

Pacific Journal of Mathematics

A NOTE ON UNCOUNTABLY MANY DISKS

JOSEPH MARTIN

A NOTE ON UNCOUNTABLY MANY DISKS

JOSEPH MARTIN

R. H. Bing has shown [2] that E^3 (Euclidean three dimensional space) does not contain uncountably many mutually disjoint wild 2-spheres. J. R. Stallings has given an example [6] to show that E^3 does contain uncountably many mutually disjoint wild disks. It is the goal of this note to show that E^3 does not contain uncountably many mutually disjoint disks each of which fails to lie on a 2-sphere in E^3 . (A disk which fails to lie on a 2-sphere is necessarily wild.) For definitions the reader is referred to [1].

THEOREM 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists a disk D of the collection V such that D lies on a 2-sphere in E^3 .*

The proof of Theorem 1 follows immediately from the following three lemmas.

LEMMA 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists an uncountable subcollection V^* of V such that if D belongs to V^* , x is an interior point of D , ax is an arc intersecting D only in the point x , and ε is a positive number then there exists an uncountable subcollection V_1 of V^* such that if D_1 is an element of V_1 then (i) $D_1 \cap ax \neq \phi$ and (ii) there is a homeomorphism of D_1 onto D which moves no point more than ε .*

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3 . Let V' denote the subcollection of V defined as follows: D is an element of V' if and only if there exist a point x of $\text{Int } D$, an arc ax intersecting D only in x , and a positive number ε such that there is no uncountable subcollection V_1 of V such that if D_1 belongs to V_1 then (i) $D_1 \cap ax \neq \phi$ and (ii) there is a homeomorphism of D_1 onto D which moves no point more than ε .

It is clear that in order to establish Lemma 1 it is sufficient to show that the collection V' is countable. Suppose that V' is uncountable.

For each element D_α of V' let an arc a_α and a positive number ε_α be chosen such that (i) the common part of D_α and a_α is an end-point of a_α which is on the interior of D_α , and (ii) a_α intersects only a countable number of elements D of V such that there is a homeomorphism of D onto D_α which moves no point by more than ε_α .

Received January 15, 1963. This paper was written while the author was a post-doctoral fellow of The National Science Foundation.

Let ε be a positive number and V'' be an uncountable subcollection of V' such that if D_α is an element of V'' then $\varepsilon < \varepsilon_\alpha$.

Let E be a disk and v be an arc such that the common part of E and v is an endpoint of v which is on the interior of E . For each element D_α of V'' let h_α be a homeomorphism of $E \cup v$ onto $D_\alpha \cup a_\alpha$. Now $\{h_\alpha; D_\alpha \in V''\}$ with the distance function

$$D(h_\alpha, h_\beta) = \max_{t \in E \cup v} \rho(h_\alpha(t), h_\beta(t))$$

is a metric space. In [3] (Theorem 2) Borsuk shows that this metric space is separable. It follows that there exists an element D_{α_0} of V'' such that if δ is a positive number then $\{h_\beta; D(h_\beta, h_{\alpha_0}) < \delta\}$ is uncountable. Let h_{α_0} be denoted by h_0 , $h_0(E)$ be denoted by D_0 , and $h_0(v)$ be denoted by a_0 .

Let the endpoints of a_0 be denoted by x and y and assume that the notation is chosen so that $y \in \text{Int } D_0$. Let zyx be an arc such that $a_0 \subset zyx$ and zyx pierces D_0 at y . Let zwx be an arc in $E^3 - D_0$ such that $zwx \cap zyx = \{z, x\}$, and let J denote the simple closed curve $zyx \cup zwx$. Since $J \cup D_0 = \{y\}$ it follows that $Bd D_0$ links J .

Now let ε_1 be a positive number such that $2\varepsilon_1$ is less than the minimum of ε , $\text{dist}(J, Bd D_0)$, and $\text{dist}(zwx, D_0)$.

Let H be $\{h_\beta; D(h_\beta, h_0) < \varepsilon_1/2\}$, and let V''' be the set of all elements of V'' such that $D \in V'''$ if and only if there exists an element h of H such that $h(E) = D$. Now if D_1 and D_2 are two elements of V''' then there exists a homeomorphism of D_1 onto D_2 that moves no point more than ε_1 .

Suppose that D is an element of V''' . Then since $2\varepsilon_1 < \text{dist}(J, Bd D_0)$, $Bd D_0$ links J , and there is a homeomorphism of D_0 onto D which moves no point more than $\varepsilon_1/2$ it follows that $Bd D$ links J , and hence that $J \cap D \neq \emptyset$. Since $2\varepsilon_1 < \text{dist}(zwx, D_0)$, $D \cap zyx \neq \emptyset$.

Now for each element D_α of V''' let P_α be the greatest point of $D_\alpha \cap zyx$ in the order from z to x on zyx . Now there exists an element D_γ of V''' such that for uncountably many elements D_α of V''' , P_α is greater than P_γ . But since $2\varepsilon_1 < \text{dist}(x, D_0)$, $2\varepsilon_1 < \text{dist}(J, Bd D_0)$, and for each element D_α of V''' there is a homeomorphism of $D_0 \cup a_0$ onto $D_\alpha \cup a_\alpha$ which moves no point more than $\varepsilon_1/2$, it follows that a_γ intersects every element D_α of V''' such that P_α is greater than P_γ . This is because a_γ may be completed to a simple closed curve J' which links $Bd D_\alpha$ and which intersects D_α only in a_γ . Hence a_γ intersects uncountably many elements of the collection V''' . This is contradictory to the way in which a_γ was chosen and it follows that the collection V' is countable. This establishes Lemma 1.

LEMMA 2. *Suppose that V is an uncountable collection of mutu-*

ally disjoint disks in E^3 . Then there exists a disk D of the collection V such that D is locally tame at each point of $\text{Int } D$.

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3 . Let V^* be an uncountable subcollection of V satisfying the conclusion of Lemma 1. Let D be an element of the collection V^* and let p be an interior point of D . By Theorem 5 of [1] there exists a subdisk D' of D and a 2-sphere S in E^3 such that $p \in \text{Int } D'$ and $D' \subset S$. Without loss of generality it may be assumed that $ap \subset \text{Int } S$ and $pb \subset \text{Ext } S$. Now there exist sequences $D_1 D_2 \cdots$ and $C_1 C_2 \cdots$ of disks of the collection V^* such that for each i , (1) $D_i \cap ap \neq \phi$, (2) $C_i \cap pb \neq \phi$, and (3) there exist homeomorphisms f_i and g_i of D_i and C_i , respectively, onto D which move no point more than $1/i$.

Let D'' be a subdisk of D' such that $p \in \text{Int } D''$ and $D'' \subset \text{Int } D'$. Now without loss of generality it may be assumed that each of $f_1^{-1}(D'')$, $f_2^{-1}(D'') \cdots$ lies in $\text{Int } S$ and that each of $g_1^{-1}(D'')$, $g_2^{-1}(D'') \cdots$ lies in $\text{Ext } S$. It follows from Theorem 9 of [1] that S is locally tame at p and hence that D is locally tame at p . This establishes Lemma 2.

LEMMA 3. *If D is a disk in E^3 and D is locally tame at each point of $\text{Int } D$ then D lies on a 2-sphere in E^3 .*

Proof. Let D be a disk in E^3 which is locally tame at each point of $\text{Int } D$. It follows from [5] that there exists a homeomorphism h of E^3 onto itself such that $h(D)$ is locally polyhedral except on $h(\text{Bd } D)$. It follows from the proof of Lemma 5.1 of [4] that there exists a 2-sphere S in E^3 such that $h(D) \subset S$. Then $h^{-1}(S)$ is a 2-sphere in E^3 such that $D \subset h^{-1}(S)$. This establishes Lemma 3.

REFERENCES

1. R. H. Bing, *A surface is tame if its complement is 1-ULC*, Trans. Amer. Math. Soc., **101** (1961), 294-305.
2. ———, *Conditions under which a surface in E^3 is tame*, Fund. Math., **47** (1959), 105-139.
3. K. Borsuk, *Sur les rétracts*, Fund. Math., **17** (1931), 152-170.
4. O. G. Harrold, H. C. Griffith and E. E. Posey, *A characterization of tame curves in three space*, Trans. Amer. Math. Soc., **79** (1955), 12-34.
5. E. E. Moise, *Affine structures in 3-manifolds, IV. Piecewise linear approximations of homeomorphisms*, Ann. of Math., **55** (1952), 215-222.
6. J. R. Stallings, *Uncountably many wild disks*, Ann. of Math., **71** (1960), 185-186.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 13, No. 4

June, 1963

Dallas O. Banks, <i>Bounds for eigenvalues and generalized convexity</i>	1031
Jerrold William Bebernes, <i>A subfunction approach to a boundary value problem for ordinary differential equations</i>	1053
Woodrow Wilson Bledsoe and A. P. Morse, <i>A topological measure construction</i>	1067
George Clements, <i>Entropies of several sets of real valued functions</i>	1085
Sandra Barkdull Cleveland, <i>Homomorphisms of non-commutative *-algebras</i>	1097
William John Andrew Culmer and William Ashton Harris, <i>Convergent solutions of ordinary linear homogeneous difference equations</i>	1111
Ralph DeMarr, <i>Common fixed points for commuting contraction mappings</i>	1139
James Robert Dorroh, <i>Integral equations in normed abelian groups</i>	1143
Adriano Mario Garsia, <i>Entropy and singularity of infinite convolutions</i>	1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., <i>Convergence of extended Bernstein polynomials in the complex plane</i>	1171
Irving Leonard Glicksberg, <i>A remark on analyticity of function algebras</i>	1181
Charles John August Halberg, Jr., <i>Semigroups of matrices defining linked operators with different spectra</i>	1187
Philip Hartman and Nelson Onuchic, <i>On the asymptotic integration of ordinary differential equations</i>	1193
Isidore Heller, <i>On a class of equivalent systems of linear inequalities</i>	1209
Joseph Hersch, <i>The method of interior parallels applied to polygonal or multiply connected membranes</i>	1229
Hans F. Weinberger, <i>An effectless cutting of a vibrating membrane</i>	1239
Melvin F. Janowitz, <i>Quantifiers and orthomodular lattices</i>	1241
Samuel Karlin and Albert Boris J. Novikoff, <i>Generalized convex inequalities</i>	1251
Tilla Weinstein, <i>Another conformal structure on immersed surfaces of negative curvature</i>	1281
Gregers Louis Krabbe, <i>Spectral permanence of scalar operators</i>	1289
Shige Toshi Kuroda, <i>Finite-dimensional perturbation and a representation of scattering operator</i>	1305
Marvin David Marcus and Afton Herbert Cayford, <i>Equality in certain inequalities</i>	1319
Joseph Martin, <i>A note on uncountably many disks</i>	1331
Eugene Kay McLachlan, <i>Extremal elements of the convex cone of semi-norms</i>	1335
John W. Moon, <i>An extension of Landau's theorem on tournaments</i>	1343
Louis Joel Mordell, <i>On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$</i>	1347
Kenneth Roy Mount, <i>Some remarks on Fitting's invariants</i>	1353
Miroslav Novotný, <i>Über Abbildungen von Mengen</i>	1359
Robert Dean Ryan, <i>Conjugate functions in Orlicz spaces</i>	1371
John Vincent Ryff, <i>On the representation of doubly stochastic operators</i>	1379
Donald Ray Sherbert, <i>Banach algebras of Lipschitz functions</i>	1387
James McLean Sloss, <i>Reflection of biharmonic functions across analytic boundary conditions with examples</i>	1401
L. Bruce Treybig, <i>Concerning homogeneity in totally ordered, connected topological space</i>	1417
John Wermer, <i>The space of real parts of a function algebra</i>	1423
James Juei-Chin Yeh, <i>Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables</i>	1427
William P. Ziemer, <i>On the compactness of integral classes</i>	1437