EXTREMAL ELEMENTS OF THE CONVEX CONE OF SEMI-NORMS

EUGENE KAY MCLACHLAN
1. **Introduction.** Let $L$ be a real linear space and let $p$ be a real function on $L$ such that (1) $p(\lambda x) = |\lambda| \cdot p(x)$ for all $x$ in $L$ and all real $\lambda$, and $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ for all $x_1$ and $x_2$ in $L$, i.e. is a semi-norm on $L$. Since the sum of two semi-norms, $p_1 + p_2$ and the positive scalar multiplication of a semi-norm, $\lambda p$, $\lambda > 0$ are semi-norms, the set of semi-norms on $L$, $C$ form a convex cone. Those $p \in C$ such that if $p = p_1 + p_2$ where $p_1$ and $p_2 \in C$ we have $p_1$ and $p_2$ proportional to $p$ are extremal elements of $C$, [1]. In this paper it is shown that $p = |f|$, where $f$ is a real linear functional of $L$ is an extremal element of $C$. For $L$, the plane it is shown that these are the only extremal elements of $C$. Since norms are semi-norms, $C$ includes this interesting class of functionals.

2. **The main results.** The convex cone $C$ and the convex cone $-C$, the negatives of the elements of $C$ have only the zero semi-norm in common since semi-norms are nonnegative. The zero semi-norm is an extremal element if one wishes to allow in the definition the vertex of a convex cone to be an extremal element. Below only the nonzero elements are considered.

The following lemma which characterizes the nature of certain semi-norms will be used in obtaining the two main theorems.

**Lemma 1.** If $p$ is a semi-norm on $L$ such that the co-dimension of $N(p) = 1$, then $p$ is of the form $p = |f|$ where $f$ is a linear functional on $L$.

Proof. Let $b \in L \setminus N(p)$, where $N(p)$ is the null space of $p$. Then any element $x \in L$ can be written $x = z + \lambda b$ where $z \in N(p)$ and $\lambda$ is real. Let $f(x) = \lambda p(b)$. Then clearly $f$ is a linear functional on $L$. It shall now be shown that $|f(x)| = p(x)$ for all $x \in L$. Notice that

$$|f(x)| = |f(z + \lambda b)| = |\lambda p(b)| = |\lambda| \cdot p(b).$$

Thus

$$|f(x)| = p(\lambda b) = p(z) + p(\lambda b) \geq p(z + \lambda b) = p(x).$$

The proof will be complete if it can be shown that the inequality...
cannot be a strict inequality for \( \lambda \neq 0 \).

Consider the case of the strict inequality occurring at \( z' - \lambda_\circ b \) where \( \lambda_\circ > 0 \) and \( z' \in N(p) \). The set \( U = \{x : p(x) \leq \lambda_\circ p(b)\} \) is a convex set containing \( N(p) \) and \( \lambda_\circ b \). It follows that there exists \( \gamma \geq 1 \) such that

\[
p(\gamma(z' + \lambda_\circ b)) = \gamma p(z' + \lambda_\circ b) = \lambda_\circ p(b)
\]

and hence \( \gamma(z' + \lambda_\circ b) \in U \). Take \( \beta = (\gamma(1 - \alpha))/\alpha \) where \( \alpha = (1 - \lambda)/(2\gamma) \). Then \( 0 < \alpha < 1 \) and

\[
\alpha[\beta(-z')] + (1 - \alpha)[\gamma(z' + \lambda_\circ b)] = (1 - \alpha)\gamma\lambda_\circ b
\]

belongs to \( U \) since \( -z' \) and \( \gamma(z' + \lambda_\circ b) \in U \) and \( U \) is convex. Now

\[
p((1 - \alpha)\gamma\lambda_\circ b) = (1 - \alpha)\gamma p(\lambda_\circ b) > \lambda_\circ p(b)
\]

since \( (1 - \alpha)\gamma = (1/2)(1 + \gamma) > 1 \), a contradiction since \( (1 - \alpha)\gamma\lambda_\circ b \in U \). Thus \( |f(x)| = p(x) \) for \( \lambda_\circ > 0 \). Now for the case \( \lambda_\circ < 0 \) it follows from the above

\[
|f(x)| = |f(z + \lambda_\circ b)| = |f(-z - \lambda_\circ b)| = |f(z - \lambda_\circ b)|
\]

and

\[
|f(-z - \lambda_\circ b)| = p(-z - \lambda_\circ b) = p(z + \lambda_\circ b).
\]

Thus \( p(x) = |f(x)| \) for all \( x \in L \).

It is now possible to prove the following theorem which shows that the absolute value of a real linear functional is an extremal element of \( C \).

**Theorem 1.** If \( f \) is a real linear functional on \( L \), then \( |f| \) is an extremal element of \( C \).

**Proof.** It is easy to check that \( |f| \) is subadditive and absolutely homogeneous and hence \( |f| \in C \).

Suppose \( |f| = p_1 + p_2 \) where \( p_1 \) and \( p_2 \in C \). Since \( p_1 \) and \( p_2 \) are nonnegative \( 0 \leq p_i \leq |f| \), \( i = 1, 2 \). Thus when \( f(x) = 0 \), \( p_i(x) = 0 \), \( i = 1, 2 \) and \( N(f) \subset N(p_i) \), \( i = 1, 2 \). Hence the co-dimension of \( p_1 \) and \( p_2 \) is less than or equal to one. If the co-dimension of \( N(p_i) \) is zero, then clearly \( p_1 \) and \( p_2 \) are proportional to \( |f| \). If the co-dimension of \( N(p_i) \) is one then by Lemma 1, there exists a real linear functional \( f_1 \) such that \( p_i = |f_1| \). Since \( N(f_1) = N(p_i) \subset N(f) \) it follows that \( \lambda_1 f = f_1 \) for some real \( \lambda_1 \neq 0 \). Hence \( |\lambda_1| |f| = p_i \). Thus \( p_i \) (and consequently \( p_2 \)) is proportioned to \( |f| \), and hence \( |f| \) is an extremal element of \( C \).
The following theorem shows that for the case $L = E^2$, the Euclidean plane, the only extremal elements for $C$ are the semi-norms given in Theorem 1.

**Theorem 2.** Let $L = E^2$, then if $p$ is an extremal element of $C$, there exists a linear functional $f$ on $L$ such that $p = |f|$. 

In order to prove this theorem it will be necessary to show that for $p$ a semi-norm on $L$ and $p$ not of the form $p = |f|$ then there exists semi-norms $p_1$ and $p_2$ on $L$ such that $p = p_1 + p_2$ and $p_1$ (and consequently $p_2$) is not proportional to $p$.

It follows from Lemma 1 that for a semi-norm $p$ on $L$ to not be of the form $|f|$, where $f$ is a linear functional on $L$ that the co-dimension of $N(p)$ must be greater than one. Hence for arbitrary $L$ and $p$ an extremal element of $C$ other than those of Theorem 1, then $p$ must have the co-dimension of $N(p) > 1$. For $L = E^2$ and $p \in C$ such that the co-dimension of $N(p) > 1$, then $p$ is a norm. Thus for the proof of Theorem 2 a non-proportional decomposition must be provided for all norms on $E^2$.

For $p$ a norm on $E^2 = \{(x_1, x_2)\}$, the unit ball $U(p) = \{x : p(x) \leq 1\}$ is a convex circled set containing the origin as a core point. There is no loss in generality in assuming that the segment $(-1, 0), (1, 0)$ is a diameter of $U(p)$. This will mean that $U(p)$ is contained in the closed unit disk with center at the origin. Let $b_p(x) = \sup \{x_2 : (x_1, x_2) \in U(p)\}$, the function giving the upper boundary of $U(p)$. Then $b_p$ is a concave function on $[-1, 1]$. The lower boundary is given by $b'_p(x) = -b_p(-x)$ since $p(-x) = p(x)$. The next lemma gives a non-proportional decomposition of norms $p$ such that the set $U(p)$ is a parallelogram.

**Lemma 2.** Let $p$ be a norm on $E^2$ such that $b_p(a_i) = b_1 > 0$ for some $a_i$, $-1 \leq a_i \leq 1$ and $b(x_1)$ is linear on $[-1, a_1]$ and on $[a_1, 1]$, then $p$ is not an extremal element of $C$.

**Proof.** Let $p_1((x_1, x_2)) = (1/b_1)|b_1x_1 - a_1x_1|$ and let $p_2((x_1, x_2)) = (1/b_1)|x_2|$. Then $p_1$ and $p_2 \in C$ since they are positive multiples of the absolute values of linear functionals. In order to see $f = p_1 + p_2$ it is sufficient to show that $p_1((x_1, b_p(x_1))) + p_2((x_1, b_p(x_1))) = 1$ for all $x_1 \in [-1, 1]$. This can be easily checked directly by substituting in the equations of the appropriate straight lines for $b_p$. Clearly $p_1$ and $p_2$ are not proportional to $p$.

The next lemma will give a non-proportional decomposition of a norm $p$ such that the set $U(p)$ is a six-sided polygon.

**Lemma 3.** Let $p$ be a norm on $E^2$ such that $b_p(a_i) = b_1 > 0$, ...
\[ i = 1, 2, \text{ where } -1 < a_1 < a_2 < 1 \text{ and } b_p \text{ is linear on } [-1, a_1], [a_1, a_2] \text{ and on } [a_2, 1], \text{ then } p \text{ is not an extremal element of } C. \]

**Proof.** Let \( p_i((x_1, x_2)) = \alpha_i \left| a_i x_2 - b_i x_1 \right|, \ i = 1, 2 \) and let \( p_3((x_1, x_2)) = \alpha_3 \left| x_2 \right| \) where

\[
\begin{align*}
\alpha_1 &= \left( b_2 / \Delta \right) \left( b_1 - b_2 + \left| b_2 a_1 - a_2 b_1 \right| \right), \\
\alpha_2 &= \left( b_1 / \Delta \right) \left( b_2 - b_1 + \left| b_2 a_1 - a_2 b_1 \right| \right), \\
\alpha_3 &= \left( \left| b_2 a_1 - a_2 b_1 \right| / \Delta \right) \left( b_1 + b_2 - \left| b_2 a_1 - a_2 b_1 \right| \right),
\end{align*}
\]

and

\[ \Delta = 2b_1 b_2 \left| b_2 a_1 - a_2 b_1 \right|. \]

Then \( p = p_1 + p_2 + p_3 \) gives a non-proportional decomposition of \( p \).

Although an extension of this method will not be used in the proof of Theorem 2 it is worth noting at this point that this method of decomposing \( p \) can be used on any norm \( p \) such that \( U(p) \) is a polygon. For a polygon with \( 2n + 2 \) sides then \( b_p(x) \) is a concave polygonal function having vertices at \( \{(a_i, b_i)\}, \ i = 1, 2, \cdots, n \) where \( b_i > 0 \) and \( -1 < a_1 < a_2 < \cdots < a_n < 1 \). In this case set

\[ p(x) = \sum_{i=1}^{n} \alpha_i \left| a_i x_2 - b_i x_1 \right| + \alpha_{n+1} \left| x_2 \right|. \]

By substituting each of the points \((a_i, b_i), \ i = 1, 2, \cdots, n \) and \((1, 0)\) in this equation we have \( n + 1 \) linear equations in \( \alpha_1, \alpha_2, \cdots, \alpha_{n+1} \) since \( p((a_i, b_i)) = p((1, 0)) = 1 \) for all \( i \). By solving for the \( \alpha_i \) and nothing that they are nonnegative we get the required decomposition of \( p \). Notice that \( p \) is a finite sum of extremal elements of \( C \).

For any norm \( p \) on \( E^2 \) such that \( U(p) \) is not a polygon of less than six sides, that is \( p \) is a norm different from those considered in Lemmas 2 and 3, then there exist points of \( E^2 \), \( x^{(1)} = (a_1, b_p(a_1)), \ x^{(2)} = (a_2, b_p(a_2)), \ -1 \leq a_1 < a_2 \leq 1, \ a_2 - a_1 < 2 \) such that \( b_p \) is not piecewise linear on \( [a_1, a_2] \) on three or fewer non-overlapping segments whose union is \( [a_1, a_2] \). This means that \( p \) restricted to the line segment \( [x^{(1)}, x^{(2)}] \) is a strictly positive convex function that is not piecewise linear on three or fewer non-overlapping segments whose union is \( [x^{(1)}, x^{(2)}] \).

Let \( C_{12} \) be the convex cone in \( E^2 \) with vertex at the origin that is generated by \([x^{(1)}, x^{(2)}]\) and let \(-C_{12}\) be the negatives of the vectors in \( C_{12} \). Let \( U(p') \) be the closed convex hull of \( U(p) \cap (C_{12} \cup (-C_{12})) \). Let \( t_1 \) and \( t_2 \) be the tangent half-lines to \( U(p) \) at \( x^{(1)} \) and \( x^{(2)} \) respectively. These tangent half-lines are to be taken from the interior of \( C_{12} \). Their intersection \( x^{(3)} \) will be a point in \( C_{12} \). Let \( U(p'') \) be the closed convex circled set whose boundary \( U(p) \cap (C_{12} \cup (-C_{12})) \) is the same as \( U(p) \) and whose boundary in \( C_{12} \) is \([x^{(1)}, x^{(3)}] \cup [x^{(3)}, x^{(2)}] \).
Let $p'$ and $p''$ be the semi-norms whose unit ball is $U(p')$ and $U(p'')$ respectively. Since $U(p') \subset U(p) \subset U(p'')$, we have $p'(x) \leq p(x) \leq p''(x)$ for all $x \in E^2$. Then if there exist semi-norms $q_1$ and $q_2$ on $E^2$ such that $p'(x) \leq q_i(x) \leq p''(x)$, $i = 1, 2$ for all $x \in E^2$ and such that on $C_{12} \cup (-C_{12})$,

$$\alpha q_1(x) + (1 - \alpha)q_2(x) = p(x),$$

$0 < \alpha < 1$, $q_i$ (and hence $q_2$) is not equal to $p$ on $C_{12} \cup (-C_{12})$, then $p_i = \alpha q_1$ and $p_2 = (1 - \alpha)q_2$ will be semi-norms on $E^2$ such that $p_i + p_2 = p$ and $p_i$, $i = 1, 2$ is not proportional to $p$. Thus the problem reduces to showing the existence of these semi-norms $q_1$ and $q_2$.

Notice that it must be that $q_i(x) = q_4(x) = p(x)$ on $E^2 \setminus (C_{12} \cup (-C_{12}))$ and hence it remains to show that the definition of $q_1$ and $q_2$ can be satisfactorily extended as required above to all of $E^2$. If $q_i$, $i = 1, 2$, restricted to the closed line segment $[x^{(1)}, x^{(2)}]$ is defined to be a convex function such that $q_i \neq p$ restricted to this same segment but agreeing with $p$ at $x^{(1)}$ and $x^{(2)}$, $q_i \geq p'$ restricted to this same segment then $q_i$ can be extended to a semi-norm on $E^2$. Consider the following: For $x \in C_{12}$, $x \neq 0$, there is a $\lambda > 0$ such that $\lambda x$ belongs to $[x^{(1)}, x^{(2)}]$. Then take $q_i(x) = (1/\lambda)q_i(\lambda x)$. For $x \in (-C_{12})$ take $q_i(x) = q_i(-x)$ and take $q_i(0) = 0$. Now $U(q_i)$ is a closed convex circled set since the central projection of a convex curve is convex. Hence $q_i$ is a semi-norm. Notice $U(p') \subset U(q_i) \subset U(p'')$ and thus $p'(x) \leq q_i(x) \leq p''(x)$, $i = 1, 2$ and $x \in E^2$. Notice also that the slopes of the half-tangents to $q_i$, $i = 1, 2$ restricted to $[x^{(1)}, x^{(2)}]$ are finite even at the end-points. The possibility of defining $q_i$, $i = 1, 2$ on $[x^{(1)}, x^{(2)}]$ as required above is assured by the following lemma.

**Lemma 4.** Let $f$ be a real convex function on $[a, b]$ such that the right-hand derivative at $a$, $f'_+(a)$ and the left-hand derivative at $b$, $f'_-(b)$ are finite. Suppose further that $f$ is not piecewise linear on three or fewer non-overlapping segments whose union is $[a, b]$. Then there exist real convex functions $f_i$ and $f_+$ on $[a, b]$ that differ from $f$ on $[a, b]$, but have the same values and derivatives as $f$ at the end-points and for some $\alpha$, $0 < \alpha < 1$, $\alpha f'_i(x) = (1 - \alpha)f'_+(x) + f(x)$ for all $x \in [a, b]$.

**Proof.** Let $h(x) = f'_+(a)(x - a) + f(a)$. Then $F = (1/m)(f - h)$, where $m$ is the left-hand derivative of $f - h$ at $b$, is a nonnegative convex function on $[a, b]$ such that $F(a) = 0$, $F'_+(a) = 0$, and $F'_-(b) = 1$. The right-hand derivative of $F$, $F'_+$ is a nondecreasing right continuous function on $[a, b]$. Let $F'_+$ be defined at $b$ by $F'_+(b) = F'_+(b)$. Since $f$ is not piecewise linear on three or fewer
non-overlapping segments whose union is \([a, b]\) then the range of \(F'_+\) has at least four values, that is two besides 0 and 1. If there exist two non-decreasing right continuous functions \(F_i, \ i = 1, 2\) on \([a, b]\) such that \(F_i(a) = 0, \ F_i(b) = 1, \ F_i \neq F'_+\) on some subinterval of \([a, b]\),

\[
\alpha F_i(x) + (1 - \alpha)F_2(x) = F'_+(x),
\]

then the required functions \(f_i\) are given by

\[
f_i(x) = h(x) + m \int_a^x F_i(t)dt,
\]

\(i = 1, 2\).

Consider first the case of \(F'_+\) having at least three discontinuities. Let \(F'_+\) have positive jump discontinuities of \(\theta_i\) at \(c_i, \ i = 1, 2, 3\) where \(a < c_1 < c_2 < c_3 < b\). Take \(\theta = (1/2) \min(\theta_1, \theta_2, \theta_3)\). Let

\[
F_i(x) = F'_+(x) - \sigma_i,
\]

when \(c_1 \leq x < c_2\),

\[
F_i(x) = F'_+(x) + \sigma_3,
\]

when \(c_2 \leq x < c_3\), and \(F_3(x) = F'_+(x)\) elsewhere; and let

\[
F_3(x) = F'_+(x) + \sigma_3,
\]

when \(c_1 \leq x < c_2\),

\[
F_3(x) = F'_+(x) - \sigma_3,
\]

when \(c_2 \leq x < c_3\), and \(F_3(x) = F'_+(x)\) elsewhere. Take \(\sigma_i, \ i = 1, 2\) such that \(0 < \sigma_i < \theta, \ \sigma_i(c_i - c) = \sigma(c_i - c)\). It follows that \(F_1\) and \(F_2\) satisfy the above requirement for \(\alpha = (1/2)\).

Now for the case where \(F'_+\) has less than three points of discontinuity it follows from the condition that \(F'_+\) has at least four range values that there exists a subinterval of \([a, b]\) on which \(F'_+\) is continuous and non-constant. If now \(F_1\) and \(F_2\) can be defined on \([a, b]\) as it was required that they be on \([a, b]\) then \(F_1\) and \(F_2\) can be extended to \([a, b]\) by taking \(F_i(x) = F'_+(x)\) for \(x \in [a, b]\) \([a, b]\). It will follow that \(F_1\) and \(F_2\) obtained in this manner satisfy the above requirements. Thus it is sufficient to show the existence of \(F_1\) and \(F_2\) where \(F'_+\) is continuous on \([a, b]\).
Let us perform one further simplification. Let \( \bar{a} = \sup \{ x : F'_+(x) = 0 \} \) and let \( \bar{b} = \inf \{ x : F'_+(x) = 1 \} \). Then \( a \leq \bar{a} < \bar{b} \leq b \). Since \( F_1 \) and \( F_2 \) are non-decreasing, \( F_i(\alpha) = 0 \), and \( F_i(b) = 1 \), and since \( \alpha F_1 + (1 - \alpha) F_2 = F'_+ \) it follows that \( F_i(x) = 0 \) on \([a, \bar{a}]\) and \( F_i(x) = 1 \) on \([\bar{b}, b]\), \( i = 1, 2 \). Thus we may assume that \( 0 < F'_+ (x) < 1 \) on the interior of the interval of definition. Take the interval \([\bar{a}, \bar{b}]\) to be \([0, 1]\) since there is no loss in generality in doing so.

The problem is now reduced to the following: Given \( F \) (instead of \( F'_+ \) for simplicity) a continuous non-decreasing function on \([0, 1]\) such that \( F(0) = 0 \), \( F(1) = 1 \) and \( 0 < F(x) < 1 \) for \( 0 < x < 1 \). Show that there exist two functions \( F_1 \) and \( F_2 \) that have the same properties as \( F \) but are not \( F \) (that is, they differ from \( F \) at one point) and such that for some \( \alpha \), \( 0 < \alpha < 1 \), \( \alpha F_1 + (1 - \alpha) F_2 = F' \) and such that

\[
\int_0^1 F_i \, dx = \int_0^1 F \, dx
\]

\( i = 1, 2 \). Take \( \eta_1, \eta_2, \eta_3 \) such that \( 0 < \eta_1 < \eta_2 < \eta_3 < 1 \) and let \( \xi_i, i = 1, 2, 3 \) be such that \( F(\xi_i) = \eta_i \). Then let

\[
F_1(x) = (\eta_3/\eta_1) \min (F(x), \eta_1),
\]

when \( 0 \leq x \leq \xi_2 \) and

\[
F_1(x) = ((1 - \eta_2)/(1 - \eta_3))(\max (F(x), \eta_3) - \eta_3) + \eta_2,
\]

when \( \xi_2 < x \leq 1 \). Let

\[
F_2(x) = (\eta_3/\eta_2)(\eta_2 - \eta_1))\max (F(x), \eta_1) - \eta_1,
\]

when \( 0 \leq x \leq \xi_3 \) and

\[
F_2(x) = ((1 - \eta_3)/(1 - \eta_2))(\min (F(x), \eta_3) - \eta_3) + \eta_2,
\]

when \( \xi_2 < x \leq 1 \). Now \( F_1 \) and \( F_2 \) are continuous non-decreasing on \([0, 1]\) such that \( F_i(0) = 0 \), \( F_i(1) = 1 \), \( i = 1, 2 \) and \( F_i \neq F \). Then

\[
(\eta_3/\eta_2)F_1 + ((\eta_2 - \eta_1)/\eta_3)F_2 = F
\]

on \([0, \xi_2]\) and

\[
((1 - \eta_3)/(1 - \eta_2))F_1 + ((\eta_3 - \eta_2)/(1 - \eta_3))F_2 = F
\]

on \((\xi_2, 1]\). Take \( \eta_1 = (1/2)\eta_2 \) and \( \eta_3 = (1/2)(1 + \eta_3) \). Then it follows that \( f = (1/2)F_1 + (1/2)F_2 \) on \([0, 1]\), with \( \eta_2 \) arbitrary. It remains only to be shown that \( \eta_2 \) can be chosen such that

\[
\int_0^1 F_i \, dx = \int_0^1 F \, dx,
\]

\( i = 1, 2 \).
i = 1, 2 but this is assured if there exists a ξ₂, 0 < ξ₂ < 1 such that

\[ G(ξ₂) = \int_{0}^{ξ₂} (F'_1 - F') \, dx = \int_{ξ₂}^{1} (F' - F') \, dx = H(ξ₂). \]

It can easily be checked that \( G(0) = H(1) = 0 \), \( G \) is a not identically zero non-decreasing continuous function on \([0, 1)\) and \( H \) is a not identically zero non-increasing continuous function on \((0, 1]\). Hence there exists \( ξ₂, 0 < ξ₂ < 1 \) such that \( G(ξ₂) = H(ξ₂) \).

3. **Remarks.** The argument in \( E² \) that shows that the norms in \( E² \) are not extremal elements of \( C \) shows also that for \( L \) general and \( p \in C \) such that the co-dimension of \( N(p) = 2 \), then \( p \) is not an extremal element of \( C \). Thus for \( L \) general any extremal element of \( C \) other than those mentioned in Theorem 1 must be such that the co-dimension of its null space is greater than or equal to two.

**REFERENCES**


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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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