AN EFFECTLESS CUTTING OF A VIBRATING MEMBRANE

HANS F. WEINBERGER
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Let $G$ be a multiply connected domain bounded by an outer boundary $\Gamma_0$, inner boundaries $\Gamma_1, \Gamma_2, \cdots$, and possibly some other inner boundaries $\gamma_1, \gamma_2, \cdots$. Let $u$ be the eigenfunction corresponding to the lowest eigenvalue $\lambda_1$ of the membrane problem

\begin{equation}
\Delta u + \lambda_1 u = 0 \quad \text{in} \ G
\end{equation}

with

\begin{equation}
u = 0 \quad \text{on} \ \Gamma_0, \Gamma_1, \cdots
\end{equation}

\begin{equation}
\frac{\partial u}{\partial n} = 0 \quad \text{on} \ \gamma_1, \gamma_2, \cdots.
\end{equation}

We shall show that there exists a cut $\tilde{\gamma}$ consisting of a finite set of analytic arcs along which $\left(\frac{\partial u}{\partial n}\right) = 0$ which separates any given one of the fixed holes, say $\Gamma_1$, from the outer boundary $\Gamma_0$ and the other holes $\Gamma_2, \Gamma_3, \cdots$. This means that the membrane $G$ may be cut in two along $\tilde{\gamma}$ without lowering its lowest eigenvalue. This fact is used in the preceding paper of J. Hersch to establish an upper bound for $\lambda_1$.

We assume that $\Gamma_0, \Gamma_1, \cdots$ have continuous normals and that $\gamma_1, \gamma_2, \cdots$ are analytic. Then it is well-known that $u$ has the following properties:

\begin{equation}
\begin{align*}
(3) \quad & (a) \quad u > 0 \text{ in } G, \quad \text{and} \quad \frac{\partial u}{\partial n} < 0 \text{ on } \Gamma_0, \Gamma_1, \cdots. \\
& (b) \quad u \text{ is analytic in } G + \gamma_1 + \gamma_2 + \cdots. \\
& (c) \quad u_{xx} \text{ and } u_{yy} \text{ do not vanish simultaneously.}
\end{align*}
\end{equation}

(The last property follows from (3a) and (1)).

We define $G_1$ to be the set of points of $G$ from which the fall lines, i.e. the trajectories of

\begin{equation}
\begin{align*}
\frac{dx}{dt} &= -u_x \\
\frac{dy}{dt} &= -u_y
\end{align*}
\end{equation}

reach $\Gamma_1$. By property (3a) $G_1$ contains a neighborhood in $G$ of $\Gamma_1$, and its exterior contains neighborhoods in $G$ of $\Gamma_0, \Gamma_2, \cdots$. Since $u_=
and \( u_x \) are continuous, \( G_1 \) is open.

Let \( \tilde{\gamma} \) be the part of the boundary of \( G_1 \) that lies in \( G \). Let \( P \) be a point of \( \tilde{\gamma} \) where the gradient of \( u \) does not vanish. Then there is a trajectory \( \gamma \) satisfying (4) through \( P \). Let \( Q \) be any other point on \( \gamma \). Since \( P \) is not in \( G_1 \), it follows from the definition that \( Q \) is not in \( G_1 \). On the other hand, if a whole neighborhood of \( Q \) were not in \( G_1 \), it would follow from the continuity of the trajectories with respect to their initial points that a whole neighborhood of \( P \) would be outside \( G_1 \). This would contradict the fact that \( P \) is a boundary point of \( G_1 \).

Thus we have shown that the whole trajectory \( \gamma \) lies in \( \tilde{\gamma} \). It cannot go to \( \Gamma_1 \). Since the set of points from which trajectories go to \( \Gamma_0, \Gamma_2, \ldots \) is also open, \( \gamma \) cannot go to these boundary components.

We note that \( u \) is monotone on \( \gamma \), and

\[
(5) \quad \frac{d u}{d s} = |\nabla u|.
\]

Thus \( \gamma \) is either of finite length, or it must contain a sequence of points \( Q_1, Q_2, \ldots \) on which \( \nabla u \) approaches zero. These will have a limit point \( Q \) at which \( \nabla u = 0 \). (It may be that \( Q \) lies on one of the \( \gamma_i \). In this case we think of \( u \) extended across \( \gamma_i \) as an analytic function by reflection).

There is a neighborhood of \( Q \) in which the trajectories can be determined by examining the first few terms of the power series for \( u \). Using property (3c), we find that \( \gamma \) is of finite length. This is, of course, true in both the \( t \) and \( -t \) directions.

The free boundary curves \( \gamma_i \) are composed of trajectories of (4) and critical points, i.e., points where \( \nabla u = 0 \). Hence it follows from the uniqueness of the initial value problem for (4) that if \( \gamma \) ends on \( \gamma_i \), the end point must again be a critical point. Thus, each trajectory \( \gamma \) in \( \tilde{\gamma} \) connects two critical points.

It follows from properties (3b) and (3c) and the implicit function theorem that a critical point \( Q \) is either an isolated critical point or lies on an analytic arc of critical points. These arcs are again isolated.

Thus we have shown that \( \tilde{\gamma} \) is composed of a finite number of analytic arcs of finite length along which \( (\partial u / \partial n) = 0 \), and a finite number of critical points. We delete any isolated points of \( \tilde{\gamma} \).

The fact that \( \tilde{\gamma} \) separates \( \Gamma_1 \) from \( \Gamma_0, \Gamma_2, \ldots \) is clear from the definition of \( G_1 \).

The above considerations apply to any function with properties (3).

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