SPECTRAL PERMANENCE OF SCALAR OPERATORS

Gregers Louis Krabbe

SPECTRAL PERMANENCE OF SCALAR OPERATORS

Gregers L. Krabbe

0. Introduction. Let $\{\mathfrak{B}_k\colon k \in K\}$ be a family of Banach spaces whose intersection D is dense in $\mathfrak{B}_k$ for each $k \in K$. Suppose that 0, $1 \in K$ and that each $\mathfrak{B}_k$ satisfies the relations $\mathfrak{B}_0 \cap \mathfrak{B}_1 \subset \mathfrak{B}_k$ and $\mathfrak{B}_0 < \mathfrak{B}_k < \mathfrak{B}_1$ (see § 1); let T be a linear operator which is simultaneously defined and bounded on each member of the family $\{\mathfrak{B}_k\colon k \in K\}$ (that is, $T \in [\mathfrak{B}_k]$; see 0.1). This paper gives a condition insuring that $\sigma(T; [\mathfrak{B}_0]) = \sigma(T; [B_k])$ for any $k \neq 1$; here $\sigma(T; [\mathfrak{B}_k])$ denotes the spectrum of T relative to $[\mathfrak{B}_k]$. In fact, this spectral equality holds whenever T has a spectral resolution which is bounded in $[\mathfrak{B}_1]$; see § 1. In this connection, it should be mentioned that the articles of Halberg and A. E. Taylor [4, 5] study relations between $\sigma(T; [\mathfrak{B}_0])$ and $\sigma(T; [\mathfrak{B}_1])$ in the particular case $K = \{0, 1\}$.

Let $\mathscr{D}_X$ be the Banach algebra of all complex-valued functions of bounded variation on a finite interval X. Our end-result depends on the fact that an operational calculus into $[\mathfrak{B}_0]$ induces a continuous representation of some $\mathscr{D}_X$ into $[\mathfrak{B}_k]$ (for $k \neq 1$); as is the case with the "spectral distributions" of Foias [3], properties of such representations may be exploited to some extent: see § 3.

Let μ be a continuous representation of $\mathscr{D}_X$ into some Banach algebra $\mathfrak{C}$, and let Σ_m be the largest open set G such that $\mu[a, b[= O$ whenever $[a, b] \subset G$; the existence of Σ_m is established in § 2. If $h \in \mathscr{D}_X$, then $\mu(h)$ is a member T of $\mathfrak{C}$, and the spectrum $\sigma(T; \mathfrak{C})$ coincides with the image $h(\Sigma_m)$ whenever h is continuous on X. Let g be a function on $h(\Sigma_m)$; as we shall see, it is natural to write $g(T) = \mu(g \circ h)$. It will be shown that

$$(1) \qquad \sigma(g(T); \mathfrak{C}) = g(\sigma(T; \mathfrak{C})) = (g \circ h)(\Sigma_m)$$

whenever the images $h(x)$ and $(g \circ h)(X)$ are plane rectifiable continuous curves: see 3.2. The first equality in (1) is well-known when g is a polynomial

$$g(\lambda) = \sum_{\nu=0}^{n} \alpha_\nu \lambda^\nu \qquad \text{(for } \lambda \in h(X));$$

note that[1], in this case

[1] The composition $g \circ h$ is defined by the relation $(g \circ h)(\lambda) = g(h(\lambda))$ for any $\lambda \in (-\infty, \infty)$.

$$g(T) = \mu(g \circ h) = \sum_{\nu=0}^{n} \alpha_\nu T^\nu \ .$$

The equality (1) is an analogue of the Dunford Mapping Theorem. Another analogue of the same theorem has been obtained by Foias: see § 3.

0.1. NOTATION. When $\mathfrak{Y}$ is a Banach space, then $[\mathfrak{Y}]$ will denote the Banach algebra of bounded endomorphisms of $\mathfrak{Y}$. From the denseness of D it follows that $[\mathfrak{B}_k]$ is identifiable with the space of all bounded linear operators on D to $\mathfrak{B}_k$.

0.2. APPLICATIONS. Set $L^p(a, \nu)$, where (a, ν) is an arbitrary measure space. Suppose that p_0, p_1, p_2 are three distinct points in the closed interval $[1, \infty]$, such that p_2 is between p_0 and p_1. Set $p_0 = 2$, and let T be a self-ajoint bounded linear operator on $L(2)$; its spectral resolution $\{\lambda \rightarrow m^T(\lambda)\}$ is here considered as a function of a real variable taking its values in $[L(2)]$. Theorem II (see § 1) implies the following

THEOREM I.   *If the function* $\{\lambda \rightarrow m^T(\lambda)\}$ *is bounded in the norm of* $[L(p_1)]$, *then*

(2) $\sigma(T; [L(2)]) = \sigma(T; [L(p_2)]) \ .$

For example, let $L(p)$ be the sequence space l_p: in [10, p. 355] it is proved that Theorem I is satisfied when T is represented by the Hilbert matrix

$$\left(\frac{\sqrt{-1}}{n - j} \right) \qquad (n, j = 0, \pm 1, \pm 2, \pm 3, \cdots) \ .$$

Theorem I is also satisfied when T is the operator Δ defined on p. 335 of [10].

Let $L(p) = L^p(-\infty, 0)$, $1 < p < \infty$. It should be noted that the conclusion (2) is false when T is the Legendre differential operator (see p. 89 in G.-C. Rota's paper [16]); consequently, Theorem I shows that $\{\lambda \rightarrow m^T(\lambda)\}$ is—in this particular case—an unbounded function in the norm of $[L(p)]$ whenever $p \neq 2$.

1. The main result. Again, let $\{\mathfrak{B}_k: k = 0, 1, 2\}$ be a family of Banach spaces whose intersection is dense in $\mathfrak{B}_k$ for each $k = 0, 1, 2$. We will write $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$ to indicate that $\mathfrak{B}_2$ is an interpolation space (in the sense of J.-L. Lions [15]) strictly intermediate between $\mathfrak{B}_0$ and some space $\mathfrak{B}_3$ such that $\mathfrak{B}_3$ is an interpolation space strictly intermediate between $\mathfrak{B}_0$ and $\mathfrak{B}_1$. The property $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$ can be

stated without using the above terminology; this is done in § 4, wherein is found two conditions whose conjunction is equivalent to the property $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$: see Remark 4.2.

THEOREM II. *Suppose that* $\mathfrak{B}_0 \cap \mathfrak{B}_1 \subset \mathfrak{B}_2$ *and* $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$. *Let* $T \in [\mathfrak{B}_0]$ *be a scalar operator whose spectral resolution is a bounded mapping into* $[\mathfrak{B}_1]$; *then*

$$\sigma(T;\ [\mathfrak{B}_0]) = \sigma(T;\ [\mathfrak{B}_2]) \ .$$

The spectral solution of T is here considered as a $[\mathfrak{B}_0]$-valued function of two variables.

THEOREM III. *Under the hypotheses of Theorem* II, *let* X *be a finite interval such that* $(-\infty, \infty) \supset X \supset \sigma(T; [\mathfrak{B}_0])$; *is* h *is a continuous function such that the image* $h(X)$ *is a plane rectifiable curve, then* $h(T) \in [\mathfrak{B}_2]$ *and*

$$\sigma(h(T);\ [\mathfrak{B}_0]) = h(\sigma(T;\ [\mathfrak{B}_0])) = \sigma(h(T);\ [\mathfrak{B}_2]) \ .$$

Theorem III is proved in § 4; since . T is scalar (i.e., "of scalar type" in the sense of Dunford [2]), $h(T)$ is defined as in [2, p. 343]; note also that $h(\sigma(T; [\mathfrak{B}_0])$ is the image $h(S_0)$ of the spectrum $S_0 = \sigma(T; [\mathfrak{B}_0])$ of T with respect to $[\mathfrak{B}_0]$. In the case where T has a real spectrum, the spectral resolution of T is considered as a mapping of $(-\infty, \infty)$ into $[\mathfrak{B}_0]$, and Theorem II is a direct consequence of Theorem III. To the reader is left the task of verifying that Theorem II can be proved by extending to two dimensions the methods of the present paper.

We now return to the setting $L(p) = L^p(a, \nu)$ of 0.2, and let p_0, p_1, p_2 be any three distinct points in the closed interval $[1, \infty]$ such that p_2 is between p_0 and p_1. If $\mathfrak{B}_k = L(p_k)$, then $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$, whence Theorem I is seen to be an immediate consequence of Theorem III when $T \in [L(p_0)]$ is a scalar operators[2].

2. **The spectrum of a vector-valued function.** Let M be a spectral (σ-additive) measure on $R = (-\infty, \infty)$; the spectrum of M (see [6, p. 62]) is a set A such that

$$T = \int_R \lambda \cdot M(d\lambda) = \int_A \lambda \cdot M(d\lambda) \ .$$

Let m be a vector-valued function on R; we shall presently introduce a set $\sum_m$ whose role will be to duplicate a theorem [6, p. 64] stating that the spectrum of M coincides with the spectrum of T; mutatis mutandis, since our case corresponds to

---

[2] In 0.2 we took the particular case $p_0 = 2$ and supposed that T is a bounded self-adjoint transformation of the Hilbert space $L(2)$.

$$T = \int_R \lambda \cdot dm(\lambda) = \int_{\Sigma_m} \lambda \cdot dm(\lambda) \, ,$$

although neither integral need exist under our assumptions.

2.1. DEFINTION. If $G \subset R$, we say that m is constant on G if $m(a) = m(b)$ whenever $[a, b]$ is a closed interval with $[a, b] \subset G$.

2.2. NOTATION. Let $\mathcal{O}_m$ be the family of all open subsets G of $R = (-\infty, \infty)$ such that m is constant on G. The complement $\{x \in R: x \notin B\}$ is denoted $R \sim B$, and we write

$$(3) \qquad \Sigma_m = R \sim \cup \mathcal{O}_m = R \sim \cup \{G: G \in \mathcal{O}_m\} \, ;$$

the set Σ_m is called the "*spectrum*" of m. It will now be proved that Σ_m is the complement of the largest open set on which m is constant. If $x, y \in R$, then $[x, y] = \{\theta \in R: x \leq \theta < y\}$.

2.3. LEMMA. *Let K be a closed nonvoid subinterval (of R) such that $K \subset \cup \mathcal{O}_m$. There exists a finite family of subintervals such that $K \subset \cup \mathcal{I}$, and*

$$(4) \qquad [x, y[\subset K \cap J_0 \quad implies \ that \quad m(x) = m(y)$$

whenever $J_0 \in \mathcal{I}$.

Proof. Take $t \in K$. The hypothesis $K \subset \cup \mathcal{O}_m$ shows that $t \in G_t$ for some $G_t \in \mathcal{O}_m$. Since G_t is an open subset of R, there exists a family $\mathcal{I}_t$ of open subintervals such that $G_t = \cup \mathcal{I}_t$; but $t \in G_t$ implies that $t \in I_t$ for some $I_t \in \mathcal{I}_t$; note that m is constant on I_t (since the property $G_t \in \mathcal{O}_m$ necessitates that m is a constant on $G_t \supset I_t$). The family $\mathcal{I}_0 = \{I_t: t \in K\}$ is therefore an open cover of the compact set K, whence the existence of a finite sub-family $\mathcal{I} = \{I_t: t \in p\} \subset \mathcal{I}_0$ such that

$$(5) \qquad K \subset \cup \mathcal{I} = \cup \{I_t: t \in p\} \, .$$

Since $[x, y[\subset K$, it follows that $y \in K$, and (5) shows the existence of $J_1 =]\alpha_1, \beta_1[\in \mathcal{I}$ with

$$(6) \qquad \alpha_1 < y < \beta_1 \, .$$

From $x < y$ and $\alpha_1 < y$ we see that there exists a number λ such that

$$(7) \qquad \alpha_1 < \lambda < y \quad and \quad x < \lambda < y \, .$$

But (6)–(7) show that $\alpha_1 < \lambda < y < \beta_1$, whence $[\lambda, y] \subset J_1$ and $[x, \lambda] \subset J_0$ (the last inequality comes from (7) and the hypothesis $[x, y[\subset J_0$).

Since m is constant on all members of $\mathscr{F}$, it follows that m is constant on J_k when $k = 0, 1$. Having thus seen that $[\lambda, y] \subset J_1$ and $[x, \lambda] \subset J_0$, we conclude that $m(x) = m(\lambda) = m(y)$.

2.4. THEOREM. *If $G_m = \cup \mathscr{O}_m$, then $G_m \in \mathscr{O}_m$ and*

$$(8) \qquad\qquad G \in \mathscr{O}_m \Longrightarrow G \subset G_m .$$

Proof. Property (8) is obvious. Since G_m is clearly an open set, it remains to show that m is constant on G_m. Take an arbitrary closed nonvoid interval $K = [a_0, b_0]$ with $K \subset G_m$. As we shall see, there exists a finite partition $a_0 = x_0 < x_1 < \cdots < x_{s+1} = b_0$ such that $m(x_{n-1}) = m(x_n)$ for $n = 1, 2, \cdots, s + 1$; therefore $m(a_0) = m(b_0)$, so that m is constant on G_m (which had to be demonstrated). Accordingly, the forthcoming construction of such a partition will conclude the proof.

In view of 2.3, our hypothesis $\dot{K} \subset \cup \mathscr{O}_m$ implies the existence of a finite family $\mathscr{F}$ such that $K \subset \cup \mathscr{F}$ and (4). If $x \in K$, then $x \in J_x$ for some $J_x \in \mathscr{F}$; set $J_x =]\alpha_x, \beta_x[$ and $f(x) = \min \{\beta_x, b_0\}$. Since $\alpha_x < x < \beta_x$ and $a_0 \leqq x \leqq b_0$, it follows that $x \leqq f(x)$ and $[x, f(x)[\subset K \cap J_x$. Consequently, we infer from (4) that

$$(9) \qquad\qquad m(x) = m(f(x)) .$$

Let H denote the set of all end-points of the members of $\mathscr{F}$. If $f(x) < b_0$, then $f(x) = \beta_x$ (since the only alternative is $f(x) = b_0$); in view of the inequality $x < \beta_x$, it follows therefore that

$$(10) \qquad\qquad f(x) < b_0 \Longrightarrow x < f(x) \in H .$$

Now to construct our partition. Define recursively a sequence $(x_n)_n$ as follows: $x_0 = a_0$, $x_n = f(x_{n-1})$ for $n \geqq 1$. From (10) we see that $\{x_n : x_n < b_0\}$ is a subset of the finite set H, whence the existence of

$$(11) \qquad\qquad x_s = \max \{x_n : x_n < b_0\} .$$

Note that $x_{s+1} = f(x_s) \leqq b_0$. I say that $x_{s+1} = b_0$: otherwise $x_{s+1} < b_0$, so that $x_{s+1} \leqq x_s$ (by (11)), which contradicts the conclusion $x_s < f(x_s) = x_{s+1}$ obtained from (10) and $f(x_s) = x_{s+1} < b_0$. Thus $x_{s+1} = b_0$, and $m(x_{n-1}) = m(x_n)$ (from (9)); the partition has all the required properties.

3. **Spectral mapping theorems.** Let X be a compact (i.e., closed and bounded) subinterval of $R = (-\infty. \infty)$, and let $\mathscr{D}_X$ be the Banach algebra of all complex-valued functions g on R such that g vanishes outside of X, and such that g is of bounded variation on X. Note

that $\mathscr{D}_x$ is identifiable with the usual algebra of all functions having finite total variation on X. Throughout, μ is a continuous representation of the Banach algebra $\mathscr{D}_x$ into some Banach algebra $\mathfrak{C}$.

If $B \subset \boldsymbol{R}$, then $[B]$ will denote the characteristic function of the set B:

$$[B](\theta) = \begin{cases} 1 & \text{if } \theta \in B \\ 0 & \text{if } \theta \notin B \text{ and } \theta \in \boldsymbol{R} . \end{cases}$$

If $\lambda \in \boldsymbol{R}$, set $e_\lambda = [(-\infty, \lambda[\cap X]$; thus, if $X = [x', x'']$, then $e_\lambda(\theta) = 1$ if $x' \leqq \theta < \lambda$ and $e_\lambda(\theta) = 0$ for all other values of θ in $\boldsymbol{R}$. Let m be the function defined on $\boldsymbol{R}$ by the relation

$$(12) \qquad\qquad m(\lambda) = \mu(e_\lambda) \qquad\qquad (\text{when } \lambda \in \boldsymbol{R}) .$$

3.1. THEOREM. *Let h be a continuous function with $h \in \mathscr{D}_x$. If $\sigma(\mu(h); \mathfrak{C})$ denotes the spectrum of $\mu(h)$, then*

$$(13) \qquad\qquad \sigma(\mu(h); \mathfrak{C}) = h(\Sigma_m) = \{h(\theta): \theta \in \Sigma_m\} .$$

The proof is given in 3.10.

It is easily seen that $\Sigma_m \subset X$. If j^1 is the function defined by $j^1(\theta) = \theta$ for $\theta \in X$ and vanishing outside of X, then (13) shows that

$$(14) \qquad\qquad \Sigma_m = \sigma(\mu(j^1); \mathfrak{C}) .$$

Foias' paper [3] deals with a continuous representation of the function-space $C^\infty(\boldsymbol{R})$; Foias' formulae (1.2)–(1.3) (see [3]) correspond to (13)–(14).

3.2. With h as in 3.1, let g be a continuous function on the image $h(X)$; if $T = \mu(h)$, write $g(T) = \mu(g \circ h)$. A double application of 3.1 now shows easily that

$$(15) \qquad \sigma(g(T); E) = g(\sigma(T; E)) = \{g(h(\theta)): \theta \in \Sigma_m\} ,$$

whenever the curve $\{g(h(\theta)): \theta \in X\}$ is rectifiable. For an analogue of (15), see formula (28) in [13, p. 426].

Schaefer [14] considers the continuous representation $\mu = \left\{ h \rightarrow \int h \cdot dM \right\}$ of the algebra of bounded Baire functions (when M is a spectral measure); his Proposition 16 [14, p. 151] asserts that (15) holds for any continuous function g on $\sigma(T; \mathfrak{C})$; he presupposes that $T = \mu(h)$ for some h in the domain of μ.

3.3. ORIENTATION. The remainder of this section is devoted to the task of proving Theorem 3.1. To that effect, we begin by introducing some notation.

3.4. NOTATION. Let $\mathfrak{A}$ be a Banach algebra with multiplicative unit e. If $a \in \mathfrak{A}$, then the resolvent set $\rho(a; \mathfrak{A})$ consists of all complex λ such that $\lambda e - a$ has an inverse $R(a, \mathfrak{A}; \lambda) \in \mathfrak{A}$. It is known [7, p. 126] that

(16) $R(a, \mathfrak{A}; \cdot)$ is analytic in $\rho(a; \mathfrak{A})$ with values in $\mathfrak{A}$;

here "analytic" is what is called "locally analytic" in [7]. The complement of $\rho(a; \mathfrak{A})$ is the spectrum $\sigma(a; \mathfrak{A})$.

If $\mathscr{F}_x$ is the family of complex-valued functions on $R = (-\infty, \infty)$ such that g vanishes on $R \sim X = \{\theta \in R: \theta \notin X\}$, then

$$\mathscr{D}_x = \{g \in \mathscr{F}_x: V(g; X) < \infty\} ,$$

where $V(g; X)$ is the total variation of g on X. Note that $\langle \mathscr{D}_x, [\phi], [x], \cdot \rangle$ is a Banach algebra under pointwise multiplication $(f, g) \to f \cdot g$, where $(f \cdot g)(\theta) = f(\theta)g(\theta)$ for $\theta \in R$; the zero-element and the identity-element are, respectively, the characteristic functions $[\phi]$ and $[X]$. We shall consistently drop the brackets and parentheses and simply write μB instead of $\mu([B])$.

3.5. HYPOTHESES. Henceforth, μ is a continuous representation (i.e., homomorphism) of the Banach algebra $\langle E, O, I, \circ \rangle$. In particular, $\mu(f \cdot g) = \mu(f) \circ \mu(g)$ and $\mu([X]) = I$, while $\mu([\phi]) = O$. Finally, suppose that h is a continuous function with $h \in \mathscr{D}_x$.

3.6. ORIENTATION. Our objective is the proof of Theorem 3.1, which will now be presented in a sequence of four steps. The complex plane is denoted C, and the complement $\{\zeta \in C: \zeta \notin B\}$ is denoted $\sim B$.

3.7. Let A and B be subintervals of R. Since $\mu(A \cap B) = \mu A \circ \mu B$, it is immediately seen that

(17) $A \subset B$ and $\mu B = O \Longrightarrow \mu A = O$,

and

(18) $S \subset B$ and $\mu S = I \Longrightarrow \mu B = I$.

On the other hand, if $B = [a, b[= \{x \in R: a \leqq x < b\}$, then $[B] = e_b - e_a$, so that

(19) $\mu[a, b[= m(b) - m(a)$;

in accordance with our notational convention, we have written $\mu[a, b[= \mu B$ instead of $\mu([B])$. It follows from (17) that

(20) $\mu[a, b] = O \Longrightarrow m(b) = m(a)$.

3.8. LEMMA. *Let K be a closed nonvoid subinterval of X. If $\lambda \in \sim h(K)$, then there exists an element $\Phi(\lambda)$ of $\mathfrak{E}$ such that*

$$(\lambda I - \mu(h)) \circ \Phi(\lambda) = \mu K .$$

Moreover, the function Φ is analytic on $\sim h(K)$.

Proof. Since K is compact, the continuity of h implies that $h(K)$ is compact. The hypothesis $\lambda \in \sim h(K)$ implies the existence of some $\delta > 0$ such that $|\lambda - h(\theta)| \geqq \delta$ whenever $\theta \in K$, whence the existence of some function f of bounded variation on K such that $f(\theta)(\lambda - h(\theta)) = 1$ for any $\theta \in K$. In other words, $\lambda[K] - h \cdot [K]$ has an inverse in the Banach algebra $\mathscr{D}_K$ whenever $\lambda \in \sim h(K)$. Consequently, $\sim h(K) \subset \rho(h \cdot [K]; \mathscr{D}_K)$; but (16) shows that $R(h \cdot]K], \mathscr{D}_K; \cdot)$ is an analytic function φ with values in the subset $\mathscr{D}_K$ of $\mathscr{D}_X$; furthermore, $(\lambda[K] - h \cdot [K]) \cdot \varphi(\lambda) = [K]$ whenever $\lambda \in \sim h(K)$. Since

$$(\lambda[K] - h \cdot [K]) \cdot \varphi(\lambda) = (\lambda[X] - h \cdot [X]) \cdot \varphi(\lambda) \cdot [K]$$

and $\varphi(\lambda) = \varphi(\lambda) \cdot [K] \in \mathscr{D}_X$, we see that

$$(\lambda[X] - h) \cdot \varphi(\lambda) = [K] \qquad \text{(when } \lambda \in \sim h(K)) .$$

Since μ is a homomorphism of $\mathscr{D}_X$, it follows that

$$(\lambda I - \mu(h)) \circ \mu(\varphi(\lambda)) = \mu([K]) = \mu K .$$

The proof is concluded by setting $\Phi(\lambda) = \mu(\varphi(\lambda))$ and nothing that the analyticity of Φ is implied by the continuity of μ (to see this, observe that $\Phi(\lambda)$ has a power series expansion with coefficients in the subset $\mathscr{D}_K$ of $\mathscr{D}_X$).

3.9. LEMMA. *If $T = \mu(h)$, then*

(21)
$$h(\Sigma_m) \subset \sigma(T; \mathfrak{E}) .$$

Proof. If S is a subset of the complex plane C, denote by $h^{-1}S$ the inverse image $\{x \in X: h(x) \in S\}$, and observe that

(22)
$$A \subset h^{-1}S \Longrightarrow h(A) \subset S .$$

Let us admit for a moment that m is constant on $h^{-1}\rho(T; \mathfrak{E})$. The continuity of h implies that $h^{-1}\rho(T; \mathfrak{E}) \in \mathscr{C}_m$ (see 2.2); from 2.4(8) it therefore results that $h^{-1}\rho(T; \mathfrak{E}) \subset \cup \mathscr{C}_m$: consequently, $R \sim \cup \mathscr{C}_m \subset \sim h^{-1}\rho(T; \mathfrak{E}) = h^{-1} \sim \rho(T; \mathfrak{E}) = h^{-1}\sigma(T; \mathfrak{E})$, so that $R \sim \cup \mathscr{C}_m \subset h^{-1}\sigma(T; \mathfrak{E})$. In view of 2.2(3), an application of (22) concludes the proof of (21).

It remains to show that m is constant on $h^{-1}\rho(T; \mathfrak{E})$. To that effect, take an arbitrary nonvoid interval $K = [a, b]$ such that

(23) $K \subset h^{-1} \rho(T; \mathfrak{G})$;

in view of (20), it will be sufficient to prove that $\mu K = O$.

From (23) and (22) it follows that $h(K) \subset \rho(T; \mathfrak{G})$, whence

(24) $C = \rho(T; \mathfrak{G}) \cup \sim h(K)$.

From 3.8 it results the existence of a function $\varPhi$ defined on $\sim h(K)$ such that

(25) $(\lambda I - T) \circ \varPhi(\lambda) = \mu K$ (when $\lambda \in \sim h(K)$) .

Take $\lambda \in \rho(T; \mathfrak{G}) \cap \sim h(K)$; the inverse $R(T, \mathfrak{G}; \lambda)$ belongs to $\mathfrak{G}$, and (25) shows that $\varPhi(\lambda) = \mu K \circ R(T, \mathfrak{G}; \lambda)$. Set $F(\lambda) = \mu K \circ R(T, \mathfrak{G}; \lambda)$. The function F is defined on $\rho(T; \mathfrak{G})$ and agrees with $\varPhi$ on $\rho(T; \mathfrak{G}) \cap \sim h(K)$; thus, we may define

(26) $\psi(\lambda) = \begin{cases} \varPhi(\lambda) & \text{if} \quad \lambda \in \sim h(K) \\ F(\lambda) & \text{if} \quad \lambda \in \rho(T; \mathfrak{G}) \end{cases}$

whenever $\lambda \in \rho(T; \mathfrak{G}) \cup \sim h(K)$. From (24) we see that ψ is defined on the complex plane C; since $\varPhi$ and F are both analytic, ψ is an entire function. On the other hand, it is easily verifiable that $O = \lim F(\lambda)$ (when $|\lambda| \to \infty$); see [7, p. 125]. Consequently, ψ is an entire function such that $O = \lim \psi(\lambda)$ (when $|\lambda| \to \infty$), and Liouville's theorem [7, 3.13.2] shows that $\psi(\lambda) = O$ for every $\lambda \in C$. The desired property $\mu K = O$ now results immediately from (25) and (26).

3.10. THEOREM. *Let μ be a continuous representation of $\mathscr{D}_x$ into some Banach algebra $\mathfrak{G}$, and let Σ_m denote the spectrum of the function m defined as in* (12). *If h is a continuous function with $h \in \mathscr{D}_x$, then*

(27) $h(\Sigma_m) = \sigma(\mu(h); \mathfrak{G})$.

Proof. Set $T = \mu(h)$, and take an arbitrary $\lambda \in \sim h(\Sigma_m)$. It will suffice to find a member Q of $\mathfrak{G}$ such that

(28) $(\lambda I - T) \circ Q = I$;

indeed, (28) shows that $\sim h(\Sigma_m) \subset \rho(T; \mathfrak{G})$, so that $h(\Sigma_m) \supset \sigma(T; \mathfrak{G})$; the conclusion (27) now comes from 3.9.

Take $K = [a_0, b_0]$ such that $R \supset [a_0, b_0[\supset X$; from 3.7 (18) we see that

(29) $\mu[a_0, b_0[= I$.

Set $g = \{\theta \in K \to |\lambda - h(\theta)|\}$; since λ does not belong to the compact

set $h(\Sigma_m)$, there exists a number $\delta > 0$ such that

$$(30) \qquad\qquad g(\theta) = |\lambda - h(\theta)| \geqq \delta \qquad\qquad \text{(for any } \theta \in \Sigma_m) .$$

Let M be the family of all $y \in R$ such that $K(g = y)$ is an infinite set (by definition, $K(g = y)$ is the set of all $\theta \in K$ such that $g(\theta) = y$). By means of the Banach indicatrix, it can be shown that M is a set of Lebesgue measure zero (see [12, p. 231]); consequently, the open interval $]0, \delta[$ cannot be included in M, and there exists a number $\varepsilon \in]0, \delta[$ such that $\varepsilon \notin M$. Accordingly, $\delta > \varepsilon > 0$ and $K(g = \varepsilon)$ is a finite set.

Now $K = [a_0, b_0]$, and the finite set $B = \{a_0, b_0\} \cup K(g = \varepsilon)$ may be indexed as follows: $B = \{x_n : n = 0, 1, \cdots, k\}$, where $a_0 = x_0 < x_1 < \cdots < x_{k-1} < x_k = b_0$. On the other hand, $u = \{\theta \to g(\theta) - \varepsilon\}$ is a continuous real-valued function vanishing on $K(g = \varepsilon)$; since u vanishes nowhere else, its sign is a constant λ_n on each open interval $]x_{n-1}, x_n[$. The following abbreviations will be used: $r = \{1, 2, \cdots, k\}$, $q = \{\nu \in r : \lambda_\nu < 0\}$, and $p = \{s \in r : \lambda_s > 0\}$; we also write

$$K(g \leqq \varepsilon) = \{\theta \in K : g(\theta) \leqq \varepsilon\} ,$$

with a similar definition for $K(g \geqq \varepsilon)$. The following two identities are immediate consequences of the preceding considerations :

(i) $\qquad\qquad K(g \leqq \varepsilon) = \cup \{[x_{\nu-1}, x_\nu] : \nu \in q\}$

(ii) $\qquad\qquad K(g \geqq \varepsilon) = \cup \{[x_{s-1}, x_s] : s \in p\} .$

Since $\delta > \varepsilon$, it follows from (30) that $g(\theta) > \varepsilon$ for any $\theta \in \Sigma_m$, so that

$$(31) \qquad\qquad K(g \leqq \varepsilon) \subset K \sim \Sigma_m \subset \cup \mathscr{O}_m ;$$

see 2.2 (3) for the last inclusion. But (i) and (31) imply that $[x_{\nu-1}, x_\nu] \subset \cup \mathscr{O}_m$; on the other hand, m is constant on $\cup \mathscr{O}_m$ (see 2.4 and 2.2), so that $m(x_{\nu-1}) - m(x_\nu) = O$; from 3.7 (19) therefore $\mu[x_{\nu-1}, x_\nu[= O$ whenever $\nu \in q$. Since $\{A \to \mu A\}$ is an additive set-function defined on all finite unions of disjoint intervals, it follows that

$$(32) \qquad\qquad \mu B_q = O ,$$

where

$$B_q = \cup \{[x_{\nu-1}, x_\nu[: \nu \in q\} .$$

Set

$$(33) \qquad\qquad B_p = \cup \{[x_{s-1}, x_s[: s \in p\} ;$$

since $[a_0, b_0[= B_p \cup B_q$, it again follows from the additivity of μ that

$\mu[a_0, b_0[= \mu B_p + \mu B_q$; from (32) and (29) it therefore results that $I = \mu B_p$; since (ii) and (33) show that $B_p \subset K(g \geqq \varepsilon)$, the conclusion

$$(34) \qquad\qquad \mu[K(g \geqq \varepsilon)] = I$$

is now an immediate consequence of 3.7(18).

Take $s \in p$, and call $A_s = [x_{s-1}, x_s]$. From (ii) we see that

$$(35) \qquad\qquad [K(g \geqq \varepsilon)] = \sum_{s \in p} [A_s]$$

and $A_s \subset K(g \geqq \varepsilon)$; in other words, $g(\theta) = |\lambda - h(\theta)| \geqq \varepsilon$ for each $\theta \in A_s$. Accordingly, there exists a function $F_s \in \mathscr{D}_x$ such that $(\lambda - h(\theta))F_s(\theta) = 1$ for all $\theta \in A_s$; that is, $(\lambda[X] - h)\cdot[A_s]\cdot F_s = [A_s]$. Set

$$f = \sum_{s \in p} [A_s]\cdot F_s$$

and note that $f \in \mathscr{D}_x$. From what has just been seen, it results that

$$(36) \qquad (\lambda[X] - h)\cdot f = \sum_{s \in p} [A_s] = [K(g \geqq \varepsilon)] ;$$

the last equality comes from (35). By applying the operator μ to both sides of (36), we obtain that $(\lambda I - \mu(h)) \circ \mu(f) = \mu[K(g \geqq \varepsilon)]$; setting $Q = \mu(f)$, a glance at (34) now establishes (28)—this concludes the proof.

4. Spectral permanence. When $\mathfrak{Y}$ and $\mathfrak{X}$ are topological spaces, we write $\mathfrak{Y} \ll \mathfrak{X}$ to indicate that $\mathfrak{Y}$ is a subset of $\mathfrak{X}$ such that the identity-mapping of $\mathfrak{Y}$ into $\mathfrak{X}$ is continuous. Let $\{\mathfrak{B}_k: k = 0, 1, 2\}$ be a family of Banach spaces such that $\mathfrak{B}_0 \cap \mathfrak{B}_1 \subset \mathfrak{B}_2$. Let $\mathfrak{E}_k$ denote the Banach algebra of all bounded endomorphisms of $\mathfrak{B}_k$ (i.e., all continuous linear mappings of $\mathfrak{B}_k$ into itself). If $k = 0, 1, 2$, let $\| x \|_k$ be the norm of a member x of $\mathfrak{B}_k$; if E is a linear subset of $\mathfrak{B}_k$, then $[E, \mathfrak{B}_k]$ will consist of all linear mappings π of E into $\mathfrak{B}_k$ such that

$$\infty \neq |\pi|_k = \sup \{\| \pi_x \|_k: x \in E \text{ and } \| x \|_k \leqq 1\} ;$$

the linear space $[E, \mathfrak{B}_k]$ is endowed with the norm $\{\pi \rightarrow |\pi|_k\}$.

4.1. HYPOTHESES. Henceforth, we presuppose the following three conditions:

(iii) *there exists a linear set D such that D is a dense subset of $\mathfrak{B}_K$ for $k = 0, 1, 2$.*

(iv) *there exists a topological space $\mathfrak{X}$ such that $\mathfrak{B}_k \ll \mathfrak{X}$ for $k = 0, 1, 2$.*

(v) *there exists a semi-norm $\mathfrak{N}[\cdot]$ on $\mathfrak{E}_0 \cap \mathfrak{E}_1$ and there exist two*

points τ, r *in* $]0, 1[$ *such that*

$$\mathfrak{N}[\pi] \leqq |\pi|_0^{1-\tau} |\pi|_1^{\tau} \quad and \quad |\pi|_2 \leqq |\pi|_0^{1-r} \mathfrak{N}[\pi]^r$$

whenever $\pi \in \mathfrak{E}_0 \cap \mathfrak{E}_1$.

4.2. REMARKS. Conditions (iv)–(v) are equivalent to the property $\mathfrak{B}_0 < \mathfrak{B}_2 < \mathfrak{B}_1$ that was defined in § 1. In view of (iii), we may identify $[D, \mathfrak{B}_k]$ with $\mathfrak{E}_k$ (for $k = 0, 1, 2$); this identification is assumed throughout. All the above conditions are satisfied in the case $\mathfrak{B}_k = L(p_k)$, where $1 \leqq p_0 < p_2 < p_1 \leqq \infty$ or $1 \leqq p_1 < p_2 < p_0 \leqq \infty$ (cf. 0.2); to see this, take $\mathfrak{X}$ to be the space of measurable functions, and take D to be the linear space of simple functions.

We now prepare to state our main theorem. Let $T \in \mathfrak{E}_0$ be a scalar operator with $\sigma(T; \mathfrak{E}_0) \subset \boldsymbol{R} = (-\infty, \infty)$. The boundedness of T insures the existence of a compact interval X containing the spectrum $\sigma(T; \mathfrak{E}_0)$. Let M be the resolution of the identity for the operator T (see [2, p. 325]). Theorem 4 in [14] states that

$$(37) \qquad \text{Supp } M = \sigma(T; \mathfrak{E}_0),$$

where Supp M denotes the support (also called the spectrum) of M. Since therefore Supp $M \subset X$, the mapping

$$U = \left\{ f \to \int_X f \cdot dM \right\}$$

is a continuous representation into $\mathfrak{E}_0$ of the Banach algebra $\mathscr{B}(X)$ of bounded Baire functions. If $\lambda \in \boldsymbol{R}$, let $\chi(-\infty, \lambda)$ be the characteristic function of the open interval $]-\infty, \lambda[$; that is:

$$\chi(-\infty, \lambda)(\theta) = \begin{cases} 1 & \text{If } -\infty < \theta < \lambda \\ 0 & \text{if } \lambda \leqq \theta < \infty. \end{cases}$$

The spectral resolution m is defined for any $\lambda \in \boldsymbol{R}$ by the identity $m(\lambda) = U\chi(-\infty, \lambda)$; the function m is clearly a bounded mapping into $\mathfrak{E}_0$. We suppose henceforth that m is a bounded mapping into $\mathfrak{E}_1$. In other words, there must exist a number $k_0 > 0$ such that

$$m(\lambda)y \in \mathfrak{B}_1 \quad and \quad \|m(\lambda)y\|_1 \leqq k_0 \|y\|_1$$

whenever $y \in D$ and $\lambda \in \boldsymbol{R}$.

4.3. THEOREM. *If h is a continuous function with $h \in \mathscr{D}_X$, then*

$$(38) \qquad \sigma(h(T); \mathfrak{E}_0) = h(\Sigma_m) = \sigma(h(T); \mathfrak{E}_2).$$

The proof is given in 4.7.

Note that the relation

$$\Sigma_m = \text{Supp } M$$

is obtained from (37) by replacing in (38) the function h by the identity-function.

4.4. LEMMA. *The mapping* U *is a continuous representation of* $\mathscr{D}_X$ *into the Banach algebra* $\mathfrak{E}_0$.

Proof. Observe that $\mathscr{D}_X \ll \mathscr{B}(X)$; since U is a continuous mapping of $\mathscr{B}(X)$, it is therefore a continuous mapping of $\mathscr{D}_X$. Since U is a homomorphism of $\mathscr{B}(X) \supset \mathscr{D}_X$, it follows that U is a homomorphism of $\mathscr{D}_X$.

4.5. LEMMA. U *is a continuous linear mapping of* $\mathscr{D}_X$ *into* $\mathfrak{E}_2$.

Proof. The space G_0 consists of all complex-valued functions g on R having only simple discontinuities and such that $|g(x + 0)| + |g(\lambda - 0)| < \infty$ whenever $-\infty \leqq x < \infty$ and $-\infty < \lambda \leqq \infty$. The space W_α consists of all $g \in G_0$ such that

$$\infty \neq V_\alpha(g) = \sup_z \left(\sum_{s=1}^{n} |g(z_{s-1}) - g(z_s)|^{1/\alpha} \right)^\alpha$$

where z runs over all partitions $-\infty < z_0 < z_1 < \cdots < z_{n-1} < z_n < \infty$. The space W_α is endowed with the norm $\{g \to \|g\| + V_\alpha(g)\}$, where

$$\|g\| = \sup \{|g(x)|: -\infty < x < \infty\}.$$

Observe that $G_0 \ll \mathscr{B}(X)$; as in 4.4, it is therefore readily inferred that U is a continuous mapping of G_0 into $\mathfrak{E}_0 = [D, \mathfrak{B}_0]$. Note that $Uf = Ug$ whenever f and g belong to $\mathscr{B}(X)$ and $f = g$ almost-everywhere; thus, we have the situation enviseaged in [8, 1.11]. Consequently, from Theorem B in [8] we conclude that the operator is a continuous mapping of $G_\tau = \cup \{W_\alpha: \tau < \alpha \leqq 1\}$ into $[D, \mathfrak{B}_2] = \mathfrak{E}_2$; since G_τ is endowed with the inductive limit-topology, it follows that U is a continuous mapping of W_α whenever $\tau < \alpha \leqq 1$. In particular, U is a continuous mapping of W_1 into $\mathfrak{E}_2$: there exists a number $c_1 > 0$ such that

$$\|Ug\|_2 \leqq c_1(\|g\| + V_1(g)).$$

Since $\{g \to \|g\| + V_1(g)\}$ is a norm of the subspace $\mathscr{D}_X$ of W_1, it results that U is a continuous linear mapping of $\mathscr{D}_X$ into $\mathfrak{E}_2$.

4.6. LEMMA. U *is a homomorphism of* $\mathscr{D}_X$ *into* $\mathfrak{E}_2$.

Proof. Take g and f in $\mathscr{D}_X$ and set $T_1 = Ug$, $T_2 = Uf$, and $T_3 = U(g \cdot f)$; from 4.4 we see that $T_u \in \mathfrak{C}_0$ $(u = 1, 2, 3)$ and that $T_1 \circ T_2 = T_3$. Set $E = B_0 \cap B_2$; consequently,

$$(39) \qquad\qquad T_1(T_2 y) = T_3 y \qquad\qquad (\text{when } y \in E) .$$

Set $i = 0, 2$. Since $\mathfrak{B}_i \supset E$ and $E \supset D$, the denseness of D in $\mathfrak{B}_i$ implies that E is dense in $\mathfrak{B}_i$; accordingly, we may identify $[E, \mathfrak{B}_i]$ with $\mathfrak{C}_i$. Set $n = 1, 2, 3$; from 4.4 and 4.5 we see that $T_n \in \mathfrak{C}_0 \cap \mathfrak{C}_2 = [E, \mathfrak{B}_0] \cap [E, \mathfrak{B}_2]$; therefore T_n is an endomorphism of E, and both $T_1 \circ T_2$ and T_3 belong to $[E, \mathfrak{B}_2]$. In view of (39), U is a homomorphism of $\mathscr{D}_X$ into $[E, \mathfrak{B}_2] = \mathfrak{C}_2$.

4.7. *Proof of 4.1.* Set $i = 0, 2$. From 4.4, 4.5 and 4.6 it follows that U is a continuous representation of $\mathscr{D}_X$ into $\mathfrak{C}_1$; 3.10 therefore shows that $\sigma(Uh; \mathfrak{C}_1) = h(\Sigma_m)$. Since M is the resolution of the identity for T, we have that

$$Uh = \int_X h \cdot dM = h(T) .$$

The identity (38) is now immediate.

Added in proof. An extension of Theorem III is found in the author's announcement *"Sur la permanence spectrale,"* C. R. Acad. Sci. Paris, **225** (1962), 1326–1328.

REFERENCES

1. N. Bourbaki, *Éléments de mathématique*, Intégration, Livre VI, Chapitre III, Actualités Scientifiques et Industrielles No. 1175, Paris, 1952.
2. N. Dunford, *Spectral operators*, Pacific J. Math., **4** (1954), 321–354.
3. C. Foias, *Une application des distributions vectorielles à la théorie spectrale*, Bull. Sci. Math., 2ᵉ serie, **84** (1960), 147–158.
4. C. J. A. Halberg, Jr. and A. E. Taylor, *On the spectra of linked operators*, Pacific J. Math., **6** (1956), 283–290.
5. C. J. A. Halberg, Jr. *The spectra of bounded linear operators on the sequence spaces,* Proceedings Amer. Math. Soc., **8** (1957), 728–732.
6. P. R. Halmos, *Introduction to Hilbert space*, New York 1951.
7. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Revised ed., Amer. Math. Soc. Colloquium Publications, vol. XXXI, Providence, 1957.
8. G. L. Krabbe, *Generalized measures whose values are operators into an intermediate space*, Bulletin Amer. Math. Soc., **68** (1962), 42–46.
9. ———, *Generalized measures whose values are operators into an intermediate space*, Mathematische Annalen to appear.
10. ———, *Réfractions non-hilbertiennes d'une transformation symétrique bornée* Studia Mathematica, **XX** (1961), 347–357.
11. ———, *Convolution operators that satisfy the spectral theorem*, Math. Zeitscher., **70** (1959), 446–462.
12. I. P. Natanson, *Theorie der Funktionen einer reellen Veränderlichen*, Berlin 1954.
13. F. Riesz et B, Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Paris, 1955.
14. H. H. Schaefer, *Spectral measures in locally convex algebras*, Acta Math., **107** (1962), 125–173.

15. J.-L. Lions, *Une construction d'espaces d'interpolation*, C. R. Acad. Sci. Paris, **251** (1960), 1853–1855.

16. G.-C. Rota, *On the spectra of singular boundary value problems*, J. of Math. and Mech., **10** (1961), 83–90.

FACULTÉ DES SCIENCES
RENNES, FRANCE

PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics

Vol. 13, No. 4   June, 1963