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CONJUGATE FUNCTIONS IN ORLICZ SPACES

ROBERT DEAN RYAN

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1. The purpose of this paper is to prove the following results:

THEOREM 1. *Let*

$$\tilde{f}(x) = -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan(1/2)t} dt = \lim_{\varepsilon \rightarrow +0} \left\{ -\frac{1}{\pi} \int_\varepsilon^\pi \right\}.$$

The mapping $f \rightarrow \tilde{f}$ is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the L_p spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116-118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of $f \rightarrow \tilde{f}$. Our results are more general than Lozinski's results since we use the definition of an Orlicz space given by A. C. Zaanan [5] which includes, for example, the space L_1 .

Section 2 contains preliminary material about Orlicz spaces. In § 3 we prove that boundedness implies reflexivity and in § 4 we prove the converse.

2. Let $v = \varphi(u)$ be a nondecreasing real valued function defined for $u \geq 0$. Assume that $\varphi(0) = 0$, that φ is left continuous and that φ does not vanish identically. Let $u = \psi(v)$ be the left continuous inverse of φ . If $\lim_{u \rightarrow \infty} \varphi(u) = l$ is finite then $\psi(v) = \infty$ for $v > l$; otherwise $\psi(v)$ is finite for all $v \geq 0$. The complementary Young's functions Φ and Ψ are defined by

$$\Phi(u) = \int_0^u \varphi(t) dt, \quad \Psi(v) = \int_0^v \psi(s) ds.$$

Φ is an absolutely continuous convex function for $0 \leq u < \infty$ and Ψ is absolutely continuous and convex in the interval where it is finite.

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If $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ this interval is $0 \leq v < \infty$. If $\lim_{u \rightarrow \infty} \varphi(u) = l$ is finite we say that Ψ jumps to infinity at $v = l$.

Φ is said to satisfy the Δ_2 -condition if there is a constant $k > 0$ and a $u_0 \geq 0$ such that $\Phi(2u) \leq k\Phi(u)$ for $u \geq u_0$. This is equivalent to satisfying the inequality $\Phi(lu) \leq kl\Phi(u)$ for all sufficiently large u , where l is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space $L_\Phi = L_\Phi(0, 2\pi)$ consists, by definition, of all measurable complex functions f defined on the unit circle for which $\|f\|_\Phi = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty$, where the supremum is taken over all functions g with $\int_0^{2\pi} \Psi |g(t)| dt \leq 1$. The space L_Ψ is defined by interchanging Φ and Ψ . The Orlicz space $L_{M\Phi}$ is defined to be the set of all measurable complex functions f for which

$$\|f\|_{M\Phi} = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty,$$

where the supremum is taken over all g with $\|g\|_\Psi \leq 1$. $L_{M\Psi}$ is similarly defined. The spaces $L_\Phi, L_\Psi, L_{M\Phi}$ and $L_{M\Psi}$ are all Banach spaces with their respective norms when functions equal almost everywhere are identified. The spaces L_Φ and $L_{M\Phi}$ consist of the same functions and $\|f\|_{M\Phi} \leq \|f\|_\Phi \leq 2\|f\|_{M\Phi}$. The same is true replacing Φ by Ψ . The space L_Φ is reflexive with dual space $L_{M\Psi}$ if and only if both Φ and Ψ satisfy the Δ_2 -condition.

Two Young's functions Φ_1 and Φ_2 are said to be equivalent ($\Phi_1 \sim \Phi_2$) if and only if there exist positive constants k_1, k_2 , and u_0 such that $\Phi_1(k_1u) \leq \Phi_2(u) \leq \Phi_1(k_2u)$ for $u \geq u_0$. It is clear that \sim is an equivalence relation and that the Δ_2 -condition is an equivalence class property. If $\Phi_1 \sim \Phi_2$ then L_{Φ_1} and L_{Φ_2} consist of the same functions and the norm $\|\cdot\|_{\Phi_1}$ and $\|\cdot\|_{\Phi_2}$ are equivalent. Conversely, if L_{Φ_1} and L_{Φ_2} have the same elements then $\Phi_1 \sim \Phi_2$ [1; p. 112].

3. In this section we will show that if $f \rightarrow \tilde{f}$ is bounded then L_Φ is reflexive. Let $S_n(f)$ denote the n th partial sum of the Fourier series of f and write $D_n(t) = \sin [n + (1/2)]t/2 \sin (1/2)t$. If $\|\tilde{f}\|_\Phi \leq C\|f\|_\Phi$ for all $f \in L_\Phi$ then it follows [6; vol. I, p. 266] that $\|S_n(f)\|_\Phi \leq A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n , where A is a positive constant independent of n and f . Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

THEOREM 2. *If $\|S_n(f)\|_\Phi \leq A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n then L_Φ is reflexive.*

The proof of Theorem 2 uses the following two lemmas given by

S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on φ than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

LEMMA 1. $(\varphi(u)/250) \log (n/u\varphi(u)) \leq \|D_n\|_\phi$ for $u\varphi(u) \geq 1$.

LEMMA 2. If $\|S_n(f)\|_\phi \leq A\|f\|_\phi$ for all $f \in L_\phi$ and all n then $\|D_n\|_\phi \leq 2\pi A(n + \Phi(u))/u$ for $0 < u < \infty$.

Proof of Theorem 2. Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

$$(1) \quad \varphi(v) \log \frac{n}{v\varphi(v)} \leq k \frac{n + \Phi(u)}{u}$$

for $v\varphi(v) \geq 1$ and $0 < u < \infty$. $k = 2\pi A/250$. Our immediate aim is to show that for all sufficiently large $\lambda > 1$

$$(2) \quad \log \left(\frac{\lambda}{2} \right) \leq 2k \frac{\varphi(v)}{\varphi\left(\frac{v}{\lambda}\right)}$$

for $v \geq v_0$, where v_0 depends upon λ .

For any

$$\lambda > 1, \Phi(u) = \int_0^u \varphi(t)dt > \int_{u/\lambda}^u \varphi(t)dt$$

and hence

$$\Phi(u) > \left(u - \frac{u}{\lambda}\right) \varphi\left(\frac{u}{\lambda}\right) = (\lambda - 1) \frac{u}{\lambda} \varphi\left(\frac{u}{\lambda}\right).$$

Thus

$$(3) \quad \log \frac{(\lambda - 1)n}{\Phi(v)} < \log \frac{n}{\frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right)}.$$

By combining (3) and (1) we see that

$$(4) \quad \varphi\left(\frac{v}{\lambda}\right) \log \frac{(\lambda - 1)n}{\Phi(v)} \leq k \frac{n + \Phi(v)}{v}$$

whenever $(v/\lambda) \varphi(v/\lambda) \geq 1$. Let $n = [\Phi(v)] =$ greatest integer in $\Phi(v)$. Then (4) becomes

$$(5) \quad \varphi\left(\frac{v}{\lambda}\right) \log \left\{ (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)} \right\} \leq k \frac{[\Phi(v)] + \Phi(v)}{v} \leq 2k \frac{\Phi(v)}{v}.$$

For every sufficiently large λ there exist a $v_0 \geq 0$ such that for $v \geq v_0$

$$(6) \quad 1 < \frac{\lambda}{2} \leq (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)}$$

and

$$(7) \quad \frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right) \geq 1.$$

Using (5), (6) and the fact that $\Phi(v) \leq v\varphi(v)$ we get inequality (2) for $v \geq v_0$. Since λ can be arbitrarily large (2) implies that $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ and hence that Ψ does not jump to infinity. We next show that Ψ satisfies the Δ_2 -condition.

Let λ be large but fixed and write $l = (1/2k) \log(\lambda/2)$. Then (2) states that

$$(8) \quad l\varphi\left(\frac{t}{\lambda}\right) \leq \varphi(t)$$

for $t \geq v_0$. This implies, on taking inverses, that there is a number s_0 such that for $s \geq s_0$

$$(9) \quad \psi(s) \leq \lambda\psi\left(\frac{s}{l}\right).$$

Thus

$$\int_{s_0}^v \psi(s) ds \leq \lambda \int_{s_0/l}^{v/l} \psi\left(\frac{s}{l}\right) ds = \lambda l \int_{s_0/l}^{v/l} \psi(s) ds$$

or

$$(10) \quad \Psi(v) - \Psi(s_0) \leq \lambda l \left[\Psi\left(\frac{v}{l}\right) - \Psi\left(\frac{s_0}{l}\right) \right].$$

This shows that for sufficiently large v

$$(11) \quad \Psi(lv) \leq 2\lambda l \Psi(v)$$

and hence proves that Ψ satisfies the Δ_2 -condition.

If $\|S_n(f)\|_\phi \leq A\|f\|_\phi$ for all $f \in L_\phi$ then it follows that $\|S_n(g)\|_{M^\Psi} \leq A\|g\|_{M^\Psi}$ for all $g \in L_{M^\Psi}$ or, equivalently, that $\|S_n(g)\|_\Psi \leq 2A\|g\|_\Psi$ for all $g \in L_\Psi$. Since we have shown that Ψ does not jump to ∞ we can interchange the rôle of Φ and Ψ in the above argument to show that Φ satisfies the Δ_2 -condition. This proves that L_ϕ is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz

spaces which combined with the classical results of M. Riesz [6; vol. I, p. 253 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

THEOREM 3. *Suppose that T is a bounded linear operator on L_p into L_p for $1 < p < \infty$. Then if L_ϕ is reflexive T is defined and bounded on L_ϕ into L_ϕ .*

Proof. The proof consists of showing that Φ can be replaced by an equivalent function Φ_1 ($\Phi \sim \Phi_1$) such that Φ_1 satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$(12) \quad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt = O\left\{\frac{\Phi_1(u)}{u^\beta}\right\}$$

and

$$(13) \quad \int_1^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt = O\left\{\frac{\Phi_1(u)}{u^\alpha}\right\}$$

for $u \rightarrow \infty$, where $1 < \alpha < \beta < \infty$.

The assumption that L_ϕ is reflexive implies that $\lim_{u \rightarrow \infty} \mathcal{P}(u) = \infty$ and hence that $\lim_{u \rightarrow \infty} \Phi(u)/u = \infty$. By [1; p. 16] Φ is equal for sufficiently large values of u to a function M of the form $M(u) = \int_0^u p(t) dt$ where p is a nondecreasing right continuous function with $\lim_{u \rightarrow 0} p(u) = 0$ and $\lim_{u \rightarrow \infty} p(u) = \infty$. Clearly $\Phi \sim M$.

By [1; p. 46] the function M_1 defined by $M_1(u) = \int_0^u (M(t)/t) dt$ is equivalent to M and hence to Φ . The derivative of M_1 is continuous and strictly increasing.

Since L_ϕ is reflexive both Φ and \mathcal{P} satisfy the Δ_2 -condition. Thus both M_1 and its conjugate Young's function N_1 satisfy the Δ_2 -condition [1; p. 23]. According to [1; pp. 26-27] this implies the existence of numbers a, b , and $u_0 \geq 0$ with $1 < a < b < \infty$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all $u \geq u_0$. If we define Φ_1 by

$$\Phi_1(u) = \begin{cases} \frac{M_1(u_0)}{u_0^a} u^a & \text{for } u \leq u_0 \\ M_1(u) & \text{for } u \geq u_0 \end{cases}$$

we obtain a function $\Phi_1 \sim \Phi$ such that

$$(14) \quad 1 < a \leq \frac{u\varphi_1(u)}{\Phi_1(u)} \leq b$$

for all $u \geq 0$.

We next show that Φ_1 satisfies (12) and (13) for suitably chosen α and β . In particular choose α and β such that $1 < \alpha < a \leq b < \beta < \infty$. This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

$$(15) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt = \beta \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt - \frac{\Phi_1(u)}{u^\beta}$$

and

$$(16) \quad \int_0^u \frac{\varphi_1(t)}{t^\alpha} dt = \alpha \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt + \frac{\Phi_1(u)}{u^\alpha}.$$

From (14) we obtain

$$(17) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt \leq b \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt$$

and

$$(18) \quad \int_0^u \frac{\varphi_1(t)}{t^\alpha} dt \geq a \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt.$$

Combining (15) with (17) and (16) with (18) shows that

$$(19) \quad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \leq \frac{1}{\beta - b} \left\{ \frac{\Phi_1(u)}{u^\beta} \right\}$$

and

$$(20) \quad \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt \leq \frac{1}{a - \alpha} \left\{ \frac{\Phi_1(u)}{u^\alpha} \right\}.$$

This shows that Φ_1 satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p. 174] there exists a constant K_1 such that $\|Tf\|_{\phi_1} \leq K_1 \|f\|_{\phi_1}$ for all $f \in L_{\phi_1}$. Since $\Phi \sim \Phi_1$ there is a constant K such that $\|Tf\|_{\phi} \leq K \|f\|_{\phi}$ for all $f \in L_{\phi}$. This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

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