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**ANOTHER CONFORMAL STRUCTURE ON IMMERSED
SURFACES OF NEGATIVE CURVATURE**

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ANOTHER CONFORMAL STRUCTURE ON IMMERSSED SURFACES OF NEGATIVE CURVATURE

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1. Everyone is familiar with the ordinary conformal structure on oriented surfaces immersed smoothly in E^3 . This standard structure is obtained by using the first fundamental form as metric tensor. It is possible, however, to define very different conformal structures which are still vitally connected with the geometry of a surface's immersion in E^3 .

Consider, for instance, the conformal structure induced upon a strictly convex surface (oriented so that mean curvature $H > 0$) by using its positive definite second fundamental form as metric tensor. (See [3] and [4].) This particular structure coincides with the usual one only on spheres.

The present paper is devoted, principally, to a description of the corresponding non-standard conformal structure on oriented surfaces of negative Gaussian curvature immersed smoothly in E^3 . This new structure is obtained by using a specific linear combination of the first and second fundamental forms as metric tensor. It will be seen that our new structure coincides with the usual one only on minimal surfaces.

Also included below is a section describing the arithmetic of certain expressions associated with the various fundamental forms on an immersed surface. These expressions become quadratic differentials whenever *any* particular conformal structure is introduced on a surface.

The paper closes with a theorem which generalizes a well known fact about minimal surfaces. For investigations related to the material which follows, see [5].

2. Consider an oriented surface S which is C^3 immersed in E^3 . We may number the principal curvatures so that

$$(1) \quad k_1 \geq k_2$$

holds over all of S . For convenience of notation, lines of curvature coordinates x, y will always be chosen so that k_1 is the principal curvature in the $y \equiv \text{constant}$ direction, while k_2 is the principal curvature in the $x \equiv \text{constant}$ direction.

We now define the function

$$H' = \frac{k_2 - k_1}{2} = -\sqrt{H^2 - K}.$$

Note that the zeros of H' coincide with the umbilics on S , that H' is never positive, and that the value of H' (unlike H) is independent of the orientation on S . If we set

$$(2) \quad H'II' = KI - HII,$$

we obtain a new "fundamental form" II' , defined at all non-umbilic points on S .¹ Of course, I is determined by II and II' wherever $K \neq 0$, and II is determined by I and II' wherever $H \neq 0$. Moreover, II' (unlike II) is independent of the orientation on S . The relationship of II to II' can be more clearly seen in the following lemma.

LEMMA 1. *If x, y are lines of curvature coordinates on S , so that*

$$(3) \quad \begin{aligned} I &= Edx^2 + Gdy^2, \\ II &= k_1Edx^2 + k_2Gdy^2, \end{aligned}$$

then, where II' is defined,

$$(4) \quad II' = k_1Edx^2 - k_2Gdy^2.$$

Proof. Suppose $II' = L'dx^2 + 2M'dxdy + M'dy^2$. Then (2) yields

$$H'L' = k_1k_2E - \left(\frac{k_1 + k_2}{2}\right)k_1E = H'k_1E.$$

If II' is defined at all, $H' \neq 0$, so that $L' = k_1E$. Similarly, $M' = 0$, and $N' = -k_2G$, as claimed.

COROLLARY. *II' is positive definite on S if and only if $K < 0$.*

Proof. Wherever II' is defined or $K < 0$, there are no umbilics to consider. Thus (4) applies, and, using (1), the result is obvious.

LEMMA 2. *Just as $H' = 0$ characterizes points where $I \propto II$, $H = 0$ characterizes points where $I \propto II'$, and $K = 0$ characterizes points where $II \propto II'$.*

Proof. The first fact merely recalls the definition of an umbilic point. The remaining facts follow easily from (2), recalling that $H' \neq 0$ wherever II' is defined.

REMARK. Elementary application of (2) reveals that lines of

¹ By continuity, II' could be sensibly defined at certain umbilics. We assume throughout this paper, however, that II' is defined only at non-umbilics.

curvature coordinates are characterized by $M' = F' = 0$ wherever $H \neq 0$, and by $M' = M = 0$ wherever $K \neq 0$.

3. Suppose now that $K < 0$ on the surface S discussed above, so that II' is a C^1 positive definite form on S . Then C^2 coordinates x, y may be found in the neighborhood of any point on S in terms of which

$$II' = \mu'(x, y)\{dx^2 + dy^2\},$$

with $\mu' > 0$. (See § 4 of [1].) Such coordinates will be called *disothermal*. It is well known that distinct pairs of disothermal coordinates are related by the Cauchy Riemann equations, and that coordinates so related to a pair of disothermals are themselves disothermal. Thus we obtain on S the structure of a Riemann surface R'_2 , with conformal parameters $z = x + iy$ corresponding to disothermals x, y .

Of course, there is still the usual structure of a Riemann surface R_1 on S , determined by using conformal parameters $z = x + iy$ corresponding to isothermal coordinates x, y in terms of which

$$I = \lambda(x, y)\{dx^2 + dy^2\}.$$

By Lemma 2, R_1 and R'_2 coincide on S if and only if S is a minimal surface. (We will also have occasion to mention below the Riemann surface R_2 determined on strictly convex surfaces by using conformal parameters $z = x + iy$ corresponding to bisothermal coordinates x, y in terms of which

$$II = \mu(x, y)\{dx^2 + dy^2\}$$

with $\mu > 0$.)

Assume now that x, y are disothermals on S . Then (2) becomes

$$(5) \quad \begin{aligned} HL + H'\mu' &= KE, \\ HN + H'\mu' &= KG, \\ HM &= KF. \end{aligned}$$

Thus, for instance, the equation for the directions of principal curvature reads

$$(6) \quad -Fdx^2 + (E - G) dx dy + Fdy^2 = 0.$$

This is, incidentally, exactly the form which the same equation takes when using bisothermal coordinates on a strictly convex surface. (See [3] or [4].) Note that (6) depends only on I .

REMARK. Recalling the remark which closes § 2, we see that all

conjugate disothermals are lines of curvature coordinates. And, where $H \neq 0$, all orthogonal disothermals are lines of curvature coordinates.

It will be helpful to note that since

$$(7) \quad \begin{aligned} K &= \frac{LN - M^2}{EG - F^2}, \\ H &= \frac{EN - 2FM + GL}{2(EG - F^2)}, \end{aligned}$$

asymptotic coordinates, which are characterized by $L = N = 0$, always yield the third equation of (5), so that $M' = 0$. Moreover, by (2), coordinates are asymptotic if and only if $H'L' = KE$, $H'N' = KG$. Thus the following can be said.

REMARK. Asymptotic coordinates are disothermal if and only if $E = G$.

The previous remark and the two results which follow characterize disothermal coordinates on S in terms of the coefficients of I and II . Lemma 3 is a trivial consequence of (2), (5), and the fact that $K < 0$ on S .

LEMMA 3. *Coordinates are disothermal on S if and only if*

$$\begin{aligned} H(L - N) &= K(E - G), \\ HM &= KF. \end{aligned}$$

LEMMA 4. *Nonasymptotic coordinates are disothermal on a C^3 immersed surface if and only if*

$$(8) \quad \begin{aligned} L &= -N \neq 0, \\ HM &= KF. \end{aligned}$$

Proof. Using (7), we obtain

$$(9) \quad 2H(LN - M^2) = K(EN - 2FM + GL).$$

If x, y are disothermals, (5) holds. Thus $HM = KF$, and we may multiply the equations of (5) by N, L and M respectively, and combine the resulting expressions so as to obtain the right side of (9). This yields

$$2H(LN - M^2) = 2H(LN - M^2) + H'\mu'(L + N).$$

Since $H'\mu' \neq 0$, assuming x, y to be nonasymptotic, we have $L = -N \neq 0$. Suppose, on the other hand, that (8) holds when using coordinates x, y . Then multiplying the equations of (2) by L, N and M respectively, and proceeding as above, we obtain

$$2H(LN - M^2) = -2H(L^2 + M^2) + LH'(N' - L').$$

Since $L = -N \neq 0$ while $H' \neq 0$, it follows that $L' = N'$. But $M' = 0$ by (8) and (2), so that x, y are isothermals.

Many formulas may be simplified, of course, by using isothermals and (8). We note here only that

$$L' = N' = \mu' = \frac{K(E + G)}{2H'},$$

while

$$L = -N = \frac{K(E - G)}{2H}.$$

4. In [2] and [6], use is made of the quadratic differential $\Omega_2 = \phi_2 dz^2$ on R_1 , where

$$\phi_2 = \frac{L - N}{2} - iM.$$

In [4], use is made of the quadratic differential $\Omega_1 = \phi_1 dz^2$ on R_2 (for a strictly convex surface), where

$$\phi_1 = \frac{E - G}{2} - iF.$$

We prove the following lemma in order to facilitate the definition of similar quadratic differentials on the various Riemann surfaces of interest here.

LEMMA 5. *Let $A = Adx^2 + 2Bdxdy + Cdy^2$ be a quadratic form on an oriented C^1 surface S . Suppose R is a Riemann surface defined on S . Then $\Omega = \phi dz^2$ with*

$$\phi = \frac{A - C}{2} - iB$$

is a quadratic differential on R .

Proof. Let $z = x + iy$ and $w = u + iv$ be conformal parameters on R . Then

$$A = \hat{A}du^2 + 2\hat{B}dudv + \hat{C}dv^2$$

where

$$\begin{aligned} \hat{A} &= Ax_u^2 + 2Bx_u y_u + Cy_u^2, \\ \hat{B} &= Ax_u x_v + B(x_u y_v + x_v y_u) + Cy_u y_v, \end{aligned}$$

$$\hat{C} = Ax_v^2 + 2Bx_vy_v + Cy_v^2.$$

Since

$$\begin{aligned}x_u &= y_v, \\x_v &= -y_u,\end{aligned}$$

while,

$$\frac{dz}{dw} = x_u - ix_v,$$

simple computation yields

$$\left\{ \frac{\hat{A} - \hat{C}}{2} - i\hat{B} \right\} = \left\{ \frac{A - C}{2} - iB \right\} \left(\frac{dz}{dw} \right)^2,$$

as required.

REMARK. The quadratic differential Ω on R associated with $A \neq 0$ on S is identically zero if and only if R is determined by choosing conformal parameters $z = x + iy$ on S corresponding to coordinates x, y in terms of which $A = \lambda(x, y)\{dx^2 + dy^2\}$. Thus, for example, there is no R on which $\Omega \equiv 0$ if A is indefinite somewhere on S .

REMARK. Let R and \hat{R} be Riemann surfaces defined on S . Suppose the identity mapping on S is not conformal from R to \hat{R} at p . Then $\Omega = 0$ at p on both R and \hat{R} if and only if $A = 0$ at p .

We are now free to discuss Ω_1 and Ω_2 on R_1, R_2 or R_2' . We may also define the quadratic differential $\Omega_2' = \phi_2' dz$ on any umbilic free portion of R_1, R_2 or R_2' with

$$\phi_2' = \frac{L' - N'}{2} - iM'.$$

And, for the sake of completeness, we will consider Ω_3 , the quadratic differential associated with

$$(10) \quad III = 2HII - KI$$

on S . As is well known, III yields, wherever $K \neq 0$, the first fundamental form on the unit spherical image of S . The relation (2) implies that

$$(11) \quad III = HII - H'II' = KI - 2H'II',$$

so that III is determined by any pair of the forms I, II and II' .

All linear relations, such as (2), (10) or (11), among the forms $I,$

II, *II'* and *III* hold also for their associated quadratic differentials. However, simplified versions of these relations are satisfied by Ω_1 , Ω_2 , Ω_2' and Ω_3 on R_1 , R_2 , R_2' , and on the Riemann surface R_3 determined on S by using the spherical image map (where $K \neq 0$) to carry back onto S the ordinary conformal structure of the sphere. For on each of the Riemann surfaces R_1 , R_2 , R_2' or R_3 , at least one of the quadratic differentials Ω_1 , Ω_2 , Ω_2' or Ω_3 vanishes identically.

LEMMA 6. *Let R be a Riemann surface on the oriented surface S immersed C^3 in E^3 . Suppose K , H , $H' \neq 0$. Then $R = R_1$ if and only if*

$$2H\Omega_2 = -2H'\Omega_2' = \Omega_3$$

on R ; $R = R_2$ if and only if

$$K\Omega_1 = H'\Omega_2' = -\Omega_3$$

on R ; $R = R_2'$ if and only if

$$K\Omega_1 = H\Omega_2 = \Omega_3$$

on R ; and $R = R_3$ if and only if

$$K\Omega_1 = 2H\Omega_2 = 2H'\Omega_2'$$

on R .

Proof. Use (2), (10), (11), and the remarks which follow Lemma 5.

REMARK. By Lemma 4, $\Omega_2 = (L - iM)dz^2$ on R_2' . By (2) and (7), $\Omega_2' = (L' - iM')dz^2$ on R_2 .

5. In [2] it is shown that Ω_2 is holomorphic on R_1 if and only if S is of constant mean curvature. In [3] it is shown that Ω_1 is holomorphic on R_2 if and only if S is of constant (positive) Gaussian curvature. We have the following (somewhat less satisfying) result of a similar nature.

THEOREM. Ω_1 is holomorphic on R_2' if and only if the vector X describing the immersion of S in E^3 is a harmonic function of isothermal coordinates.

Proof. By the definition of I , it is easily checked, using

$$\frac{\partial}{\partial z} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right\},$$

that

$$(12) \quad \phi_1 = 2X_z \cdot X_{\bar{z}}.$$

But Ω_1 is holomorphic on R'_2 if and only if

$$(\phi_1)_{\bar{z}} = 0,$$

that is, by (12), if and only if

$$(13) \quad X_z \cdot X_{\bar{z}\bar{z}} = 0,$$

where $z = x + iy$ is any conformal parameter on R'_2 , and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right\}.$$

And (13) is equivalent to

$$(14) \quad (X_{xx} + X_{yy}) \cdot X_x = (X_{xx} + X_{yy}) \cdot X_y = 0.$$

On the other hand, by Lemma 4, the Gauss equations in disothermals yield

$$\begin{aligned} X_{xx} &= \Gamma_{11}^1 X_x + \Gamma_{11}^2 X_y + L \bar{X}^\perp, \\ X_{yy} &= \Gamma_{22}^1 X_x + \Gamma_{22}^2 X_y - L \bar{X}^\perp, \end{aligned}$$

where \bar{X}^\perp is the unit normal to S . Thus (14) holds if and only if X is a harmonic function of disothermal coordinates.

Note, however, that disothermal coordinates are isothermal on a minimal surface. Thus $\Omega_1 \equiv 0$ is trivially holomorphic on R'_2 if S is minimal. Our theorem therefore includes the well known fact that the vector immersing a minimal surface is a harmonic function of isothermal coordinates. It would be nice, of course, to have a geometric characterization of all surfaces for which X is a harmonic function of disothermals.

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Dallas O. Banks, <i>Bounds for eigenvalues and generalized convexity</i>	1031
Jerrold William Bebernes, <i>A subfunction approach to a boundary value problem for ordinary differential equations</i>	1053
Woodrow Wilson Bledsoe and A. P. Morse, <i>A topological measure construction</i>	1067
George Clements, <i>Entropies of several sets of real valued functions</i>	1085
Sandra Barkdull Cleveland, <i>Homomorphisms of non-commutative *-algebras</i>	1097
William John Andrew Culmer and William Ashton Harris, <i>Convergent solutions of ordinary linear homogeneous difference equations</i>	1111
Ralph DeMarr, <i>Common fixed points for commuting contraction mappings</i>	1139
James Robert Dorroh, <i>Integral equations in normed abelian groups</i>	1143
Adriano Mario Garsia, <i>Entropy and singularity of infinite convolutions</i>	1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., <i>Convergence of extended Bernstein polynomials in the complex plane</i>	1171
Irving Leonard Glicksberg, <i>A remark on analyticity of function algebras</i>	1181
Charles John August Halberg, Jr., <i>Semigroups of matrices defining linked operators with different spectra</i>	1187
Philip Hartman and Nelson Onuchic, <i>On the asymptotic integration of ordinary differential equations</i>	1193
Isidore Heller, <i>On a class of equivalent systems of linear inequalities</i>	1209
Joseph Hersch, <i>The method of interior parallels applied to polygonal or multiply connected membranes</i>	1229
Hans F. Weinberger, <i>An effectless cutting of a vibrating membrane</i>	1239
Melvin F. Janowitz, <i>Quantifiers and orthomodular lattices</i>	1241
Samuel Karlin and Albert Boris J. Novikoff, <i>Generalized convex inequalities</i>	1251
Tilla Weinstein, <i>Another conformal structure on immersed surfaces of negative curvature</i>	1281
Gregers Louis Krabbe, <i>Spectral permanence of scalar operators</i>	1289
Shige Toshi Kuroda, <i>Finite-dimensional perturbation and a representation of scattering operator</i>	1305
Marvin David Marcus and Afton Herbert Cayford, <i>Equality in certain inequalities</i>	1319
Joseph Martin, <i>A note on uncountably many disks</i>	1331
Eugene Kay McLachlan, <i>Extremal elements of the convex cone of semi-norms</i>	1335
John W. Moon, <i>An extension of Landau's theorem on tournaments</i>	1343
Louis Joel Mordell, <i>On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$</i>	1347
Kenneth Roy Mount, <i>Some remarks on Fitting's invariants</i>	1353
Miroslav Novotný, <i>Über Abbildungen von Mengen</i>	1359
Robert Dean Ryan, <i>Conjugate functions in Orlicz spaces</i>	1371
John Vincent Ryff, <i>On the representation of doubly stochastic operators</i>	1379
Donald Ray Sherbert, <i>Banach algebras of Lipschitz functions</i>	1387
James McLean Sloss, <i>Reflection of biharmonic functions across analytic boundary conditions with examples</i>	1401
L. Bruce Treybig, <i>Concerning homogeneity in totally ordered, connected topological space</i>	1417
John Wermer, <i>The space of real parts of a function algebra</i>	1423
James Juei-Chin Yeh, <i>Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables</i>	1427
William P. Ziemer, <i>On the compactness of integral classes</i>	1437