

# Pacific Journal of Mathematics

**CONJUGATE FUNCTIONS IN ORLICZ SPACES**

ROBERT DEAN RYAN

# CONJUGATE FUNCTIONS IN ORLICZ SPACES

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1. The purpose of this paper is to prove the following results:

THEOREM 1. *Let*

$$\tilde{f}(x) = -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan(1/2)t} dt = \lim_{\varepsilon \rightarrow +0} \left\{ -\frac{1}{\pi} \int_\varepsilon^\pi \right\}.$$

*The mapping  $f \rightarrow \tilde{f}$  is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.*

Beginning with the classical result by M. Riesz for the  $L_p$  spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116-118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of  $f \rightarrow \tilde{f}$ . Our results are more general than Lozinski's results since we use the definition of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space  $L_1$ .

Section 2 contains preliminary material about Orlicz spaces. In § 3 we prove that boundedness implies reflexivity and in § 4 we prove the converse.

2. Let  $v = \varphi(u)$  be a nondecreasing real valued function defined for  $u \geq 0$ . Assume that  $\varphi(0) = 0$ , that  $\varphi$  is left continuous and that  $\varphi$  does not vanish identically. Let  $u = \psi(v)$  be the left continuous inverse of  $\varphi$ . If  $\lim_{u \rightarrow \infty} \varphi(u) = l$  is finite then  $\psi(v) = \infty$  for  $v > l$ ; otherwise  $\psi(v)$  is finite for all  $v \geq 0$ . The complementary Young's functions  $\Phi$  and  $\Psi$  are defined by

$$\Phi(u) = \int_0^u \varphi(t) dt, \quad \Psi(v) = \int_0^v \psi(s) ds.$$

$\Phi$  is an absolutely continuous convex function for  $0 \leq u < \infty$  and  $\Psi$  is absolutely continuous and convex in the interval where it is finite.

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If  $\lim_{u \rightarrow \infty} \varphi(u) = \infty$  this interval is  $0 \leq v < \infty$ . If  $\lim_{u \rightarrow \infty} \varphi(u) = l$  is finite we say that  $\Psi$  jumps to infinity at  $v = l$ .

$\Phi$  is said to satisfy the  $\Delta_2$ -condition if there is a constant  $k > 0$  and a  $u_0 \geq 0$  such that  $\Phi(2u) \leq k\Phi(u)$  for  $u \geq u_0$ . This is equivalent to satisfying the inequality  $\Phi(lu) \leq kl\Phi(u)$  for all sufficiently large  $u$ , where  $l$  is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space  $L_\Phi = L_\Phi(0, 2\pi)$  consists, by definition, of all measurable complex functions  $f$  defined on the unit circle for which  $\|f\|_\Phi = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty$ , where the supremum is taken over all functions  $g$  with  $\int_0^{2\pi} \Psi |g(t)| dt \leq 1$ . The space  $L_\Psi$  is defined by interchanging  $\Phi$  and  $\Psi$ . The Orlicz space  $L_{M\Phi}$  is defined to be the set of all measurable complex functions  $f$  for which

$$\|f\|_{M\Phi} = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty,$$

where the supremum is taken over all  $g$  with  $\|g\|_\Psi \leq 1$ .  $L_{M\Psi}$  is similarly defined. The spaces  $L_\Phi, L_\Psi, L_{M\Phi}$  and  $L_{M\Psi}$  are all Banach spaces with their respective norms when functions equal almost everywhere are identified. The spaces  $L_\Phi$  and  $L_{M\Phi}$  consist of the same functions and  $\|f\|_{M\Phi} \leq \|f\|_\Phi \leq 2\|f\|_{M\Phi}$ . The same is true replacing  $\Phi$  by  $\Psi$ . The space  $L_\Phi$  is reflexive with dual space  $L_{M\Psi}$  if and only if both  $\Phi$  and  $\Psi$  satisfy the  $\Delta_2$ -condition.

Two Young's functions  $\Phi_1$  and  $\Phi_2$  are said to be equivalent ( $\Phi_1 \sim \Phi_2$ ) if and only if there exist positive constants  $k_1, k_2$ , and  $u_0$  such that  $\Phi_1(k_1u) \leq \Phi_2(u) \leq \Phi_1(k_2u)$  for  $u \geq u_0$ . It is clear that  $\sim$  is an equivalence relation and that the  $\Delta_2$ -condition is an equivalence class property. If  $\Phi_1 \sim \Phi_2$  then  $L_{\Phi_1}$  and  $L_{\Phi_2}$  consist of the same functions and the norm  $\|\cdot\|_{\Phi_1}$  and  $\|\cdot\|_{\Phi_2}$  are equivalent. Conversely, if  $L_{\Phi_1}$  and  $L_{\Phi_2}$  have the same elements then  $\Phi_1 \sim \Phi_2$  [1; p. 112].

3. In this section we will show that if  $f \rightarrow \tilde{f}$  is bounded then  $L_\Phi$  is reflexive. Let  $S_n(f)$  denote the  $n$ th partial sum of the Fourier series of  $f$  and write  $D_n(t) = \sin [n + (1/2)]t/2 \sin (1/2)t$ . If  $\|\tilde{f}\|_\Phi \leq C\|f\|_\Phi$  for all  $f \in L_\Phi$  then it follows [6; vol. I, p. 266] that  $\|S_n(f)\|_\Phi \leq A\|f\|_\Phi$  for all  $f \in L_\Phi$  and all  $n$ , where  $A$  is a positive constant independent of  $n$  and  $f$ . Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

**THEOREM 2.** *If  $\|S_n(f)\|_\Phi \leq A\|f\|_\Phi$  for all  $f \in L_\Phi$  and all  $n$  then  $L_\Phi$  is reflexive.*

The proof of Theorem 2 uses the following two lemmas given by

S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on  $\varphi$  than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

**LEMMA 1.**  $(\varphi(u)/250) \log (n/u\varphi(u)) \leq \|D_n\|_\phi$  for  $u\varphi(u) \geq 1$ .

**LEMMA 2.** If  $\|S_n(f)\|_\phi \leq A\|f\|_\phi$  for all  $f \in L_\phi$  and all  $n$  then  $\|D_n\|_\phi \leq 2\pi A(n + \Phi(u))/u$  for  $0 < u < \infty$ .

*Proof of Theorem 2.* Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

$$(1) \quad \varphi(v) \log \frac{n}{v\varphi(v)} \leq k \frac{n + \Phi(u)}{u}$$

for  $v\varphi(v) \geq 1$  and  $0 < u < \infty$ .  $k = 2\pi A/250$ . Our immediate aim is to show that for all sufficiently large  $\lambda > 1$

$$(2) \quad \log \left( \frac{\lambda}{2} \right) \leq 2k \frac{\varphi(v)}{\varphi\left(\frac{v}{\lambda}\right)}$$

for  $v \geq v_0$ , where  $v_0$  depends upon  $\lambda$ .

For any

$$\lambda > 1, \Phi(u) = \int_0^u \varphi(t)dt > \int_{u/\lambda}^u \varphi(t)dt$$

and hence

$$\Phi(u) > \left(u - \frac{u}{\lambda}\right) \varphi\left(\frac{u}{\lambda}\right) = (\lambda - 1) \frac{u}{\lambda} \varphi\left(\frac{u}{\lambda}\right).$$

Thus

$$(3) \quad \log \frac{(\lambda - 1)n}{\Phi(v)} < \log \frac{n}{\frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right)}.$$

By combining (3) and (1) we see that

$$(4) \quad \varphi\left(\frac{v}{\lambda}\right) \log \frac{(\lambda - 1)n}{\Phi(v)} \leq k \frac{n + \Phi(v)}{v}$$

whenever  $(v/\lambda) \varphi(v/\lambda) \geq 1$ . Let  $n = [\Phi(v)] =$  greatest integer in  $\Phi(v)$ . Then (4) becomes

$$(5) \quad \varphi\left(\frac{v}{\lambda}\right) \log \left\{ (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)} \right\} \leq k \frac{[\Phi(v)] + \Phi(v)}{v} \leq 2k \frac{\Phi(v)}{v}.$$

For every sufficiently large  $\lambda$  there exist a  $v_0 \geq 0$  such that for  $v \geq v_0$

$$(6) \quad 1 < \frac{\lambda}{2} \leq (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)}$$

and

$$(7) \quad \frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right) \geq 1.$$

Using (5), (6) and the fact that  $\Phi(v) \leq v\varphi(v)$  we get inequality (2) for  $v \geq v_0$ . Since  $\lambda$  can be arbitrarily large (2) implies that  $\lim_{u \rightarrow \infty} \varphi(u) = \infty$  and hence that  $\Psi$  does not jump to infinity. We next show that  $\Psi$  satisfies the  $\Delta_2$ -condition.

Let  $\lambda$  be large but fixed and write  $l = (1/2k) \log(\lambda/2)$ . Then (2) states that

$$(8) \quad l\varphi\left(\frac{t}{\lambda}\right) \leq \varphi(t)$$

for  $t \geq v_0$ . This implies, on taking inverses, that there is a number  $s_0$  such that for  $s \geq s_0$

$$(9) \quad \psi(s) \leq \lambda\psi\left(\frac{s}{l}\right).$$

Thus

$$\int_{s_0}^v \psi(s) ds \leq \lambda \int_{s_0}^v \psi\left(\frac{s}{l}\right) ds = \lambda l \int_{s_0/l}^{v/l} \psi(s) ds$$

or

$$(10) \quad \Psi(v) - \Psi(s_0) \leq \lambda l \left[ \Psi\left(\frac{v}{l}\right) - \Psi\left(\frac{s_0}{l}\right) \right].$$

This shows that for sufficiently large  $v$

$$(11) \quad \Psi(lv) \leq 2\lambda l \Psi(v)$$

and hence proves that  $\Psi$  satisfies the  $\Delta_2$ -condition.

If  $\|S_n(f)\|_\phi \leq A\|f\|_\phi$  for all  $f \in L_\phi$  then it follows that  $\|S_n(g)\|_{M^v} \leq A\|g\|_{M^v}$  for all  $g \in L_{M^v}$  or, equivalently, that  $\|S_n(g)\|_\psi \leq 2A\|g\|_\psi$  for all  $g \in L_\psi$ . Since we have shown that  $\Psi$  does not jump to  $\infty$  we can interchange the rôle of  $\Phi$  and  $\Psi$  in the above argument to show that  $\Phi$  satisfies the  $\Delta_2$ -condition. This proves that  $L_\phi$  is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz

spaces which combined with the classical results of M. Riesz [6; vol. I, p. 253 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

**THEOREM 3.** *Suppose that  $T$  is a bounded linear operator on  $L_p$  into  $L_p$  for  $1 < p < \infty$ . Then if  $L_\phi$  is reflexive  $T$  is defined and bounded on  $L_\phi$  into  $L_\phi$ .*

*Proof.* The proof consists of showing that  $\phi$  can be replaced by an equivalent function  $\phi_1$  ( $\phi \sim \phi_1$ ) such that  $\phi_1$  satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$(12) \quad \int_u^\infty \frac{\phi_1(t)}{t^{\beta+1}} dt = O\left\{\frac{\phi_1(u)}{u^\beta}\right\}$$

and

$$(13) \quad \int_1^u \frac{\phi_1(t)}{t^{\alpha+1}} dt = O\left\{\frac{\phi_1(u)}{u^\alpha}\right\}$$

for  $u \rightarrow \infty$ , where  $1 < \alpha < \beta < \infty$ .

The assumption that  $L_\phi$  is reflexive implies that  $\lim_{u \rightarrow \infty} \phi(u) = \infty$  and hence that  $\lim_{u \rightarrow \infty} \phi(u)/u = \infty$ . By [1; p. 16]  $\phi$  is equal for sufficiently large values of  $u$  to a function  $M$  of the form  $M(u) = \int_0^u p(t) dt$  where  $p$  is a nondecreasing right continuous function with  $\lim_{u \rightarrow 0} p(u) = 0$  and  $\lim_{u \rightarrow \infty} p(u) = \infty$ . Clearly  $\phi \sim M$ .

By [1; p. 46] the function  $M_1$  defined by  $M_1(u) = \int_0^u (M(t)/t) dt$  is equivalent to  $M$  and hence to  $\phi$ . The derivative of  $M_1$  is continuous and strictly increasing.

Since  $L_\phi$  is reflexive both  $\phi$  and  $\psi$  satisfy the  $\Delta_2$ -condition. Thus both  $M_1$  and its conjugate Young's function  $N_1$  satisfy the  $\Delta_2$ -condition [1; p. 23]. According to [1; pp. 26-27] this implies the existence of numbers  $a, b$ , and  $u_0 \geq 0$  with  $1 < a < b < \infty$  such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all  $u \geq u_0$ . If we define  $\phi_1$  by

$$\phi_1(u) = \begin{cases} \frac{M_1(u_0)}{u_0^a} u^a & \text{for } u \leq u_0 \\ M_1(u) & \text{for } u \geq u_0 \end{cases}$$

we obtain a function  $\phi_1 \sim \phi$  such that

$$(14) \quad 1 < a \leq \frac{u\varphi_1(u)}{\Phi_1(u)} \leq b$$

for all  $u \geq 0$ .

We next show that  $\Phi_1$  satisfies (12) and (13) for suitably chosen  $\alpha$  and  $\beta$ . In particular choose  $\alpha$  and  $\beta$  such that  $1 < \alpha < a \leq b < \beta < \infty$ . This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

$$(15) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt = \beta \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt - \frac{\Phi_1(u)}{u^\beta}$$

and

$$(16) \quad \int_0^u \frac{\varphi_1(t)}{t^\alpha} dt = \alpha \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt + \frac{\Phi_1(u)}{u^\alpha}.$$

From (14) we obtain

$$(17) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt \leq b \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt$$

and

$$(18) \quad \int_0^u \frac{\varphi_1(t)}{t^\alpha} dt \geq a \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt.$$

Combining (15) with (17) and (16) with (18) shows that

$$(19) \quad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \leq \frac{1}{\beta - b} \left\{ \frac{\Phi_1(u)}{u^\beta} \right\}$$

and

$$(20) \quad \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt \leq \frac{1}{a - \alpha} \left\{ \frac{\Phi_1(u)}{u^\alpha} \right\}.$$

This shows that  $\Phi_1$  satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p. 174] there exists a constant  $K_1$  such that  $\|Tf\|_{\phi_1} \leq K_1 \|f\|_{\phi_1}$  for all  $f \in L_{\phi_1}$ . Since  $\Phi \sim \Phi_1$  there is a constant  $K$  such that  $\|Tf\|_\Phi \leq K \|f\|_\Phi$  for all  $f \in L_\Phi$ . This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

#### REFERENCES

1. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Groningen, 1961.

2. J. Lamperti, *A note on conjugate functions*, Proc. Amer. Math. Soc., **10** (1959), 71-76.
3. S. Lozinski, *On convergence and summability of Fourier series and interpolation processes*, Mat. Sbornik N.S., **14** (1944) 175-262.
4. ———. *On convergence in mean of Fourier series* Doklady, **51** (1949), 7-10.
5. A. C. Zaanen, *Linear Analysis*, New York-Amsterdam-Groningen, 1953.
6. A. Zygmund, *Trigonometric Series*, Vols. I, II, Cambridge, 1959.





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