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**EXTREMAL ELEMENTS OF THE CONVEX CONE OF  
SEMI-NORMS**

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# EXTREMAL ELEMENTS OF THE CONVEX CONE OF SEMI-NORMS

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1. **Introduction.** Let  $L$  be a real linear space and let  $p$  be a real function on  $L$  such that (1)  $p(\lambda x) = |\lambda| p(x)$  for all  $x$  in  $L$  and all real  $\lambda$ , and  $p(x_1 + x_2) \leq p(x_1) + p(x_2)$  for all  $x_1$  and  $x_2$  in  $L$ , i.e. is a *semi-norm* on  $L$ . Since the sum of two semi-norms,  $p_1 + p_2$  and the positive scalar multiplication of a semi-norm,  $\lambda p$ ,  $\lambda > 0$  are semi-norms, the set of semi-norms on  $L$ ,  $C$  form a convex cone. Those  $p \in C$  such that if  $p = p_1 + p_2$  where  $p_1$  and  $p_2 \in C$  we have  $p_1$  and  $p_2$  proportional to  $p$  are *extremal element* of  $C$ , [1]. In this paper it is shown that  $p = |f|$ , where  $f$  is a real linear functional of  $L$  is an extremal element of  $C$ . For  $L$ , the plane it is shown that these are the only extremal elements of  $C$ . Since norms are semi-norms,  $C$  includes this interesting class of functionals.

2. **The main results.** The convex cone  $C$  and the convex cone  $-C$ , the negatives of the elements of  $C$  have only the zero semi-norm in common since semi-norms are nonnegative. The zero semi-norm is an extremal element if one wishes to allow in the definition the vertex of a convex cone to be an extremal element. Below only the nonzero elements are considered.

The following lemma which characterizes the nature of certain semi-norms will be used in obtaining the two main theorems.

LEMMA 1. *If  $p$  is a semi-norm on  $L$  such that the co-dimension of  $N(p) = 1$ , then  $p$  is of the form  $p = |f|$  where  $f$  is a linear functional on  $L$ .*

*Proof.* Let  $b \in L \setminus N(p)$ , where  $N(p)$  is the *null space* of  $p$ . Then any element  $x \in L$  can be written  $x = z + \lambda b$  where  $z \in N(p)$  and  $\lambda$  is real. Let  $f(x) = \lambda p(b)$ . Then clearly  $f$  is a linear functional on  $L$ . It shall now be shown that  $|f(x)| = p(x)$  for all  $x \in L$ . Notice that

$$|f(x)| = |f(z + \lambda b)| = |\lambda p(b)| = |\lambda| p(b).$$

Thus

$$|f(x)| = p(\lambda b) = p(z) + p(\lambda b) \geq p(z + \lambda b) = p(x).$$

The proof will be complete if it can be shown that the inequality

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cannot be a strict inequality for  $\lambda \neq 0$ .

Consider the case of the strict inequality occurring at  $z' + \lambda_0 b$  where  $\lambda_0 > 0$  and  $z' \in N(p)$ . The set  $U = \{x : p(x) \leq \lambda_0 p(b)\}$  is a convex circled set containing  $N(p)$  and  $\lambda_0 b$ . It follows that there exists  $\gamma \geq 1$  such that

$$p(\gamma(z' + \lambda_0 b)) = \gamma p(z' + \lambda_0 b) = \lambda_0 p(b)$$

and hence  $\gamma(z' + \lambda_0 b) \in U$ . Take  $\beta = (\gamma(1 - \alpha))/\alpha$  where  $\alpha = (\gamma - 1)/(2\gamma)$ . Then  $0 < \alpha < 1$  and

$$\alpha[\beta(-z')] + (1 - \alpha)[\gamma(z' + \lambda_0 b)] = (1 - \alpha)\gamma\lambda_0 b$$

belongs to  $U$  since  $-z'$  and  $\gamma(z' + \lambda_0 b) \in U$  and  $U$  is convex. Now

$$p((1 - \alpha)\gamma\lambda_0 b) = (1 - \alpha)\gamma p(\lambda_0 b) > \lambda_0 p(b)$$

since  $(1 - \alpha)\gamma = (1/2)(1 + \gamma) > 1$ , a contradiction since  $(1 - \alpha)\gamma\lambda_0 b \in U$ . Thus  $|f(x)| = p(x)$  for  $\lambda_0 > 0$ . Now for the case  $\lambda_0 < 0$  it follows from the above

$$|f(x)| = |f(z + \lambda_0 b)| = |-f(-z - \lambda_0 b)| = |f(-z - \lambda_0 b)|$$

and

$$|f(-z - \lambda_0 b)| = p(-z - \lambda_0 b) = p(z + \lambda_0 b).$$

Thus  $p(x) = |f(x)|$  for all  $x \in L$ .

It is now possible to prove the following theorem which shows that the absolute value of a real linear functional is an extremal element of  $C$ .

**THEOREM 1.** *If  $f$  is a real linear functional on  $L$ , then  $|f|$  is an extremal element of  $C$ .*

*Proof.* It is easy to check that  $|f|$  is subadditive and absolutely homogeneous and hence  $|f| \in C$ .

Suppose  $|f| = p_1 + p_2$  where  $p_1$  and  $p_2 \in C$ . Since  $p_1$  and  $p_2$  are nonnegative  $0 \leq p_i \leq |f|$ ,  $i = 1, 2$ . Thus when  $f(x) = 0$ ,  $p_i(x) = 0$ ,  $i = 1, 2$  and  $N(f) \subset N(p_i)$ ,  $i = 1, 2$ . Hence the co-dimension of  $p_1$  and  $p_2$  is less than or equal to one. If the co-dimension of  $N(p_i)$  is zero, then clearly  $p_1$  and  $p_2$  are proportional to  $|f|$ . If the co-dimension of  $N(p_1)$  is one then by Lemma 1, there exists a real linear functional  $f_1$  such that  $p_1 = |f_1|$ . Since  $N(f_1) = N(p_1) \supset N(f)$  it follows that  $\lambda_1 f = f_1$  for some real  $\lambda_1 \neq 0$ . Hence  $|\lambda_1| |f| = p_1$ . Thus  $p_1$  (and consequently  $p_2$ ) is proportioned to  $|f|$ , and hence  $|f|$  is an extremal element of  $C$ .

The following theorem shows that for the case  $L = E^2$ , the Euclidean plane, the only extremal elements for  $C$  are the semi-norms given in Theorem 1.

**THEOREM 2.** *Let  $L = E^2$ , then if  $p$  is an extremal element of  $C$ , there exists a linear functional  $f$  on  $L$  such that  $p = |f|$ .*

In order to prove this theorem it will be necessary to show that for  $p$  a semi-norm on  $L$  and  $p$  not of the form  $p = |f|$  then there exists semi-norms  $p_1$  and  $p_2$  on  $L$  such that  $p = p_1 + p_2$  and  $p_1$  (and consequently  $p_2$ ) is not proportional to  $p$ .

It follows from Lemma 1 that for a semi-norm  $p$  on  $L$  to not be of the form  $|f|$ , where  $f$  is a linear functional on  $L$  that the co-dimension of  $N(p)$  must be greater than one. Hence for arbitrary  $L$  and  $p$  an extremal element of  $C$  other than those of Theorem 1, then  $p$  must have the co-dimension of  $N(p) > 1$ . For  $L = E^2$  and  $p \in C$  such that the co-dimension of  $N(p) > 1$ , then  $p$  is a norm. Thus for the proof of Theorem 2 a non-proportional decomposition must be provided for all norms on  $E^2$ .

For  $p$  a norm on  $E^2 = \{(x_1, x_2)\}$ , the unit ball  $U(p) = \{x : p(x) \leq 1\}$  is a convex circled set containing the origin as a core point. There is no loss in generality in assuming that the segment  $(-1, 0), (1, 0)$  is a diameter of  $U(p)$ . This will mean that  $U(p)$  is contained in the closed unit disk with center at the origin. Let  $b_p(x_1) = \sup \{x_2 : (x_1, x_2) \in U(p)\}$ , the function giving the upper boundary of  $U(p)$ . Then  $b_p$  is a concave function on  $[-1, 1]$  and  $b_p(+1) = 0$ . The lower boundary is given by  $b'_p(x_1) = -b_p(-x_1)$  since  $p(-x) = p(x)$ . The next lemma gives a non-proportional decomposition of norms  $p$  such that the set  $U(p)$  is a parallelogram.

**LEMMA 2.** *Let  $p$  be a norm on  $E^2$  such that  $b_p(a_1) = b_1 > 0$  for some  $a_1$ ,  $-1 \leq a_1 \leq 1$  and  $b(x_1)$  is linear on  $[-1, a_1]$  and on  $[a_1, 1]$ , then  $p$  is not an extremal element of  $C$ .*

*Proof.* Let  $p_1((x_1, x_2)) = (1/b_1) |b_1 x_1 - a_1 x_1|$  and let  $p_2((x_1, x_2)) = (1/b_1) |x_2|$ . Then  $p_1$  and  $p_2 \in C$  since they are positive multiples of the absolute values of linear functionals. In order to see  $f = p_1 + p_2$  it is sufficient to show that  $p_1((x_1, b_p(x_1))) + p_2((x_1, b_p(x_1))) = 1$  for all  $x_1 \in [-1, 1]$ . This can be easily checked directly by substituting in the equations of the appropriate straight lines for  $b_p$ . Clearly  $p_1$  and  $p_2$  are not proportional to  $p$ .

The next lemma will give a non-proportional decomposition of a norm  $p$  such that the set  $U(p)$  is a six-sided polygon.

**LEMMA 3.** *Let  $p$  be a norm on  $E^2$  such that  $b_p(a_i) = b_i > 0$ ,*

$i = 1, 2$ , where  $-1 < a_1 < a_2 < 1$  and  $b_p$  is linear on  $[-1, a_1]$ ,  $[a_1, a_2]$  and on  $[a_2, 1]$ , then  $p$  is not an extremal element of  $C$ .

*Proof.* Let  $p_i((x_1, x_2)) = \alpha_i |a_i x_2 - b_i x_1|$ ,  $i = 1, 2$  and let  $p_3((x_1, x_2)) = \alpha_3 |x_2|$  where

$$\begin{aligned}\alpha_1 &= (b_2/\Delta) (b_1 - b_2 + |b_2 a_1 - a_2 b_1|), \\ \alpha_2 &= (b_1/\Delta) (b_2 - b_1 + |b_2 a_1 - a_2 b_1|), \\ \alpha_3 &= ((|b_2 a_1 - a_2 b_1|)/\Delta) (b_1 + b_2 - |b_2 a_1 - a_2 b_1|),\end{aligned}$$

and

$$\Delta = 2b_1 b_2 |b_2 a_1 - a_2 b_1|.$$

Then  $p = p_1 + p_2 + p_3$  gives a non-proportional decomposition of  $p$ .

Although an extension of this method will not be used in the proof of Theorem 2 it is worth noting at this point that this method of decomposing  $p$  can be used on any norm  $p$  such that  $U(p)$  is a polygon. For a polygon with  $2n + 2$  sides then  $b_p(x)$  is a concave polygonal function having vertices at  $\{(a_i, b_i)\}$ ,  $i = 1, 2, \dots, n$  where  $b_i > 0$  and  $-1 < a_1 < a_2 < \dots < a_n < 1$ . In this case set.

$$p(x) = \sum_{i=1}^n \alpha_i |a_i x_2 - b_i x_1| + \alpha_{n+1} |x_2|.$$

By substituting each of the points  $(a_i, b_i)$ ,  $i = 1, 2, \dots, n$  and  $(1, 0)$  in this equation we have  $n + 1$  linear equations in  $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$  since  $p((a_i, b_i)) = p((1, 0)) = 1$  for all  $i$ . By solving for the  $\alpha_i$  and noting that they are nonnegative we get the required decomposition of  $p$ . Notice that  $p$  is a finite sum of extremal elements of  $C$ .

For any norm  $p$  on  $E^2$  such that  $U(p)$  is not a polygon of less than six sides, that is  $p$  is a norm different from those considered in Lemmas 2 and 3, then there exist points of  $E^2$ ,  $x^{(1)} = (a_1, b_p(a_1))$ ,  $x^{(2)} = (a_2, b_p(a_2))$ ,  $-1 \leq a_1 < a_2 \leq 1$ ,  $a_2 - a_1 < 2$  such that  $b_p$  is not piecewise linear on  $[a_1, a_2]$  on three or fewer non-overlapping segments whose union is  $[a_1, a_2]$ . This means that  $p$  restricted to the line segment  $[x^{(1)}, x^{(2)}]$  is a strictly positive convex function that is not piecewise linear on three or fewer non-overlapping segments whose union is  $[x^{(1)}, x^{(2)}]$ .

Let  $C_{12}$  be the convex cone in  $E^2$  with vertex at the origin that is generated by  $[x^{(1)}, x^{(2)}]$  and let  $-C_{12}$  be the negatives of the vectors in  $C_{12}$ . Let  $U(p')$  be the closed convex hull of  $U(p) \setminus (C_{12} \cup (-C_{12}))$ . Let  $t_1$  and  $t_2$  be the tangent half-lines to  $U(p)$  at  $x^{(1)}$  and  $x^{(2)}$  respectively. These tangent half-lines are to be taken from the interior of  $C_{12}$ . Their intersection  $x^{(3)}$  will be a point in  $C_{12}$ . Let  $U(p'')$  be the closed convex circled set whose boundary  $U(p) \setminus (C_{12} \cup (-C_{12}))$  is the same as  $U(p)$  and whose boundary in  $C_{12}$  is  $[x^{(1)}, x^{(3)}] \cup [x^{(3)}, x^{(2)}]$ .

Let  $p'$  and  $p''$  be the semi-norms whose unit ball is  $U(p')$  and  $U(p'')$  respectively. Since  $U(p') \subset U(p) \subset U(p'')$  we have  $p'(x) \leq p(x) \leq p''(x)$  for all  $x \in E^2$ . Then if there exist semi-norms  $q_1$  and  $q_2$  on  $E^2$  such that  $p'(x) \leq q_i(x) \leq p''(x)$ ,  $i = 1, 2$  for all  $x \in E^2$  and such that on  $C_{12} \cup (-C_{12})$ ,

$$\alpha q_1(x) + (1 - \alpha)q_2(x) = p(x),$$

$0 < \alpha < 1$ ,  $q_1$  (and hence  $q_2$ ) is not equal to  $p$  on  $C_{12} \cup (-C_{12})$ , then  $p_1 = \alpha q_1$  and  $p_2 = (1 - \alpha)q_2$  will be semi-norms on  $E^2$  such that  $p_1 + p_2 = p$  and  $p_i$ ,  $i = 1, 2$  is not proportional to  $p$ . Thus the problem reduces to showing the existence of these semi-norms  $q_1$  and  $q_2$ .

Notice that it must be that  $q_1(x) = q_2(x) = p(x)$  on  $E^2 \setminus (C_{12} \cup (-C_{12}))$  and hence it remains to show that the definition of  $q_1$  and  $q_2$  can be satisfactorily extended as required above to all of  $E^2$ . If  $q_i$ ,  $i = 1, 2$ , restricted to the closed line segment  $[x^{(1)}, x^{(2)}]$  is defined to be a convex function such that  $q_i \neq p$  restricted to this same segment but agreeing with  $p$  at  $x^{(1)}$  and  $x^{(2)}$  and  $q_i \geq p'$  restricted to this same segment then  $q_i$  can be extended to a semi-norm on  $E^2$ . Consider the following: For  $x \in C_{12}$ ,  $x \neq 0$ , there is a  $\lambda > 0$  such that  $\lambda x$  belongs to  $[x^{(1)}, x^{(2)}]$ . Then take  $q_i(x) = (1/\lambda)q_i(\lambda x)$ . For  $x \in (-C_{12})$  take  $q_i(x) = q_i(-x)$  and take  $q_i(0) = 0$ . Now  $U(q_i)$  is a closed convex circled set since the central projection of a convex curve is convex. Hence  $q_i$  is a semi-norm. Notice  $U(p') \subset U(q_i) \subset U(p'')$  and thus  $p'(x) \leq q_i(x) \leq p''(x)$ ,  $i = 1, 2$  and  $x \in E^2$ . Notice also that the slopes of the half-tangents to  $q_i$ ,  $i = 1, 2$  restricted to  $[x^{(1)}, x^{(2)}]$  are finite even at the end-points. The possibility of defining  $q_i$ ,  $i = 1, 2$  on  $[x^{(1)}, x^{(2)}]$  as required above is assured by the following lemma.

**LEMMA 4.** *Let  $f$  be a real convex function on  $[a, b]$  such that the right-hand derivative at  $a, f'_+(a)$  and the left-hand derivative at  $b, f'_-(b)$  are finite. Suppose further that  $f$  is not piecewise linear on three or fewer non-overlapping segments whose union is  $[a, b]$ . Then there exist real convex functions  $f_1$  and  $f_2$  on  $[a, b]$  that differ from  $f$  on  $[a, b]$ , but have the same values and derivatives as  $f$  at the end-points and for some  $\alpha, 0 < \alpha < 1$ ,  $\alpha f_1(x) = (1 - \alpha)f_2(x) + f(x)$  for all  $x \in [a, b]$*

*Proof.* Let  $h(x) = f'_+(a)(x - a) + f(a)$ . Then  $F = (1/m)(f - h)$ , where  $m$  is the left-hand derivative of  $f - h$  at  $b$ , is a nonnegative convex function on  $[a, b]$  such that  $F(a) = 0$ ,  $F'_+(a) = 0$ , and  $F'_-(b) = 1$ . The right-hand derivative of  $F, F'_+$  is a nondecreasing right continuous function on  $[a, b]$ . Let  $F'_+$  be defined at  $b$  by  $F'_+(b) = F'_-(b)$ . Since  $f$  is not piecewise linear on three or fewer

non-overlapping segments whose union is  $[a, b]$  then the range of  $F'_+$  has at least four values, that is two besides 0 and 1. If there exist two non-decreasing right continuous functions  $F_i$ ,  $i = 1, 2$  on  $[a, b]$  such that  $F_i(a) = 0$ ,  $F_i(b) = 1$ ,  $F_i \neq F'_+$  on some subinterval of  $[a, b]$ ,

$$\alpha F_1(x) + (1 - \alpha)F_2(x) = F'_+(x),$$

$0 < \alpha < 1$  on  $[a, b]$ , and

$$\int_a^b F_i(x)dx = \int_a^b F'_+(x)dx$$

then the required functions  $f_i$  are given by

$$f_i(x) = h(x) + m \int_a^x F_i(t)dt,$$

$i = 1, 2$ .

Consider first the case of  $F'_+$  having at least three discontinuities. Let  $F'_+$  have positive jump discontinuities of  $\theta_i$  at  $c_i$ ,  $i = 1, 2, 3$  where  $a < c_1 < c_2 < c_3 < b$ . Take  $\theta = (1/2) \min(\theta_1, \theta_2, \theta_3)$ . Let

$$F_1(x) = F'_+(x) - \sigma_1,$$

when  $c_1 \leq x < c_2$ ,

$$F_1(x) = F'_+(x) + \sigma_2,$$

when  $c_2 \leq x < c_3$ , and  $F_1(x) = F'_+(x)$  elsewhere; and let

$$F_2(x) = F'_+(x) + \sigma_1,$$

when  $c_1 \leq x < c_2$ ,

$$F_2(x) = F'_+(x) - \sigma_2,$$

when  $c_2 \leq x < c_3$ , and  $F_2(x) = F'_+(x)$  elsewhere. Take  $\sigma_i$ ,  $i = 1, 2$  such that  $0 < \sigma_i < \theta$ ,  $\sigma_1(c_2 - c_1) = \sigma_2(c_3 - c_2)$ . It follows that  $F_1$  and  $F_2$  satisfy the above requirement for  $\alpha = (1/2)$ .

Now for the case where  $F'_+$  has less than three points of discontinuity it follows from the condition that  $F'_+$  has at least four range values that there exists a subinterval of  $[a, b]$  on which  $F'_+$  is continuous and non-constant. If now  $F_1$  and  $F_2$  can be defined on  $[a_1, b_1]$  as it was required that they be on  $[a, b]$  then  $F_1$  and  $F_2$  can be extended to  $[a, b]$  by taking  $F_1(x) = F_2(x) = F'_+(x)$  for  $x \in [a, b] \setminus [a_1, b_1]$ . It will follow that  $F_1$  and  $F_2$  obtained in this manner satisfy the above requirements. Thus it is sufficient to show the existence of  $F_1$  and  $F_2$  where  $F'_+$  is continuous on  $[a, b]$ .

Let us perform one further simplification. Let  $\bar{a} = \sup\{x : F'_+(x) = 0\}$  and let  $\bar{b} = \inf\{x : F'_+(x) = 1\}$ . Then  $a \leq \bar{a} < \bar{b} \leq b$ . Since  $F_1$  and  $F_2$  are non-decreasing,  $F_i(a) = 0$ , and  $F_i(b) = 1$ , and since  $\alpha F_1 + (1 - \alpha)F_2 = F'_+$  it follows that  $F_i(x) = 0$  on  $[a, \bar{a}]$  and  $F_i(x) = 1$  on  $[\bar{b}, b]$ ,  $i = 1, 2$ . Thus we may assume that  $0 < F'_+(x) < 1$  on the interior of the interval of definition. Take the interval  $[\bar{a}, \bar{b}]$  to be  $[0, 1]$  since there is no loss in generality in doing so.

The problem is now reduced to the following: Given  $F$  (instead of  $F'_+$  for simplicity) a continuous non-decreasing function on  $[0, 1]$  such that  $F(0) = 0$ ,  $F(1) = 1$  and  $0 < F(x) < 1$  for  $0 < x < 1$ . Show that there exist two functions  $F_1$  and  $F_2$  that have the same properties as  $F$  but are not  $F$  (that is, they differ from  $F$  at one point) and such that for some  $\alpha$ ,  $0 < \alpha < 1$ ,  $\alpha F_1 + (1 - \alpha)F_2 = F$  and such that

$$\int_0^1 F_i dx = \int_0^1 F dx$$

$i = 1, 2$ . Take  $\eta_1, \eta_2, \eta_3$  such that  $0 < \eta_1 < \eta_2 < \eta_3 < 1$  and let  $\xi_i, i = 1, 2, 3$  be such that  $F(\xi_i) = \eta_i$ . Then let

$$F_1(x) = (\eta_2/\eta_1) \min(F(x), \eta_1),$$

when  $0 \leq x \leq \xi_2$  and

$$F_1(x) = ((1 - \eta_2)/(1 - \eta_3))(\max(F(x), \eta_3) - \eta_3) + \eta_2,$$

when  $\xi_2 < x \leq 1$ . Let

$$F_2(x) = (\eta_2/(\eta_2 - \eta_1))(\max(F(x), \eta_1) - \eta_1),$$

when  $0 \leq x \leq \xi_2$  and

$$F_2(x) = ((1 - \eta_2)/(\eta_3 - \eta_2))(\min(F(x), \eta_3) - \eta_2) + \eta_2,$$

when  $\xi_2 < x \leq 1$ . Now  $F_1$  and  $F_2$  are continuous non-decreasing on  $[0, 1]$  such that  $F_i(0) = 0$ ,  $F_i(1) = 1$ ,  $i = 1, 2$  and  $F_i \neq F$ . Then

$$(\eta_1/\eta_2)F_1 + ((\eta_2 - \eta_1)/\eta_2)F_2 = F$$

on  $[0, \xi_2]$  and

$$((1 - \eta_3)/(1 - \eta_2))F_1 + ((\eta_3 - \eta_2)/(1 - \eta_2))F_2 = F$$

on  $(\xi_2, 1)$ . Take  $\eta_1 = (1/2)\eta_2$  and  $\eta_3 = (1/2)(1 + \eta_2)$ . Then it follows that  $f = (1/2)F_1 + (1/2)F_2$  on  $[0, 1]$ , with  $\eta_2$  arbitrary. It remains only to be shown that  $\eta_2$  can be chosen such that

$$\int_0^1 F_i dx = \int_0^1 F dx,$$



$i = 1, 2$  but this is assured if there exists a  $\xi_2$ ,  $0 < \xi_2 < 1$  such that

$$G(\xi_2) = \int_0^{\xi_2} (F_1 - F) dx = \int_{\xi_2}^1 (F - F_1) dx = H(\xi_2).$$

It can easily be checked that  $G(0) = H(1) = 0$ ,  $G$  is a not identically zero non-decreasing continuous function on  $[0, 1)$  and  $H$  is a not identically zero non-increasing continuous function on  $(0, 1]$ . Hence there exists  $\xi_2$ ,  $0 < \xi_2 < 1$  such that  $G(\xi_2) = H(\xi_2)$ .

3. **Remarks.** The argument in  $E^2$  that shows that the norms in  $E^2$  are not extremal elements of  $C$  shows also that for  $L$  general and  $p \in C$  such that the co-dimension of  $N(p) = 2$ , then  $p$  is not an extremal element of  $C$ . Thus for  $L$  general any extremal element of  $C$  other than those mentioned in Theorem 1 must be such that the co-dimension of its null space is greater than or equal to two.

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