

Pacific Journal of Mathematics

A REMARK ON ANALYTICITY OF FUNCTION ALGEBRAS

IRVING LEONARD GLICKSBERG

A REMARK ON ANALYTICITY OF FUNCTION ALGEBRAS

I. GLICKSBERG

1. Let A be a closed separating subalgebra of $C(X)$, X compact, with maximal ideal space \mathfrak{M}_A and Šilov boundary ∂_A . Naturally A can also be viewed as a closed subalgebra of $C(\mathfrak{M}_A)$ or $C(\partial_A)$.

Call A *analytic on X* if the vanishing of $f \in A$ on a non-void open subset of X implies $f \equiv 0$, or simply *analytic* if this holds for $X = \mathfrak{M}_A$. Recently Kenneth Hoffman asked if the analyticity of A on ∂_A implied analyticity on \mathfrak{M}_A ; the present note is devoted to a counterexample.¹ Evidently such an example, analytic on its Šilov boundary, must be an integral domain, so our algebra is a non-analytic integral domain.

The example was suggested by, and utilizes, an interpolation theorem of Rudin and Carleson [5, 9], recently generalized by Bishop [3], which in fact permits the construction of a variety of unfamiliar tractable subalgebras of familiar algebras; consequently we shall discuss the construction in more generality than is absolutely necessary. Finally we give a slightly more complicated example which is also dirichlet.

NOTATION. $M(X)$ will denote the space of (finite complex regular Borel) measures μ on X ; for such a μ , μ is orthogonal to A ($\mu \perp A$) if $\mu(f) = \int f d\mu = 0$, f in A . And μ_F will denote the usual restriction of μ to $F \subset X$, while $f|F$ will be the restriction of a function f , $A|F$ the set $\{f|F : f \in A\}$. An algebra A will always be assumed to contain the constants.

2. Our construction is based on the following fact.

(2.1) Suppose F is a closed subset of X , and $\mu_F = 0$ for all μ in $M(X)$ orthogonal to A . Then²

(2.1.1) $A|F = C(F)$ [3]

(2.1.2) if X is metric, F is a peak set of A , i.e., there is an f in

Received January 7, 1963. Supported in part by the National Science Foundation through Grant G22052 and in part by the Air Force Office of Scientific Research.

¹ After this note was completed, I found that analyticity of A on \mathfrak{M}_A implies analyticity on ∂_A ; this will appear in a subsequent paper.

² (2.11) is Bishop's generalization of the Rudin-Carleson result mentioned before, which applies to the special case in which A is the "disc algebra" and F a subset of measure zero of the unit circle. (2.12) will actually be avoided in the specific examples we construct.

A with $f(F) = 1$ and $|f| < 1$ on $X \setminus F$ [7, 4.8].

Now suppose we are given two uniformly closed algebras A_1, A_2 , as subalgebras of $C(\mathfrak{M}_1), C(\mathfrak{M}_2)$, where $\mathfrak{M}_i = \mathfrak{M}_{A_i}$ is metric, $i = 1, 2$. Further suppose $\partial_2 = \partial_{A_2}$ is homeomorphic to a (compact) subset F of ∂_1 , satisfying the hypothesis of (2.1) with $A = A_1, X = \partial_1$, so that $A_1|F = C(F)$. Identifying F and ∂_2 (via some homeomorphism) we may form a compact metric space $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ containing each \mathfrak{M}_i as a subspace, with $\mathfrak{M}_1 \cap \mathfrak{M}_2 = F = \partial_2$. Now form the closed subalgebra A of $C(\mathfrak{M})$ consisting of those f with $f|_{\mathfrak{M}_i}$ in $A_i, i = 1, 2$. (Since $\partial_2 \subset \partial_1, A$ may also be viewed as a closed subalgebra of A_1 .)

The consequences of (2.1) for A are the following facts.

(2.2)
$$\mathfrak{M}_A = \mathfrak{M}$$

(2.3)
$$\partial_A = \partial_1$$

(2.4)
$$k\mathfrak{M}_2 = \{f \in A : f(\mathfrak{M}_2) = 0\} \text{ separates the points of } \mathfrak{M} \setminus \mathfrak{M}_2.$$

In particular (2.4) implies there are many functions in A vanishing on the (possibly void) open subset $\mathfrak{M} \setminus \mathfrak{M}_1 = \mathfrak{M}_2 \setminus \partial_2$ of $\mathfrak{M} = \mathfrak{M}_A$.

Note that since $A_1|F = C(F)$, for any f in $A_2, f|_{\partial_2} = f|_F$ has an extension to \mathfrak{M}_1 in A_1 ; consequently f itself has an extension to \mathfrak{M} in A . Thus

(2.5)
$$A|_{\mathfrak{M}_2} = A_2,$$

and A separates the points of \mathfrak{M}_2 . On the other hand trivially

(2.6)
$$f \text{ in } A_1 \text{ and } f(F) = f(\partial_2) = 0 \text{ imply } f \text{ has an extension } (\equiv 0 \text{ on } \mathfrak{M}_2) \text{ in } A.$$

Now the f in A_1 satisfying the hypothesis of (2.6) form an ideal kF of A_1 , and of course the quotient algebra A_1/kF has the hull of kF as its maximal ideal space. But A_1/kF is naturally isomorphic to $A_1|F = C(F)$, so that F is the maximal ideal space, hence the hull of kF . So (as is well known and easily proved) the Banach algebra kF has

(2.7)
$$\partial_{kF} = \partial_1 \setminus F = \partial_1 \setminus \partial_2, \quad \mathfrak{M}_{kF} = \mathfrak{M}_1 \setminus F.$$

Hence from the trivial relation (2.6), $k\mathfrak{M}_2 = \{f \in A : f(\mathfrak{M}_2) = 0\}$ separates the points of $\mathfrak{M}_1 \setminus F = \mathfrak{M} \setminus \mathfrak{M}_2$, yielding (2.4), and separates any element of $\mathfrak{M} \setminus \mathfrak{M}_2$ from one of \mathfrak{M}_2 . Since A separates the points of \mathfrak{M}_2 by (2.5), A separates \mathfrak{M} , and \mathfrak{M} is a subspace of \mathfrak{M}_A . Moreover by (2.6) kF and $k\mathfrak{M}_2$ are isomorphic, whence $\partial_{k\mathfrak{M}_2} = \partial_1 \setminus \partial_2$, so that

(2.8)
$$\partial_1 \setminus \partial_2 \subset \partial_A.$$

The remainder of (2.2) now follows by a standard argument: if a multiplicative linear functional φ on A vanishes on $k\mathfrak{M}_2$, hence corresponds to an element of $\mathfrak{M}_{A/k\mathfrak{M}_2}$, then the isomorphism of $A/k\mathfrak{M}_2$ and $A|_{\mathfrak{M}_2} = A_2$ shows φ arises from a point in $\mathfrak{M}_2 \subset \mathfrak{M}$. But if φ does not vanish on $k\mathfrak{M}_2$ it provides a nonzero functional on this algebra,

hence on kF , and (since $\mathfrak{M}_{kF} = \mathfrak{M}_1 \setminus F$) we have some x in \mathfrak{M}_1 for which $\varphi(f) = f(x)$, f in $k\mathfrak{M}_2$. Choosing f in $k\mathfrak{M}_2$ with $f(x) = \varphi(f) = 1$, we have fg in $k\mathfrak{M}_2$ for any g in A , so $\varphi(g) = \varphi(fg) = fg(x) = g(x)$.

For (2.3), we already have $\partial_A \subset \partial_1$ (since $f \in A$ assumes its maximum modulus on ∂_1 by the definition of A) and $\partial_1 \setminus \partial_2 \subset \partial_A$ by (2.8). Consequently (2.3) follows immediately if $F = \partial_2$ is nowhere dense in ∂_1 (as in the case of our examples to follow) since $\partial_1 = (\partial_1 \setminus \partial_2)^- \subset \partial_A$.

For the general case we need only show x in ∂_2 lies in ∂_A , and for this part of the argument we shall restrict our attention to ∂_1 and regard A and A_1 as subalgebras of $C(\partial_1)$, A_2 as one of $C(\partial_2)$. By (2.12) (with $X = \partial_1$, $F = \partial_2$ and A_1 our algebra) we have an element f of A_1 peaking on F , so $f(F) = 1$, $|f| < 1$ on $\partial_1 \setminus F$; and of course $f \in A$. For our x in ∂_2 and any open neighborhood U of x in ∂_1 we know there is a g_2 in A_2 assuming its maximum modulus over $\partial_2 - 1$ say—only within $\partial_2 \cap U$, and by (2.5) g_2 has an extension g in A . Moreover for some $\varepsilon > 0$, $|g_2| < 1 - \varepsilon$ on $\partial_2 \setminus U$, so $|g| < 1 - \varepsilon$ on some open subset V of ∂_1 containing $\partial_2 \setminus U$. Since ∂_2 is contained in the open subset $U \cup V$ of ∂_1 , $\sup |f(\partial_1 \setminus (U \cup V))| < 1$, so $|f^n g| < 1 - \varepsilon$ on $\partial_1 \setminus (U \cup V)$ for some n , while $|f^n g| \leq |g| < 1 - \varepsilon$ on V . Thus $|f^n g| < 1 - \varepsilon$ on $\partial_1 \setminus U$; since $f^n g = g$ on ∂_2 the element $f^n g$ of A assumes its maximum modulus 1 only within U , whence $x \in \partial_A$ and $\partial_2 \subset \partial_A$ as desired.

2.2 REMARK. (2.2)–(2.4) apply to a more general construction; for with $F \subset \partial_1$ having $\mu_F = 0$ for all μ in $M(\partial_1)$ orthogonal to A_1 as before, and ρ any (not one-to-one) continuous map of F onto ∂_2 we can set

$$A = \{f \in A_1 : f|_F \in A_2 \circ \rho\}$$

and again arrive at the same conclusions. Here, of course, in forming \mathfrak{M} there is some identification of points in F , while ∂_A is ∂_1 with just such identifications. (An appropriate modification of (4.1) below can also be obtained in this setting.)

3. We can now write down our example. Let A_1 be the disc algebra of all functions continuous in the disc $D = \{z : |z| \leq 1\}$ and analytic on $|z| < 1$. Let A_2 be Rudin's algebra [10] of all functions continuous on the Riemann sphere S and analytic off a compact perfect 0-dimensional subset E of the plane with $E \cap U$ void or of positive plane measure for each open U . Then³ $E = \partial_2$ and $\mathfrak{M}_2 = S$ [2].

³ This follows from the argument of [10, p. 826]. For if U is open in S and $E \cap U \neq \emptyset$ is open and closed in E then—with $E \cap U$ in place of E —[10] shows there are non-constant f in $C(S)$ analytic off $E \cap U$, hence elements of A assuming their maximum modulus only within $E \cap U$.

Now pick a Cantor set F of measure 0 on the unit circle $T^1 = \partial_1$ so $\mu_F = 0$ for each μ in $M(T^1)$ orthogonal to A_1 by the F. and M. Riesz theorem [8]. $E = \partial_2$ and F are homeomorphic so we may identify these sets as before, in effect tacking S onto D along F . Our algebra A on the resulting space $\mathfrak{M} = D \cup S$ consists of all functions continuous on an open subset of $\partial_A = \partial_1 = T^1$ must vanish on \mathfrak{M} and analytic off T^1 .

Now $S \setminus E = \mathfrak{M}_2 \setminus F$ is a non-void open subset of $\mathfrak{M}_A = \mathfrak{M}$ on which nonzero elements of A do vanish by (2.4); but an f in A which vanishes on all of T^1 , being analytic on the interior of D , whence $f \equiv 0$.

4. We conclude with a modification of our example in which our nonanalytic integral domain is also a dirichlet algebra on its Šilov boundary [8]. In order to see the example is dirichlet, we require the following additional information, which holds in the context of § 2.

Let A, A_1, A_2 again be as in § 2. Let A_i^\perp denote the measures on ∂_i orthogonal to A_i , and A^\perp those on $\partial_A = \partial_1$ orthogonal to A . (Since $\partial_2 \subset \partial_1$, we shall view A_2^\perp as consisting of measures on ∂_1 .) Then

$$(4.1) \quad A^\perp = A_1^\perp + A_2^\perp .$$

(4.1) is a consequence of an argument of Browder and Wermer [4]. To obtain it, consider the weak* closed subspaces A^\perp, A_i^\perp of the dual $M(\partial_1)$ of $C(\partial_1)$. Clearly $A_i^\perp \subset A^\perp$, so $A_1^\perp + A_2^\perp \subset A^\perp$. On the other hand any f in $C(\partial_1)$ orthogonal to $A_1^\perp + A_2^\perp$ has $f|_{\partial_i}$ in $A_i |_{\partial_i}$, so $f|_{\partial_i}$ has an extension g_i in $A_i, i = 1, 2$; and evidently g_1 and g_2 combine to yield an extension g of $f, g \in A$. So $f \in A |_{\partial_1}$, which shows $A_1^\perp + A_2^\perp$ is weak* dense in A^\perp .

So it suffices to prove $A_1^\perp + A_2^\perp$ is weak* closed in $M(\partial_1)$. But by hypothesis $\mu_{\partial_2} = 0$ for all μ in A_1^\perp , so μ in A_1^\perp and ν in A_2^\perp are mutually singular, and $\|\mu + \nu\| = \|\mu\| + \|\nu\|$. Consequently the argument of Browder and Wermer [4] applies to complete the proof of (4.1).

Now let Z^2 be the lattice points in the plane, α an irrational real number, and H the half-space of Z^2 of all (m, n) with

$$m\alpha + n \geq 0 .$$

Let A_1 be the closed algebra of continuous functions on the torus T^2 spanned by the characters of T^2 corresponding to the elements of the semigroup H ; alternatively A_1 consists of those f in $C(T^2)$ with Fourier coefficients vanishing off H . A description of \mathfrak{M}_1 can be found in [1]; but here we only need the fact that $\partial_1 = T^2$ [1], and that A_1 is a dirichlet algebra on T^2 .

Let F be the subset $T^1 \times \{1\}$ of T^2 . Then from an extension of the F. and M. Riesz theorem obtained recently by K. de Leeuw and the

author [6] we have⁴ (i) $\mu_F = 0$ for all μ in $M(T^2)$ orthogonal to A_1 [6, Th. 3.1], while (ii) any f in A_1 which vanishes on an open subset of T^2 vanishes identically [6, Th. 4.1]. From (i) we can apply our construction, identifying F with the boundary of the disc D , taking A_2 as the disc algebra. The resulting algebra A again contains nonzero elements vanishing on an open subset of \mathfrak{M}_A —the interior of D —and again is analytic on $\partial_A = T^2$ by (ii).

And A is dirichlet on T^2 by (4.1): for if λ is any real measure in $M(T^2)$ orthogonal to A , so that $\lambda = \mu_1 + \mu_2$, μ_i in A_i^\perp , then $\mu_2 = \lambda_F$, $\mu_1 = \lambda_{F'}$, by (i). Consequently μ_i is a real measure on ∂_i orthogonal to A_i , hence zero since A_i is dirichlet on ∂_i .

Finally, note that A has a simple description as a subalgebra of $C(T^2)$: viewing T^1 as the reals mod 2π , A consists of all f with

$$\int_0^{2\pi} \int_0^{2\pi} f(\theta, \varphi) e^{-i(m\theta + n\varphi)} d\theta d\varphi = 0, \quad m\alpha + n < 0,$$

$$\int_0^{2\pi} f(0, \varphi) e^{-in\varphi} d\varphi = 0, \quad n < 0.$$

REFERENCES

1. R. Arens and I. M. Singer, *Generalized analytic functions*, Trans. Amer. Math. Soc., **81** (1956), 379-393.
2. R. Arens, *The maximal ideals of certain function algebras*, Pacific J. Math., **8** (1958), 641-648.
3. E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc., **13** (1962), **13** (1962), 140-143.
4. A. Browder and J. Wermer, *Some algebras of functions on an arc*, J. Math. Mech., **12** (1963), 119-130.
5. L. Carleson, *Representations of continuous functions*, Math. Z. **66** (1957), 447-451.
6. K. de Leeuw and I. Glicksberg, *Analytic measures on compact groups, Quasi-invariance and analyticity of measures on compact groups*, Acta Math., **109** (1963), 179-205.
7. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc., **105** (1962), 415-435.
8. K. Hoffman, *Banach Spaces of Analytic Functions*, Englewood Cliffs, N.J., 1962.
9. W. Rudin, *Boundary values of continuous analytic functions*, Proc. Amer. Math. Soc., **7** (1956), 808-811.
10. ———, *Subalgebras of spaces of continuous functions*, Proc. Amer. Math. Soc., **7** (1956), 825-830.

UNIVERSITY OF WASHINGTON

⁴ Here the map ψ of [6] taking Z^2 into R is $(m, n) \rightarrow m\alpha + n$.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Dallas O. Banks, <i>Bounds for eigenvalues and generalized convexity</i>	1031
Jerrold William Bebernes, <i>A subfunction approach to a boundary value problem for ordinary differential equations</i>	1053
Woodrow Wilson Bledsoe and A. P. Morse, <i>A topological measure construction</i>	1067
George Clements, <i>Entropies of several sets of real valued functions</i>	1085
Sandra Barkdull Cleveland, <i>Homomorphisms of non-commutative *-algebras</i>	1097
William John Andrew Culmer and William Ashton Harris, <i>Convergent solutions of ordinary linear homogeneous difference equations</i>	1111
Ralph DeMarr, <i>Common fixed points for commuting contraction mappings</i>	1139
James Robert Dorroh, <i>Integral equations in normed abelian groups</i>	1143
Adriano Mario Garsia, <i>Entropy and singularity of infinite convolutions</i>	1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., <i>Convergence of extended Bernstein polynomials in the complex plane</i>	1171
Irving Leonard Glicksberg, <i>A remark on analyticity of function algebras</i>	1181
Charles John August Halberg, Jr., <i>Semigroups of matrices defining linked operators with different spectra</i>	1187
Philip Hartman and Nelson Onuchic, <i>On the asymptotic integration of ordinary differential equations</i>	1193
Isidore Heller, <i>On a class of equivalent systems of linear inequalities</i>	1209
Joseph Hersch, <i>The method of interior parallels applied to polygonal or multiply connected membranes</i>	1229
Hans F. Weinberger, <i>An effectless cutting of a vibrating membrane</i>	1239
Melvin F. Janowitz, <i>Quantifiers and orthomodular lattices</i>	1241
Samuel Karlin and Albert Boris J. Novikoff, <i>Generalized convex inequalities</i>	1251
Tilla Weinstein, <i>Another conformal structure on immersed surfaces of negative curvature</i>	1281
Gregers Louis Krabbe, <i>Spectral permanence of scalar operators</i>	1289
Shige Toshi Kuroda, <i>Finite-dimensional perturbation and a representation of scattering operator</i>	1305
Marvin David Marcus and Afton Herbert Cayford, <i>Equality in certain inequalities</i>	1319
Joseph Martin, <i>A note on uncountably many disks</i>	1331
Eugene Kay McLachlan, <i>Extremal elements of the convex cone of semi-norms</i>	1335
John W. Moon, <i>An extension of Landau's theorem on tournaments</i>	1343
Louis Joel Mordell, <i>On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$</i>	1347
Kenneth Roy Mount, <i>Some remarks on Fitting's invariants</i>	1353
Miroslav Novotný, <i>Über Abbildungen von Mengen</i>	1359
Robert Dean Ryan, <i>Conjugate functions in Orlicz spaces</i>	1371
John Vincent Ryff, <i>On the representation of doubly stochastic operators</i>	1379
Donald Ray Sherbert, <i>Banach algebras of Lipschitz functions</i>	1387
James McLean Sloss, <i>Reflection of biharmonic functions across analytic boundary conditions with examples</i>	1401
L. Bruce Treybig, <i>Concerning homogeneity in totally ordered, connected topological space</i>	1417
John Wermer, <i>The space of real parts of a function algebra</i>	1423
James Juei-Chin Yeh, <i>Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables</i>	1427
William P. Ziemer, <i>On the compactness of integral classes</i>	1437