A REMARK ON ANALYTICITY OF FUNCTION ALGEBRAS

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FUNCTION ALGEBRAS

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1. Let $A$ be a closed separating subalgebra of $C(X)$, $X$ compact, with maximal ideal space $\mathcal{M}_A$ and Šilov boundary $\partial_A$. Naturally $A$ can also be viewed as a closed subalgebra of $C(\mathcal{M}_A)$ or $C(\partial_A)$.

Call $A$ analytic on $X$ if the vanishing of $f \in A$ on a non-void open subset of $X$ implies $f \equiv 0$, or simply analytic if this holds for $X = \mathcal{M}_A$. Recently Kenneth Hoffman asked if the analyticity of $A$ on $\partial_A$ implied analyticity on $\mathcal{M}_A$; the present note is devoted to a counterexample.1 Evidently such an example, analytic on its Šilov boundary, must be an integral domain, so our algebra is a non-analytic integral domain.

The example was suggested by, and utilizes, an interpolation theorem of Rudin and Carleson [5, 9], recently generalized by Bishop [3], which in fact permits the construction of a variety of unfamiliar tractable subalgebras of familiar algebras; consequently we shall discuss the construction in more generality than is absolutely necessary. Finally we give a slightly more complicated example which is also dirichlet.

NOTATION. $M(X)$ will denote the space of (finite complex regular Borel) measures $\mu$ on $X$; for such a $\mu$, $\mu$ is orthogonal to $A(\mu \perp A)$ if $\mu(f) = \int f d\mu = 0$, $f$ in $A$. And $\mu_{\mathcal{F}}$ will denote the usual restriction of $\mu$ to $\mathcal{F} \subset X$, while $f | F$ will be the restriction of a function $f$, $A | F$ the set $\{f | F : f \in A\}$. An algebra $A$ will always be assumed to contain the constants.

2. Our construction is based on the following fact.

(2.1) Suppose $F$ is a closed subset of $X$, and $\mu_{\mathcal{F}} = 0$ for all $\mu$ in $M(X)$ orthogonal to $A$. Then

(2.1.1) $A | F = C(F)$ [3]

(2.1.2) if $X$ is metric, $F$ is a peak set of $A$, i.e., there is an $f$ in

1 After this note was completed, I found that analyticity of $A$ on $\mathcal{M}_A$ implies analyticity on $\partial_A$; this will appear in a subsequent paper.

2 (2.11) is Bishop's generalization of the Rudin-Carleson result mentioned before, which applies to the special case in which $A$ is the "disc algebra" and $F$ a subset of measure zero of the unit circle. (2.12) will actually be avoided in the specific examples we construct.

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Now suppose we are given two uniformly closed algebras $A_1$, $A_2$, as subalgebras of $C(M_1)$, $C(M_2)$, where $M_i = M_{A_i}$ is metric, $i = 1, 2$. Further suppose $\vartheta_2 = \vartheta_{A_2}$ is homeomorphic to a (compact) subset $F$ of $\vartheta_1$ satisfying the hypothesis of (2.1) with $A = A_1$, $X = \vartheta_1$, so that $A_1|F = C(F)$. Identifying $F$ and $\vartheta_2$ (via some homeomorphism) we may form a compact metric space $M = M_1 \cup M_2$ containing each $M_i$ as a subspace, with $M_1 \cap M_2 = F = \vartheta_2$. Now form the closed subalgebra $A$ of $C(M)$ consisting of those $f$ with $f|_M_{A_i}$ in $A_i$, $i = 1, 2$. (Since $\vartheta_2 \subset \vartheta_1$, $A$ may also be viewed as a closed subalgebra of $A_1$.)

The consequences of (2.1) for $A$ are the following facts.

(2.2) $M_A = M$
(2.3) $\vartheta_A = \vartheta_1$
(2.4) $kM_2 = \{f \in A : f(M_2) = 0\}$ separates the points of $M \setminus M_2$.

In particular (2.4) implies there are many functions in $A$ vanishing on the (possibly void) open subset $M \setminus M_1 = M \setminus \vartheta_2$ of $M = M_A$.

Note that since $A_1|F = C(F)$, for any $f$ in $A_2$, $f|F = f|F$ has an extension to $M_1$ in $A_1$; consequently $f$ itself has an extension to $M$ in $A$. Thus

(2.5) $A|\mathcal{M}_2 = A_2$,

and $A$ separates the points of $\mathcal{M}_2$. On the other hand trivially

(2.6) $f$ in $A_1$ and $f(F) = f(\vartheta_2) = 0$ imply $f$ has an extension ($= 0$ on $M_2$) in $A$.

Now the $f$ in $A_1$ satisfying the hypothesis of (2.6) form an ideal $kF$ of $A_1$, and of course the quotient algebra $A_1/kF$ has the hull of $kF$ as its maximal ideal space. But $A_1/kF$ is naturally isomorphic to $A_1|F = C(F)$, so that $F$ is the maximal ideal space, hence the hull of $kF$. So (as is well known and easily proved) the Banach algebra $kF$ has

(2.7) $\vartheta_{kF} = \vartheta_1|F = \vartheta_1|\vartheta_2$, $M_{kF} = M_1\setminus F$.

Hence from the trivial relation (2.6), $kM_2 = \{f \in A : f(M_2) = 0\}$ separates the points of $M_1 \setminus F = M \setminus M_2$, yielding (2.4), and separates any element of $M \setminus M_2$ from one of $M_2$. Since $A$ separates the points of $M_2$ by (2.5), $A$ separates $\mathcal{M}$ and $\mathcal{M}$ is a subspace of $M_A$. Moreover by (2.6) $kF$ and $kM_2$ are isomorphic, whence $\vartheta_{kM_2} = \vartheta_1|\vartheta_2$, so that

(2.8) $\vartheta_1|\vartheta_2 \subset \vartheta_A$.

The remainder of (2.2) now follows by a standard argument: if a multiplicative linear functional $\varphi$ on $A$ vanishes on $kM_2$, hence corresponds to an element of $M_{A_1/kM_2}$, then the isomorphism of $A/kM_2$ and $A|\mathcal{M}_2 = A_2$ shows $\varphi$ arises from a point in $\mathcal{M}_2 \subset \mathcal{M}$. But if $\varphi$ does not vanish on $kM_2$ it provides a nonzero functional on this algebra,
hence on $kF$, and (since $\mathcal{M}_{kF} = \mathcal{M}_i \setminus F$) we have some $x$ in $\mathcal{M}_i$ for which $\varphi(f) = f(x)$, $f$ in $k\mathcal{M}_i$. Choosing $f$ in $k\mathcal{M}_i$ with $f(x) = \varphi(f) = 1$, we have $fg$ in $k\mathcal{M}_i$ for any $g$ in $A$, so $\varphi(g) = \varphi(fg) = fg(x) = g(x)$.

For (2.3), we already have $\partial_A \subset \partial_i$ (since $f \in A$ assumes its maximum modulus on $\partial_i$ by the definition of $A$) and $\partial_i \setminus \partial_A \subset \partial_A$ by (2.8). Consequently (2.3) follows immediately if $F = \partial_2$ is nowhere dense in $\partial_2$ and $\partial_2 \setminus \partial_1$ (as in the case of our examples to follow) since $\partial_i = (\partial_i \setminus \partial_0)^{-1} \subset \partial_A$.

For the general case we need only show $x$ in $\partial_2$ lies in $\partial_A$, and for this part of the argument we shall restrict our attention to $\partial_1$ and regard $A$ and $A_1$ as subalgebras of $C(\partial_1)$, $A_2$ as one of $C(\partial_2)$. By (2.12) (with $X = \partial_i$, $F = \partial_2$ and $A_1$ our algebra) we have an element $f$ of $A_1$ peaking on $F$, so $f(F) = 1$, $|f| < 1$ on $\partial_i \setminus F$; and of course $f \in A$. For our $x$ in $\partial_2$ and any open neighborhood $U$ of $x$ in $\partial_1$ we know there is a $g_2$ in $A_2$ assuming its maximum modulus over $\partial_2 - 1$ say—only within $\partial_2 \setminus U$, and by (2.5) $g_2$ has an extension $g$ in $A$. Moreover for some $\varepsilon > 0$, $|g_2| < 1 - \varepsilon$ on $\partial_2 \setminus U$, so $|g| < 1 - \varepsilon$ on some open subset $V$ of $\partial_i$ containing $\partial_2 \setminus U$. Since $\partial_2$ is contained in the open subset $U \cup V$ of $\partial_i$, sup $|f(\partial_2 \setminus (U \cup V))| < 1$, so $|f^*g| < 1 - \varepsilon$ on $\partial_2 \setminus (U \cup V)$ for some $n$, while $|f^*g| \leq |g| < 1 - \varepsilon$ on $V$. Thus $|f^*g| < 1 - \varepsilon$ on $\partial_2 \setminus U$; since $f^*g = g$ on $\partial_2$ the element $f^*g$ of $A$ assumes its maximum modulus 1 only within $U$, whence $x \in \partial_A$ and $\partial_2 \subset \partial_A$ as desired.

2.2 Remark. (2.2)-(2.4) apply to a more general construction; for with $F \subset \partial_1$ having $\mu_F = 0$ for all $\mu$ in $M(\partial_2)$ orthogonal to $A_1$ as before, and $\rho$ any (not one-to-one) continuous map of $F$ onto $\partial_2$ we can set

$$A = \{f \in A_1 : f|F \in A_2 \circ \rho\}$$

and again arrive at the same conclusions. Here, of course, in forming $\mathcal{M}$ there is some identification of points in $F$, while $\partial_A$ is $\partial_1$ with just such identifications. (An appropriate modification of (4.1) below can also be obtained in this setting.)

3. We can now write down our example. Let $A_1$ be the disc algebra of all functions continuous in the disc $D = \{z : |z| \leq 1\}$ and analytic on $|z| < 1$. Let $A_2$ be Rudin's algebra [10] of all functions continuous on the Riemann sphere $S$ and analytic off a compact perfect 0-dimensional subset $E$ of the plane with $E \cap U$ void or of positive plane measure for each open $U$. Then $E = \partial_2$ and $\mathcal{M}_2 = S$ [2].

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3 This follows from the argument of [10, p. 826]. For if $U$ is open in $S$ and $E \cap U \equiv \phi$ is open and closed in $E$ then—with $E \cap U$ in place of $E$—[10] shows there are non-constant $f$ in $C(S)$ analytic off $E \cap U$, hence elements of $A$ assuming their maximum modulus only within $E \cap U$. 
Now pick a Cantor set $F$ of measure 0 on the unit circle $T^1 = \partial_1$ so $\mu_F = 0$ for each $\mu$ in $M(T^1)$ orthogonal to $A_1$ by the F. and M. Riesz theorem [8]. $E = \partial_1$ and $F$ are homeomorphic so we may identify these sets as before, in effect tacking $S$ onto $D$ along $F$. Our algebra $A$ on the resulting space $\mathfrak{M} = D \cup S$ consists of all functions continuous on an open subset of $\partial_A = \partial_1 = T^1$ must vanish on $\mathfrak{M}$ and analytic off $T^1$.

Now $S \setminus E = \mathfrak{M}_1 \setminus F$ is a non-void open subset of $\mathfrak{M}_d = \mathfrak{M}$ on which nonzero elements of $A$ do vanish by (2.4); but an $f$ in $A$ which vanishes on all of $T^1$, being analytic on the interior of $D$, whence $f \equiv 0$.

4. We conclude with a modification of our example in which our nonanalytic integral domain is also a dirichlet algebra on its Šilov boundary [8]. In order to see the example is dirichlet, we require the following additional information, which holds in the context of § 2.

Let $A, A_1, A_2$ again be as in § 2. Let $A_i^\perp$ denote the measures on $\partial_i$ orthogonal to $A_i$, and $A^\perp$ those on $\partial_A = \partial_1$ orthogonal to $A$. (Since $\partial_2 \subset \partial_1$, we shall view $A_2^\perp$ as consisting of measures on $\partial_1$.)

Then

$$A^\perp = A_1^\perp + A_2^\perp.$$  

(4.1)

(4.1) is a consequence of an argument of Browder and Wermer [4]. To obtain it, consider the weak* closed subspaces $A^\perp, A_i^\perp$ of the dual $M(\partial_i)$ of $C(\partial_i)$. Clearly $A_1^\perp \subset A^\perp$, so $A_1^\perp + A_2^\perp \subset A^\perp$. On the other hand any $f$ in $C(\partial_i)$ orthogonal to $A_1^\perp + A_2^\perp$ has $f \mid \partial_i$ in $A_i \mid \partial_i$, so $f \mid \partial_i$ has an extension $g_i$ in $A_i, i = 1, 2$; and evidently $g_1$ and $g_2$ combine to yield an extension $g$ of $f, g \in A$. So $f \in A \mid \partial_1$, which shows $A_1^\perp + A_2^\perp$ is weak* dense in $A^\perp$.

So it suffices to prove $A_1^\perp + A_2^\perp$ is weak* closed in $M(\partial_1)$. But by hypothesis $\mu_{\partial_2} = 0$ for all $\mu$ in $A_1^\perp$, so $\mu$ in $A_1^\perp$ and $\nu$ in $A_2^\perp$ are mutually singular, and $\| \mu + \nu \| = \| \mu \| + \| \nu \|$. Consequently the argument of Browder and Wermer [4] applies to complete the proof of (4.1).

Now let $Z^2$ be the lattice points in the plane, $\alpha$ an irrational real number, and $H$ the half-space of $Z^2$ of all $(m, n)$ with $m\alpha + n \geq 0$.

Let $A_i$ be the closed algebra of continuous functions on the torus $T^2$ spanned by the characters of $T^2$ corresponding to the elements of the semigroup $H$; alternatively $A_i$ consists of those $f$ in $C(T^2)$ with Fourier coefficients vanishing off $H$. A description of $\mathfrak{M}_2$ can be found in [1]; but here we only need the fact that $\partial_1 = T^2$ [1], and that $A_i$ is a dirichlet algebra on $T^2$.

Let $F$ be the subset $T^1 \times \{1\}$ of $T^2$. Then from an extension of the F. and M. Riesz theorem obtained recently by K. de Leeuw and the
author [6] we have\((i)\) \(\mu F = 0\) for all \(\mu\) in \(M(T^2)\) orthogonal to \(A_i\) [6, Th. 3.1], while \((ii)\) any \(f\) in \(A_i\) which vanishes on an open subset of \(T^2\) vanishes identically [6, Th. 4.1]. From \((i)\) we can apply our construction, identifying \(F\) with the boundary of the disc \(D\), taking \(A_2\) as the disc algebra. The resulting algebra \(A\) again contains nonzero elements vanishing on an open subset of \(\mathcal{M}_A\)—the interior of \(D\)— and again is analytic on \(\partial_A = T^2\) by \((ii)\).

And \(A\) is dirichlet on \(T^2\) by \((4.1)\): for if \(\lambda\) is any real measure in \(M(T^2)\) orthogonal to \(A\), so that \(\lambda = \mu_i + \mu_i\), \(\mu_i\) in \(A_i^+\), then \(\mu_i = \lambda_F\), \(\mu_i = \lambda_F\), by \((i)\). Consequently \(\mu_i\) is a real measure on \(\partial_i\) orthogonal to \(A_i\), hence zero since \(A_i\) is dirichlet on \(\partial_i\).

Finally, note that \(A\) has a simple description as a subalgebra of \(C(T^2)\): viewing \(T^1\) as the reals mod \(2\pi\), \(A\) consists of all \(f\) with

\[
\int_0^{2\pi} \int_0^{2\pi} f(\theta, \varphi)e^{-i(m\theta + n\varphi)}d\theta d\varphi = 0, \quad m\alpha + n < 0, \\
\int_0^{2\pi} f(0, \varphi)e^{-in\varphi}d\varphi = 0, \quad n < 0.
\]

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\textsuperscript{4} Here the map \(\psi\) of [6] taking \(Z^2\) into \(R\) is \((m, n) \to m\alpha + n\).
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