers and orthomodular lattices</i>	1241
Samuel Karlin and Albert Boris J. Novikoff, <i>Generalized convex inequalities</i>	1251
Tilla Weinstein, <i>Another conformal structure on immersed surfaces of negative curvature</i>	1281
Gregers Louis Krabbe, <i>Spectral permanence of scalar operators</i>	1289
Shige Toshi Kuroda, <i>Finite-dimensional perturbation and a representation of scattering operator</i>	1305
Marvin David Marcus and Afton Herbert Cayford, <i>Equality in certain inequalities</i>	1319
Joseph Martin, <i>A note on uncountably many disks</i>	1331
Eugene Kay McLachlan, <i>Extremal elements of the convex cone of semi-norms</i>	1335
John W. Moon, <i>An extension of Landau's theorem on tournaments</i>	1343
Louis Joel Mordell, <i>On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$</i>	1347
Kenneth Roy Mount, <i>Some remarks on Fitting's invariants</i>	1353
Miroslav Novotný, <i>Über Abbildungen von Mengen</i>	1359
Robert Dean Ryan, <i>Conjugate functions in Orlicz spaces</i>	1371
John Vincent Ryff, <i>On the representation of doubly stochastic operators</i>	1379
Donald Ray Sherbert, <i>Banach algebras of Lipschitz functions</i>	1387
James McLean Sloss, <i>Reflection of biharmonic functions across analytic boundary conditions with examples</i>	1401
L. Bruce Treybig, <i>Concerning homogeneity in totally ordered, connected topological space</i>	1417
John Wermer, <i>The space of real parts of a function algebra</i>	1423
James Juei-Chin Yeh, <i>Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables</i>	1427
William P. Ziemer, <i>On the compactness of integral classes</i>	1437