

# Pacific Journal of Mathematics

**A NOTE ON UNCOUNTABLY MANY DISKS**

JOSEPH MARTIN

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R. H. Bing has shown [2] that  $E^3$  (Euclidean three dimensional space) does not contain uncountably many mutually disjoint wild 2-spheres. J. R. Stallings has given an example [6] to show that  $E^3$  does contain uncountably many mutually disjoint wild disks. It is the goal of this note to show that  $E^3$  does not contain uncountably many mutually disjoint disks each of which fails to lie on a 2-sphere in  $E^3$ . (A disk which fails to lie on a 2-sphere is necessarily wild.) For definitions the reader is referred to [1].

**THEOREM 1.** *If  $V$  is an uncountable collection of mutually disjoint disks in  $E^3$  then there exists a disk  $D$  of the collection  $V$  such that  $D$  lies on a 2-sphere in  $E^3$ .*

The proof of Theorem 1 follows immediately from the following three lemmas.

**LEMMA 1.** *If  $V$  is an uncountable collection of mutually disjoint disks in  $E^3$  then there exists an uncountable subcollection  $V^*$  of  $V$  such that if  $D$  belongs to  $V^*$ ,  $x$  is an interior point of  $D$ ,  $ax$  is an arc intersecting  $D$  only in the point  $x$ , and  $\varepsilon$  is a positive number then there exists an uncountable subcollection  $V_1$  of  $V^*$  such that if  $D_1$  is an element of  $V_1$  then (i)  $D_1 \cap ax \neq \phi$  and (ii) there is a homeomorphism of  $D_1$  onto  $D$  which moves no point more than  $\varepsilon$ .*

*Proof.* Let  $V$  be an uncountable collection of mutually disjoint disks in  $E^3$ . Let  $V'$  denote the subcollection of  $V$  defined as follows:  $D$  is an element of  $V'$  if and only if there exist a point  $x$  of  $\text{Int } D$ , an arc  $ax$  intersecting  $D$  only in  $x$ , and a positive number  $\varepsilon$  such that there is no uncountable subcollection  $V_1$  of  $V$  such that if  $D_1$  belongs to  $V_1$  then (i)  $D_1 \cap ax \neq \phi$  and (ii) there is a homeomorphism of  $D_1$  onto  $D$  which moves no point more than  $\varepsilon$ .

It is clear that in order to establish Lemma 1 it is sufficient to show that the collection  $V'$  is countable. Suppose that  $V'$  is uncountable.

For each element  $D_\alpha$  of  $V'$  let an arc  $a_\alpha$  and a positive number  $\varepsilon_\alpha$  be chosen such that (i) the common part of  $D_\alpha$  and  $a_\alpha$  is an end-point of  $a_\alpha$  which is on the interior of  $D_\alpha$ , and (ii)  $a_\alpha$  intersects only a countable number of elements  $D$  of  $V$  such that there is a homeomorphism of  $D$  onto  $D_\alpha$  which moves no point by more than  $\varepsilon_\alpha$ .

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Let  $\varepsilon$  be a positive number and  $V''$  be an uncountable subcollection of  $V'$  such that if  $D_\alpha$  is an element of  $V''$  then  $\varepsilon < \varepsilon_\alpha$ .

Let  $E$  be a disk and  $v$  be an arc such that the common part of  $E$  and  $v$  is an endpoint of  $v$  which is on the interior of  $E$ . For each element  $D_\alpha$  of  $V''$  let  $h_\alpha$  be a homeomorphism of  $E \cup v$  onto  $D_\alpha \cup a_\alpha$ . Now  $\{h_\alpha; D_\alpha \in V''\}$  with the distance function

$$D(h_\alpha, h_\beta) = \max_{t \in E \cup v} \rho(h_\alpha(t), h_\beta(t))$$

is a metric space. In [3] (Theorem 2) Borsuk shows that this metric space is separable. It follows that there exists an element  $D_{\alpha_0}$  of  $V''$  such that if  $\delta$  is a positive number then  $\{h_\beta; D(h_\beta, h_{\alpha_0}) < \delta\}$  is uncountable. Let  $h_{\alpha_0}$  be denoted by  $h_0$ ,  $h_0(E)$  be denoted by  $D_0$ , and  $h_0(v)$  be denoted by  $a_0$ .

Let the endpoints of  $a_0$  be denoted by  $x$  and  $y$  and assume that the notation is chosen so that  $y \in \text{Int } D_0$ . Let  $zyx$  be an arc such that  $a_0 \subset zyx$  and  $zyx$  pierces  $D_0$  at  $y$ . Let  $zwx$  be an arc in  $E^3 - D_0$  such that  $zwx \cap zyx = \{z, x\}$ , and let  $J$  denote the simple closed curve  $zyx \cup zwx$ . Since  $J \cup D_0 = \{y\}$  it follows that  $Bd D_0$  links  $J$ .

Now let  $\varepsilon_1$  be a positive number such that  $2\varepsilon_1$  is less than the minimum of  $\varepsilon$ ,  $\text{dist}(J, Bd D_0)$ , and  $\text{dist}(zwx, D_0)$ .

Let  $H$  be  $\{h_\beta; D(h_\beta, h_0) < \varepsilon_1/2\}$ , and let  $V'''$  be the set of all elements of  $V''$  such that  $D \in V'''$  if and only if there exists an element  $h$  of  $H$  such that  $h(E) = D$ . Now if  $D_1$  and  $D_2$  are two elements of  $V'''$  then there exists a homeomorphism of  $D_1$  onto  $D_2$  that moves no point more than  $\varepsilon_1$ .

Suppose that  $D$  is an element of  $V'''$ . Then since  $2\varepsilon_1 < \text{dist}(J, Bd D_0)$ ,  $Bd D_0$  links  $J$ , and there is a homeomorphism of  $D_0$  onto  $D$  which moves no point more than  $\varepsilon_1/2$  it follows that  $Bd D$  links  $J$ , and hence that  $J \cap D \neq \phi$ . Since  $2\varepsilon_1 < \text{dist}(zwx, D_0)$ ,  $D \cap zyx \neq \phi$ .

Now for each element  $D_\alpha$  of  $V'''$  let  $P_\alpha$  be the greatest point of  $D_\alpha \cap zyx$  in the order from  $z$  to  $x$  on  $zyx$ . Now there exists an element  $D_\gamma$  of  $V'''$  such that for uncountably many elements  $D_\alpha$  of  $V'''$ ,  $P_\alpha$  is greater than  $P_\gamma$ . But since  $2\varepsilon_1 < \text{dist}(x, D_0)$ ,  $2\varepsilon_1 < \text{dist}(J, Bd D_0)$ , and for each element  $D_\alpha$  of  $V'''$  there is a homeomorphism of  $D_0 \cup a_0$  onto  $D_\alpha \cup a_\alpha$  which moves no point more than  $\varepsilon_1/2$ , it follows that  $a_\gamma$  intersects every element  $D_\alpha$  of  $V'''$  such that  $P_\alpha$  is greater than  $P_\gamma$ . This is because  $a_\gamma$  may be completed to a simple closed curve  $J'$  which links  $Bd D_\alpha$  and which intersects  $D_\alpha$  only in  $a_\gamma$ . Hence  $a_\gamma$  intersects uncountably many elements of the collection  $V'''$ . This is contradictory to the way in which  $a_\gamma$  was chosen and it follows that the collection  $V'$  is countable. This establishes Lemma 1.

LEMMA 2. *Suppose that  $V$  is an uncountable collection of mutu-*

ally disjoint disks in  $E^3$ . Then there exists a disk  $D$  of the collection  $V$  such that  $D$  is locally tame at each point of  $\text{Int } D$ .

*Proof.* Let  $V$  be an uncountable collection of mutually disjoint disks in  $E^3$ . Let  $V^*$  be an uncountable subcollection of  $V$  satisfying the conclusion of Lemma 1. Let  $D$  be an element of the collection  $V^*$  and let  $p$  be an interior point of  $D$ . By Theorem 5 of [1] there exists a subdisk  $D'$  of  $D$  and a 2-sphere  $S$  in  $E^3$  such that  $p \in \text{Int } D'$  and  $D' \subset S$ . Without loss of generality it may be assumed that  $ap \subset \text{Int } S$  and  $pb \subset \text{Ext } S$ . Now there exist sequences  $D_1 D_2 \cdots$  and  $C_1 C_2 \cdots$  of disks of the collection  $V^*$  such that for each  $i$ , (1)  $D_i \cap ap \neq \phi$ , (2)  $C_i \cap pb \neq \phi$ , and (3) there exist homeomorphisms  $f_i$  and  $g_i$  of  $D_i$  and  $C_i$ , respectively, onto  $D$  which move no point more than  $1/i$ .

Let  $D''$  be a subdisk of  $D'$  such that  $p \in \text{Int } D''$  and  $D'' \subset \text{Int } D'$ . Now without loss of generality it may be assumed that each of  $f_1^{-1}(D'')$ ,  $f_2^{-1}(D'')$ ,  $\cdots$  lies in  $\text{Int } S$  and that each of  $g_1^{-1}(D'')$ ,  $g_2^{-1}(D'')$ ,  $\cdots$  lies in  $\text{Ext } S$ . It follows from Theorem 9 of [1] that  $S$  is locally tame at  $p$  and hence that  $D$  is locally tame at  $p$ . This establishes Lemma 2.

**LEMMA 3.** *If  $D$  is a disk in  $E^3$  and  $D$  is locally tame at each point of  $\text{Int } D$  then  $D$  lies on a 2-sphere in  $E^3$ .*

*Proof.* Let  $D$  be a disk in  $E^3$  which is locally tame at each point of  $\text{Int } D$ . It follows from [5] that there exists a homeomorphism  $h$  of  $E^3$  onto itself such that  $h(D)$  is locally polyhedral except on  $h(\text{Bd } D)$ . It follows from the proof of Lemma 5.1 of [4] that there exists a 2-sphere  $S$  in  $E^3$  such that  $h(D) \subset S$ . Then  $h^{-1}(S)$  is a 2-sphere in  $E^3$  such that  $D \subset h^{-1}(S)$ . This establishes Lemma 3.

## REFERENCES

1. R. H. Bing, *A surface is tame if its complement is 1-ULC*, Trans. Amer. Math. Soc., **101** (1961), 294-305.
2. ———, *Conditions under which a surface in  $E^3$  is tame*, Fund. Math., **47** (1959), 105-139.
3. K. Borsuk, *Sur les rétracts*, Fund. Math., **17** (1931), 152-170.
4. O. G. Harrold, H. C. Griffeth and E. E. Posey, *A characterization of tame curves in three space*, Trans. Amer. Math. Soc., **79** (1955), 12-34.
5. E. E. Moise, *Affine structures in 3-manifolds, IV. Piecewise linear approximations of homeomorphisms*, Ann. of Math., **55** (1952), 215-222.
6. J. R. Stallings, *Uncountably many wild disks*, Ann. of Math., **71** (1960), 185-186.



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