Pacific Journal of Mathematics

A NOTE ON UNCOUNTABLY MANY DISKS

JOSEPH MARTIN

Vol. 13, No. 4 June 1963

A NOTE ON UNCOUNTABLY MANY DISKS

JOSEPH MARTIN

R. H. Bing has shown [2] that E^3 (Euclidean three dimensional space) does not contain uncountably many mutually disjoint wild 2-spheres. J. R. Stallings has given an example [6] to show that E^3 does contain uncountably many mutually disjoint wild disks. It is the goal of this note to show that E^3 does not contain uncountably many mutually disjoint disks each of which fails to lie on a 2-sphere in E^3. (A disk which fails to lie on a 2-sphere is necessarily wild.) For definitions the reader is referred to [1].

THEOREM 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists a disk D of the collection V such that D lies on a 2-sphere in E^3.*

The proof of Theorem 1 follows immediately from the following three lemmas.

LEMMA 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists an uncountable subcollection V^* of V such that if D belongs to V^*, x is an interior point of D, ax is an arc intersecting D only in the point x, and ε is a positive number then there exists an uncountable subcollection V_1 of V^* such that if D_1 is an element of V_1 then* (i) $D_1 \cap ax \neq \phi$ *and* (ii) *there is a homeomorphism of D_1 onto D which moves no point more than ε.*

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3. Let V' denote the subcollection of V defined as follows: D is an element of V' if and only if there exist a point x of Int D, an arc ax intersecting D only in x, and a positive number ε such that there is no uncountable subcollection V_1 of V such that if D_1 belongs to V_1 then (i) $D_1 \cap ax \neq \phi$ and (ii) there is a homeomorphism of D_1 onto D which moves no point more than ε.

It is clear that in order to establish Lemma 1 it is sufficient to show that the collection V' is countable. Suppose that V' is uncountable.

For each element D_α of V' let an arc a_α and a positive number ε_α be chosen such that (i) the common part of D_α and a_α is an end-point of a_α which is on the interior of D_α, and (ii) a_α intersects only a countable number of elements D of V such that there is a homeomorphism of D onto D_α which moves no point by more than ε_α.

Received January 15, 1963. This paper was witten while the author was a post-doctoral fellow of The National Science Foundation.

Let ε be a positive number and V'' be an uncountable subcollection of V' such that if D_α is an element of V'' then $\varepsilon < \varepsilon_\alpha$.

Let E be a disk and v be an arc such that the common part of E and v is an endpoint of v which is on the interior of E. For each element D_α of V'' let h_α be a homeomorphism of $E \cup v$ onto $D_\alpha \cup a_\alpha$. Now $\{h_\alpha; D_\alpha \in V''\}$ with the distance function

$$D(h_\alpha, h_\beta) = \max_{t \in E \cup v} \rho\big(h_\alpha(t), h_\beta(t)\big)$$

is a metric space. In [3] (Theorem 2) Borsuk shows that this metric space is separable. It follows that there exists an element D_{α_0} of V'' such that if δ is a positive number then $\{h_\beta;\ D(h_\beta, h_\alpha) < \delta\}$ is uncountable. Let h_{α_0} be denoted by h_0, $h_0(E)$ be denoted by D_0, and $h_0(v)$ be denoted by a_0.

Let the endpoints of a_0 be denoted by x and y and assume that the notation is chosen so that $y \in \operatorname{Int} D_0$. Let zyx be an arc such that $a_0 \subset zyx$ and zyx pierces D_0 at y. Let zwx be an arc in $E^3 - D_0$ such that $zwx \cap zyx = \{z, x\}$, and let J denote the simple closed curve $zyx \cup zwx$. Since $J \cup D_0 = \{y\}$ it follows that $Bd\ D_0$ links J.

Now let ε_1 be a positive number such that $2\varepsilon_1$ is less than the minimum of ε, dist $(J, Bd\ D_0)$, and dist (zwx, D_0).

Let H be $\{h_\beta;\ D(h_\beta, h_0) < \varepsilon_1/2\}$, and let V''' be the set of all elements of V'' such that $D \in V'''$ if and only if there exists an element h of H such that $h(E) = D$. Now if D_1 and D_2 are two elements of V''' then there exists a homeomorphism of D_1 onto D_2 that moves no point more than ε_1.

Suppose that D is an element of V'''. Then since $2\varepsilon_1 <$ dist $(J, Bd\ D_0)$, $Bd\ D_0$ links J, and there is a homeomorphism of D_0 onto D which moves no point more than $\varepsilon_1/2$ it follows that $Bd\ D$ links J, and hence that $J \cap D \neq \phi$. Since $2\varepsilon_1 <$ dist (zwx, D_0), $D \cap zyx \neq \phi$.

Now for each element D_α of V''' let P_α be the greatest point of $D_\alpha \cap zyx$ in the order from z to x on zyx. Now there exists an element D_γ of V''' such that for uncountably many elements D_α of V''', P_α is greater than P_γ. But since $2\varepsilon_1 <$ dist (x, D_0), $2\varepsilon_1 <$ dist $(J, Bd\ D_0)$, and for each element D_α of V''' there is a homeomorphism of $D_0 \cup a_0$ onto $D_\alpha \cup a_\alpha$ which moves no point more than $\varepsilon_1/2$, it follows that a_γ intersects every element D_α of V''' such that P_α is greater than P_γ. This is because a_γ may be completed to a simple closed curve J' which links $Bd\ D_\alpha$ and which intersects D_α only in a_γ. Hence a_γ intersects uncountably many elements of the collection V'''. This is contradictory to the way in which a_γ was chosen and it follows that the collection V' is countable. This establishes Lemma 1.

LEMMA 2. *Suppose that V is an uncountable collection of mutu-*

ally disjoint disks in E^3. Then there exists a disk D of the collection V such that D is locally tame at each point of Int D.

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3. Let V^* be an uncountable subcollection of V satisfying the conclusion of Lemma 1. Let D be an element of the collection V^* and let p be an interior point of D. By Theorem 5 of [1] there exists a subdisk D' of D and a 2-sphere S in E^3 such that $p \in \operatorname{Int} D'$ and $D' \subset S$. Without loss of generality it may be assumed that $ap \subset \operatorname{Int} S$ and $pb \subset \operatorname{Ext} S$. Now there exist sequences $D_1 D_2 \cdots$ and $C_1 C_2 \cdots$ of disks of the collection V^* such that for each i, (1) $D_i \cap ap \neq \phi$, (2) $C_i \cap pb \neq \phi$, and (3) there exist homeomorphisms f_i and g_i of D_i and C_i, respectively, onto D which move no point more than $1/i$.

Let D'' be a subdisk of D' such that $p \in \operatorname{Int} D''$ and $D'' \subset \operatorname{Int} D'$. Now without loss of generality it may be assumed that each of $f_1^{-1}(D'')$, $f_2^{-1}(D'') \cdots$ lies in $\operatorname{Int} S$ and that each of $g_1^{-1}(D'')$, $g_2^{-1}(D'') \cdots$ lies in $\operatorname{Ext} S$. It follows from Theorem 9 of [1] that S is locally tame at p and hence that D is locally tame at p. This establishes Lemma 2.

LEMMA 3. *If D is a disk in E^3 and D is locally tame at each point of* Int D *then D lies on a 2-sphere in E^3.*

Proof. Let D be a disk in E^3 which is locally tame at each point of Int D. It follows from [5] that there exists a homeomorphism h of E^3 onto itself such that $h(D)$ is locally polyhedral except on $h(Bd\ D)$. It follows from the proof of Lemma 5.1 of [4] that there exists a 2-sphere S in E^3 such that $h(D) \subset S$. Then $h^{-1}(S)$ is a 2-sphere in E^3 such that $D \subset h^{-1}(S)$. This establishes Lemma 3.

REFERENCES

1. R. H. Bing, *A surface is tame if its complement is* 1-*ULC*, Trans. Amer. Math. Soc., **101** (1961), 294–305.
2. ———, *Conditions under which a surface in E^3 is tame*, Fund. Math., **47** (1959), 105–139.
3. K. Borsuk, *Sur les rêtracts*, Fund. Math., **17** (1931), 152–170.
4. O. G. Harrold, H. C. Griffeth and E. E. Posey, *A characterization of tame curves in three space*, Trans. Amer. Math. Soc., **79** (1955), 12–34.
5. E. E. Moise, *Affine structures in 3-manifolds, IV. Piecewise linear approximations of homeomorphisms*, Ann. of Math., **55** (1952), 215–222.
6. J. R. Stallings, *Uncountably many wild disks*, Ann. of Math., **71** (1960), 185–186.

THE INSTITUTE FOR ADVANCED STUDY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 13, No. 4 June, 1963