CONJUGATE FUNCTIONS IN ORLICZ SPACES

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1. The purpose of this paper is to prove the following results:

**Theorem 1.** Let

\[ \tilde{f}(x) = -\frac{1}{\pi} \int_0^\pi \frac{f(x + t) - f(x - t)}{2 \tan (1/2)t} \, dt = \lim_{\varepsilon \to 0} \left\{ -\frac{1}{\pi} \int_0^\varepsilon \right\} . \]

The mapping \( f \to \tilde{f} \) is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the \( L_p \) spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund’s book [6; vol. II, pp. 116-118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of \( f \to \tilde{f} \). Our results are more general than Lozinski’s results since we use the definition of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space \( L_1 \).

Section 2 contains preliminary material about Orlicz spaces. In § 3 we prove that boundedness implies reflexivity and in § 4 we prove the converse.

2. Let \( v = \varphi(u) \) be a nondecreasing real valued function defined for \( u \geq 0 \). Assume that \( \varphi(0) = 0 \), that \( \varphi \) is left continuous and that \( \varphi \) does not vanish identically. Let \( u = \psi(v) \) be the left continuous inverse of \( \varphi \). If \( \lim_{u \to \infty} \varphi(u) = l \) is finite then \( \psi(v) = \infty \) for \( v > l \); otherwise \( \psi(v) \) is finite for all \( v \geq 0 \). The complementary Young’s functions \( \Phi \) and \( \Psi \) are defined by

\[ \Phi(u) = \int_0^u \varphi(t) \, dt \quad \text{and} \quad \Psi(v) = \int_0^v \psi(s) \, ds . \]

\( \Phi \) is an absolutely continuous convex function for \( 0 \leq u < \infty \) and \( \Psi \) is absolutely continuous and convex in the internal where it is finite.

If \( \lim_{u \to \infty} \mathcal{P}(u) = \infty \) this internal is \( 0 \leq v < \infty \). If \( \lim_{u \to \infty} \mathcal{P}(u) = l \) is finite we say that \( \Psi \) jumps to infinity at \( v = l \).

\( \Phi \) is said to satisfy the \( J_2 \)-condition if there is a constant \( k > 0 \) and a \( u_0 \geq 0 \) such that \( \Phi(2u) \leq k\Phi(u) \) for \( u \geq u_0 \). This is equivalent to satisfying the inequality \( \Phi(lu) \leq kl\Phi(u) \) for all sufficiently large \( u \), where \( l \) is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space \( L_\phi = L_\phi(0, 2\pi) \) consists, by definition, of all measurable complex functions \( \varphi \) defined on the unit circle for which \( \sup_{t} \int_{0}^{2\pi} |f(t)g(t)| dt < \infty \), where the supremum is taken over all functions \( g \) with \( \int_{0}^{2\pi} \Phi |g(t)| dt \leq 1 \). The space \( L_\varphi \) is defined by interchanging \( \Phi \) and \( \Psi \). The Orlicz space \( L_{M_\phi} \) is defined to be the set of all measurable complex functions \( f \) for which

\[
\|f\|_{M_\phi} = \sup_{t} \int_{0}^{2\pi} |f(t)g(t)| dt < \infty ,
\]

where the supremum is taken over all \( g \) with \( \|g\|_{\varphi} \leq 1 \). \( L_{M_\varphi} \) is similarly defined. The spaces \( L_\phi, L_\varphi, L_{M_\phi} \) and \( L_{M_\varphi} \) are all Banach spaces with their respective norms when functions equal almost everywhere are identified. The spaces \( L_\phi \) and \( L_{M_\phi} \) consist of the same functions and \( \|f\|_{M_\phi} \leq \|f\|_{\phi} \leq 2\|f\|_{M_\phi} \). The same is true replacing \( \Phi \) by \( \Psi \). The space \( L_\phi \) is reflexive with dual space \( L_{M_\varphi} \) if and only if both \( \Phi \) and \( \Psi \) satisfy the \( J_2 \)-condition.

Two Young's functions \( \Phi_1 \) and \( \Phi_2 \) are said to be equivalent \((\Phi_1 \sim \Phi_2)\) if and only if there exist positive constants \( k_1, k_2, \) and \( u_0 \) such that \( \Phi_1(k_1u) \leq \Phi_2(u) \leq \Phi_1(k_2u) \) for \( u \geq u_0 \). It is clear that \( \sim \) is an equivalence relation and that the \( J_2 \)-condition is an equivalence class property. If \( \Phi_1 \sim \Phi_2 \) then \( L_{\phi_1} \) and \( L_{\phi_2} \) consist of the same functions and the norm \( \| \|_{\phi_1} \) and \( \| \|_{\phi_2} \) are equivalent. Conversely, if \( L_{\phi_1} \) and \( L_{\phi_2} \) have the same elements then \( \Phi_1 \sim \Phi_2 \) [1; p. 112].

3. In this section we will show that if \( f \to \tilde{f} \) is bounded then \( L_\phi \) is reflexive. Let \( S_n(f) \) denote the \( n \)th partial sum of the Fourier series of \( f \) and write \( D_n(t) = \sin [n + (1/2)]t/2 \sin (1/2)t \). If \( \|\tilde{f}\|_{\phi} \leq C \|f\|_{\phi} \) for all \( f \in L_\phi \) then it follows [6; vol. I, p. 266] that \( \|S_n(f)\|_{\phi} \leq A \|f\|_{\phi} \) for all \( f \in L_\phi \) and all \( n \), where \( A \) is a positive constant independent of \( n \) and \( f \). Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

**Theorem 2.** If \( \|S_n(f)\|_{\phi} \leq A \|f\|_{\phi} \) for all \( f \in L_\phi \) and all \( n \) then \( L_\phi \) is reflexive.

The proof of Theorem 2 uses the following two lemmas given by
S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on \( \phi \) than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

**Lemma 1.**  
\[
(\phi(u)/250) \log (n/u\phi(u)) \leq \| D_n \|_\phi \text{ for } u\phi(u) \geq 1.
\]

**Lemma 2.**  
If \( \| S_n(f) \|_\phi \leq A\| f \|_\phi \) for all \( f \in L_\phi \) and all \( n \) then \( \| D_n \|_\phi \leq 2\pi A(n + \phi(u))/u \) for \( 0 < u < \infty \).

**Proof of Theorem 2.** Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

\[
\phi(v) \log \frac{n}{v\phi(v)} \leq k \frac{n + \phi(u)}{u}
\]

for \( v\phi(v) \geq 1 \) and \( 0 < u < \infty \). \( k = 2\pi A/250 \). Our immediate aim is to show that for all sufficiently large \( \lambda > 1 \)

\[
\log \left( \frac{\lambda}{2} \right) \leq 2k \frac{\phi(v)}{\phi\left( \frac{v}{\lambda} \right)}
\]

for \( v \geq v_0 \), where \( v_0 \) depends upon \( \lambda \).

For any \( \lambda > 1 \),

\[
\phi(u) = \int_0^u \phi(t)dt > \int_{u/\lambda}^u \phi(t)dt
\]

and hence

\[
\phi(u) > \left( u - \frac{u}{\lambda} \right) \phi\left( \frac{u}{\lambda} \right) = (\lambda - 1) \frac{u}{\lambda} \phi\left( \frac{u}{\lambda} \right).
\]

Thus

\[
\log \left( \frac{\lambda - 1}{\lambda} \right) \frac{n}{\phi(v)} < \log \frac{n}{v} \phi\left( \frac{v}{\lambda} \right).
\]

By combining (3) and (1) we see that

\[
\phi\left( \frac{v}{\lambda} \right) \log \left( \frac{\lambda - 1}{\phi(v)} \right) \frac{n}{\phi(v)} \leq k \frac{n + \phi(v)}{v}
\]

whenever \( (v/\lambda) \phi(v/\lambda) \geq 1 \). Let \( n = \lfloor \phi(v) \rfloor = \text{greatest integer in } \phi(v) \).

Then (4) becomes

\[
\phi\left( \frac{v}{\lambda} \right) \log \left( \lambda - 1 \right) \frac{\lfloor \phi(v) \rfloor}{\phi(v)} \leq k \frac{\lfloor \phi(v) \rfloor + \phi(v)}{v} \leq 2k \frac{\phi(v)}{v}.
\]
For every sufficiently large $\lambda$ there exist a $v_0 \geq 0$ such that for $v \geq v_0$

$$1 < \frac{\lambda}{2} \leq (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)}$$

and

$$\frac{v}{\lambda} \varphi \left( \frac{v}{\lambda} \right) \geq 1.$$  

Using (5), (6) and the fact that $\Phi(v) \leq v \varphi(v)$ we get inequality (2) for $v \geq v_0$. Since $\lambda$ can be arbitrarily large (2) implies that $\lim_{u \to \infty} \varphi(u) = \infty$ and hence that $\Psi$ does not jump to infinity. We next show that $\Psi$ satisfies the $\Delta_2$-condition.

Let $\lambda$ be large but fixed and write $l = (1/2k) \log (\lambda/2)$. Then (2) states that

$$l \varphi \left( \frac{t}{\lambda} \right) \leq \varphi(t)$$

for $t \geq v_0$. This implies, on taking inverses, that there is a number $s_0$ such that for $s \geq s_0$

$$\psi(s) \leq \lambda \psi \left( \frac{s}{l} \right).$$

Thus

$$\int_{s_0}^{v} \psi(s) \, ds \leq \lambda \int_{s_0}^{v} \psi \left( \frac{s}{l} \right) \, ds = \lambda l \int_{s_0/\lambda}^{\psi^{-1}(s)} \psi(s) \, ds$$

or

$$\Psi(v) - \Psi(s_0) \leq \lambda l \left[ \Psi \left( \frac{v}{l} \right) - \Psi \left( \frac{s_0}{l} \right) \right].$$

This shows that for sufficiently large $v$

$$\Psi(lv) \leq 2\lambda l \Psi(v)$$

and hence proves that $\Psi$ satisfies the $\Delta_2$-condition.

If $\| S_n(f) \|_\phi \leq A \| f \|_\phi$ for all $f \in L_\phi$ then it follows that $\| S_n(g) \|_{\mathcal{M}_\Psi} \leq A \| g \|_{\mathcal{M}_\Psi}$ for all $g \in L_{\mathcal{M}_\Psi}$ or, equivalently, that $\| S_n(g) \|_\Psi \leq 2A \| g \|_\Psi$ for all $g \in L_\Psi$. Since we have shown that $\Psi$ does not jump to $\infty$ we can interchange the rôle of $\Phi$ and $\Psi$ in the above argument to show that $\Phi$ satisfies the $\Delta_2$-condition. This proves that $L_\phi$ is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz
spaces which combined with the classical results of M. Riesz [6; vol. I, p. 253 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

**Theorem 3.** Suppose that $T$ is a bounded linear operator on $L_p$ into $L_p$ for $1 < p < \infty$. Then if $L_\phi$ is reflexive $T$ is defined and bounded on $L_\phi$ into $L_\phi$.

**Proof.** The proof consists of showing that $\Phi$ can be replaced by an equivalent function $\Phi_1(\Phi \sim \Phi_1)$ such that $\Phi_1$ satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$(12) \quad \int_{u_0}^\infty \frac{\Phi(t)}{t^{\beta+1}} \, dt = O\left\{ \frac{\Phi(u)}{u^\beta} \right\}$$

and

$$(13) \quad \int_1^u \frac{\Phi_1(t)}{t^{\alpha+1}} \, dt = O\left\{ \frac{\Phi_1(u)}{u^\alpha} \right\}$$

for $u \to \infty$, where $1 < \alpha < \beta < \infty$.

The assumption that $L_\phi$ is reflexive implies that $\lim_{u \to \infty} p(u) = \infty$ and hence that $\lim_{u \to \infty} \Phi(u)/u = \infty$. By [1; p. 16] $\Phi$ is equal for sufficiently large values of $u$ to a function $M$ of the form $M(u) = \int_0^u p(t) \, dt$ where $p$ is a nondecreasing right continuous function with $\lim_{u \to 0} p(u) = 0$ and $\lim_{u \to \infty} p(u) = \infty$. Clearly $\Phi \sim M$.

By [1; p. 46] the function $M_1$ defined by $M_1(u) = \int_0^u (M(t)/t) \, dt$ is equivalent to $M$ and hence to $\Phi$. The derivative of $M_1$ is continuous and strictly increasing.

Since $L_\phi$ is reflexive both $\Phi$ and $\Psi$ satisfy the $\Delta_2$-condition. Thus both $M_1$ and its conjugate Young’s function $N$ satisfy the $\Delta_2$-condition [1; p. 23]. According to [1; pp. 26-27] this implies the existence of numbers $a, b$, and $u_0 \geq 0$ with $1 < a < b < \infty$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all $u \geq u_0$. If we define $\Phi_1$ by

$$\Phi_1(u) = \begin{cases} \frac{M_1(u_0)}{u_0^a} & \text{for } u \leq u_0 \\ \frac{M_1(u)}{u} & \text{for } u \geq u_0 \end{cases}$$

we obtain a function $\Phi_1 \sim \Phi$ such that
for all $u \geq 0$.

We next show that $\Phi_x$ satisfies (12) and (13) for suitably chosen $\alpha$ and $\beta$. In particular choose $\alpha$ and $\beta$ such that $1 < \alpha < a < b < \beta < \infty$. This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

\begin{equation}
\int_{u}^{\infty} \frac{\Phi_x(t)}{t^\beta} \, dt = \beta \int_{u}^{\infty} \frac{\Phi_x(t)}{t^{\beta+1}} \, dt - \frac{\Phi_x(u)}{u^\beta}
\end{equation}

and

\begin{equation}
\int_{0}^{u} \frac{\Phi_x(t)}{t^\alpha} \, dt = \alpha \int_{0}^{u} \frac{\Phi_x(t)}{t^{\alpha+1}} \, dt + \frac{\Phi_x(u)}{u^\alpha}.
\end{equation}

From (14) we obtain

\begin{equation}
\int_{u}^{\infty} \frac{\Phi_x(t)}{t^\beta} \, dt \leq b \int_{u}^{\infty} \frac{\Phi_x(t)}{t^{\beta+1}} \, dt
\end{equation}

and

\begin{equation}
\int_{0}^{u} \frac{\Phi_x(t)}{t^\alpha} \, dt \geq a \int_{0}^{u} \frac{\Phi_x(t)}{t^{\alpha+1}} \, dt.
\end{equation}

Combining (15) with (17) and (16) with (18) shows that

\begin{equation}
\int_{u}^{\infty} \frac{\Phi_x(t)}{t^{\beta+1}} \, dt \leq \frac{1}{\beta - b} \left\{ \frac{\Phi_x(u)}{u^\beta} \right\}
\end{equation}

and

\begin{equation}
\int_{0}^{u} \frac{\Phi_x(t)}{t^{\alpha+1}} \, dt \leq \frac{1}{a - \alpha} \left\{ \frac{\Phi_x(u)}{u^\alpha} \right\}.
\end{equation}

This shows that $\Phi_x$ satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p. 174] there exists a constant $K_x$ such that $\| T f \|_{\Phi_x} \leq K_x \| f \|_{\Phi_x}$ for all $f \in L_{\Phi_x}$. Since $\Phi \sim \Phi_x$ there is a constant $K$ such that $\| T f \|_{\Phi} \leq K \| f \|_{\Phi}$ for all $f \in L_{\Phi}$. This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

REFERENCES

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