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AN EFFECTLESS CUTTING OF A VIBRATING MEMBRANE

HANS F. WEINBERGER

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Let G be a multiply connected domain bounded by an outer boundary Γ_0 , inner boundaries $\Gamma_1, \Gamma_2, \dots$, and possibly some other inner boundaries $\gamma_1, \gamma_2, \dots$. Let u be the eigenfunction corresponding to the lowest eigenvalue λ_1 of the membrane problem

$$(1) \quad \Delta u + \lambda_1 u = 0 \quad \text{in } G$$

with

$$(2) \quad \begin{aligned} u &= 0 \quad \text{on } \Gamma_0, \Gamma_1, \dots \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \gamma_1, \gamma_2, \dots \end{aligned}$$

We shall show that there exists a cut $\tilde{\gamma}$ consisting of a finite set of analytic arcs along which $(\partial u / \partial n) = 0$ which separates any given one of the fixed holes, say Γ_1 , from the outer boundary Γ_0 and the other holes $\Gamma_2, \Gamma_3, \dots$. This means that the membrane G may be cut in two along $\tilde{\gamma}$ without lowering its lowest eigenvalue. This fact is used in the preceding paper of J. Hersch to establish an upper bound for λ_1 .

We assume that $\Gamma_0, \Gamma_1, \dots$ have continuous normals and that $\gamma_1, \gamma_2, \dots$ are analytic. Then it is well-known that u has the following properties:

- $$(3) \quad \begin{aligned} (a) \quad &u > 0 \text{ in } G, \quad \text{and} \quad \frac{\partial u}{\partial n} < 0 \text{ on } \Gamma_0, \Gamma_1, \dots \\ (b) \quad &u \text{ is analytic in } G + \gamma_1 + \gamma_2 + \dots \\ (c) \quad &u_{xx} \text{ and } u_{yy} \text{ do not vanish simultaneously.} \end{aligned}$$

(The last property follows from (3a) and (1)).

We define G_1 to be the set of points of G from which the fall lines, i.e. the trajectories of

$$(4) \quad \begin{aligned} \frac{dx}{dt} &= -u_x \\ \frac{dy}{dt} &= -u_y \end{aligned}$$

reach Γ_1 . By property (3a) G_1 contains a neighborhood in G of Γ_1 , and its exterior contains neighborhoods in G of $\Gamma_0, \Gamma_2, \dots$. Since u_x

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and u_y are continuous, G_1 is open.

Let $\tilde{\gamma}$ be the part of the boundary of G_1 that lies in G . Let P be a point of $\tilde{\gamma}$ where the gradient of u does not vanish. Then there is a trajectory γ satisfying (4) through P . Let Q be any other point on γ . Since P is not in G_1 , it follows from the definition that Q is not in G_1 . On the other hand, if a whole neighborhood of Q were not in G_1 , it would follow from the continuity of the trajectories with respect to their initial points that a whole neighborhood of P would be outside G_1 . This would contradict the fact that P is a boundary point of G_1 .

Thus we have shown that the whole trajectory γ lies in $\tilde{\gamma}$. It cannot go to Γ_1 . Since the set of points from which trajectories go to $\Gamma_0, \Gamma_2, \dots$ is also open, γ cannot go to these boundary components.

We note that u is monotone on γ , and

$$(5) \quad \left| \frac{du}{ds} \right| = |\text{grad } u|.$$

Thus γ is either of finite length, or it must contain a sequence of points Q_1, Q_2, \dots on which $\text{grad } u$ approaches zero. These will have a limit point Q at which $\text{grad } u = 0$. (It may be that Q lies on one of the γ_i . In this case we think of u extended across γ_i as an analytic function by reflection).

There is a neighborhood of Q in which the trajectories can be determined by examining the first few terms of the power series for u . Using property (3c), we find that γ is of finite length. This is, of course, true in both the t and $-t$ directions.

The free boundary curves γ_i are composed of trajectories of (4) and critical points, i.e., points where $\text{grad } u = 0$. Hence it follows from the uniqueness of the initial value problem for (4) that if γ ends on γ_i , the end point must again be a critical point. Thus, each trajectory γ in $\tilde{\gamma}$ connects two critical points.

It follows from properties (3b) and (3c) and the implicit function theorem that a critical point Q is either an isolated critical point or lies on an analytic arc of critical points. These arcs are again isolated.

Thus we have shown that $\tilde{\gamma}$ is composed of a finite number of analytic arcs of finite length along which $(\partial u / \partial n) = 0$, and a finite number of critical points. We delete any isolated points of $\tilde{\gamma}$.

The fact that $\tilde{\gamma}$ separates Γ_1 from $\Gamma_0, \Gamma_2, \dots$ is clear from the definition of G_1 .

The above considerations apply to any function with properties (3).

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