CONCERNING HOMOGENEITY IN TOTALLY ORDERED, CONNECTED TOPOLOGICAL SPACE

L. Bruce Treybig
Throughout this paper suppose that $L$ denotes a connected, totally ordered topological space in which there is no first or last point, and whose topology is that induced by the order.

A topological space $S$ is said to be homogeneous provided it is true that if $(x, y) \in S \times S$, there is a homeomorphism $f$ from $S$ onto $S$ such that $f(x) = y$. Let $H$ denote the set of all homeomorphisms from $L$ onto $L$, and let $I$ denote the set of all homeomorphisms which map a closed interval of $L$ onto a closed interval of $L$. Let $H_0(I_0)$ denote the set of all elements of $H(I)$ which preserve order.

**Theorem 1.** If $L$ is homogeneous, then $L$ satisfies the first axiom of countability.

**Proof.** It suffices to show that for some point $z$ of $L$ there exists an increasing sequence $x_1, x_2, \ldots$ and a decreasing sequence $y_1, y_2, \ldots$ such that each of these sequences converges to $z$. Suppose there is no such point. Let $P_1, P_2, \ldots$ denote an increasing sequence which converges to a point $P$ and $Q_1, Q_2, \ldots$ a decreasing sequence which converges to a point $Q$. There is an element $g$ in $H$ such that $g(P) = Q$. In view of the preceding supposition, $g$ is order reversing. There is a point $R$ such that $g(R) = R$, and $R$ is the limit of a sequence $R_1, R_2, \ldots$ which is either increasing or decreasing. Suppose the sequence is decreasing. The sequence $g(R_1), g(R_2), \ldots$ is increasing and converges to $R$. This yields a contradiction. The case where $R_1, R_2, \ldots$ is increasing is similar.

**Theorem 2.** The space $L$ is homogeneous if and only if each pair of closed subintervals of $L$ are topologically equivalent.

**Proof.** Part 1. Suppose each pair of closed subintervals of $L$ are topologically equivalent and $(x, y) \in L \times L$. There exist elements $z$ and $w$ of $L$ such that $z < x < w$ and $z < y < w$, and an element $g$ of $I$ from $[z, x]$ onto $[z, y]$. If $g$ is order reversing there is an element $g'$ of $I_0$ from $[z, x]$ onto $[z, y]$ which may be constructed as follows: Let $t$ denote the point of $[z, x]$ such that $g(t) = t$. $g'$ is defined by
\[ g'(u) = \{ u, \quad z \leq u \leq t \} \]. In any event, let \( g' \) and \( h' \) denote elements of \( I_o \) which map \([z, x]\) and \([x, w]\), respectively, onto \([z, y]\) and \([y, w]\), respectively. The function \( f \) defined by

\[
 f(u) = \begin{cases} 
 u, & u < z \text{ or } u > w \\
 g'(u), & z \leq u \leq x \\
h'(u), & x < u \leq w 
\end{cases}
\]

is an element of \( H_o \) such that \( f(x) = y \).

**Part 2.** Suppose \( L \) is homogeneous.

**Lemma 1.** If \((x, y) \in L \times L\), there is an element \( f \) of \( H_o \) such that \( f(x) = y \). Furthermore, if \( f \in I \) there is an element \( g \) of \( I_o \) having the same domain and range, respectively, as \( f \).

Proof. Suppose \( g \in H \) and \( g(x) = y \), but \( g \) is not in \( H_o \). There is a point \( b \) such that \( b = \#(\delta) \) and an element \( h \in H \) such that \( \Lambda(b) = \emptyset \). The function \( f = gh^{-1}g^{-1}h \) is in \( H_o \) and \( f(x) = y \). The proof of the second part of Lemma 1 follows easily from the first part and the proof of Part 1 of Theorem 2.

**Lemma 2.** Suppose \([a, b]\) is a closed interval and \( f \) and \( g \) are elements of \( I_o \) defined on \([a, b]\) such that \( f(a) = g(a) \) (\( f(b) = g(b) \)), but that \( f(x) < g(x) \) for \( a < x \leq b \) (\( a \leq x < b \)). If \( f(a) < x_0 < f(b) \) (\( g(a) < x_0 < g(b) \)) and \( x_1, x_2, \ldots \) is a sequence such that \( x_n = fg^{-1}(x_{n-1}) \) \((x_n = g^{-1}(x_{n-1}))\) for \( n \geq 1 \), then \( x_0, x_1, x_2, \ldots \) is a decreasing (increasing) sequence which converges to \( f(a) \) (\( f(b) \)).

Proof of first part. The inequality \( a < g^{-1}(x_0) < f^{-1}(x_0) < b \) implies that \( f(a) < x_1 = fg^{-1}(x_0) < x_0 < f(b) \). Suppose it has been established that \( f(a) < x_n < x_{n-1} < f(b) \). The preceding implies that \( a < g^{-1}(x_n) < f^{-1}(x_n) < b \), which implies that \( f(a) < x_{n+1} = fg^{-1}(x_n) < x_n < f(b) \). Therefore, \( x_0, x_1, x_2, \ldots \) is a decreasing sequence bounded below by \( f(a) \), and thus converges to a point \( x \geq f(a) \). Suppose \( x > f(a) \). Since \( gf^{-1}(x) > x \), there is a positive integer \( n \) such that \( gf^{-1}(x) > x_n > x \), which implies that \( x > fg^{-1}(x_n) = x_{n+1} \). This yields a contradiction, so \( x = f(a) \).

**Lemma 3.** If \( c \in L \) there exist an interval \([a, b]\) and elements \( f \) and \( g \) of \( I_o \) with domain \([a, b]\) such that \( f(a) = g(a) = c \) and \( f(x) < g(x) \), for \( a < x \leq b \); or if \( c \in L \) there exists an interval \([a, b]\) and elements \( f \) and \( g \) of \( I \) with domain \([a, b]\) such that \( f(b) = g(b) = c \) and \( f(x) < g(x) \), for \( a \leq x < b \).
Proof. Suppose that for each element \((x, y)\) of \(L \times L\) there is a unique element \(f\) of \(H_0\) such that \(f(x) = y\). Let \(u_1, u_2, \ldots\) denote an increasing sequence converging to a point \(u\), and for each \(n\), let \(f_n\) denote the element of \(H_0\) such that \(f_n(u) = u_n\). If \(x\) is an element of \(L\) and \(n\) a positive integer, then \(f_n(x) < f_{n+1}(x) < x\); for if this is not the case, the graph of \(f_n\) intersects the graph of \(f_{n+1}\), or the graph of \(f_{n+1}\) intersects the graph of the identity homeomorphism, and in either event there is a contradiction to the unique homeomorphism hypothesis. If for some \(x\), the sequence \(f_1(x), f_2(x), \ldots\) converges to a point \(y < x\), the element \(g\) of \(H_0\) such that \(g(x) = y\) has the property that its graph either intersects the graph of the identity function or the graph of \(f_n\), for some \(n\). Therefore, for any \(x\) in \(L\), the sequence \(f_1(x), f_2(x), \ldots\) is increasing and converges to \(x\).

For each positive integer \(j\), let \(a_{j1}, a_{j2}, \ldots\) and \(b_{j1}, b_{j2}, \ldots\) denote sequences such that (1) \(a_{j1} = f_{j-1}^{-1}(u)\) and \(b_{j1} = f_j(u)\), and (2) \(a_{jn} = f_{j-1}^{-1}(a_{j, n-1})\) and \(b_{jn} = f_j(b_{j, n-1})\), for \(n > 1\). Suppose \(u < x\) and \((r, s)\) is an open interval containing \(x\). Let \(n\) denote an integer such that \(r < f_n(x)\) and \(x < f_n(s)\). Since \(u < x < f_n(s)\), it follows that \(a_{n1} = f_{n-1}^{-1}(u) < s\). If \(a_{n1}\) is not in \((r, s)\), let \(K\) denote the set of all \(a_{nj}\) such that \(a_{nj} < x\) and let \(z = \text{l.u.b. } K\). If \(z \leq r\), there is an element \(a_{nj}\) of \(K\) such that \(f_n(z) < a_{nj} \leq z < f_n(x)\), which implies that \(z < f_n^{-1}(a_{nj}) = a_{nj+1} < x\), which is a contradiction. In any event, some \(a_{nj}\) is an element of \((r, s)\). The preceding argument clearly indicates that \(\sum (a_{ij} + b_{ij})\) is a countable set dense in \(L\), so \(L\) is a real line and the unique homeomorphism hypothesis is contradicted.

There exist elements \(h\) and \(k\) of \(H_0\) and points \(s\) and \(t\) of \(L\) such that \(h(s) = k(s)\), but \(h(t) < k(t)\). Suppose \(s < t\). Let a denote the largest element \(x\) of \(L\) such that \(h(x) = k(x)\) and \(x < t\). There is an element \(p\) of \(I_0\) with domain \([k(a), k(t)]\) such that \(p(k(a)) = c\). The functions \(f = p(h)\) and \(g = p(k)\) and the interval \([a, t]\) satisfy the first conclusion of the lemma. The case \(t < s\) yields the second conclusion.

**Lemma 4.** Suppose \([a, b]\) is a closed interval and \(c\) is a point. If \(x > c\), there is a point \(y\) in \((c, x)\) and an element \(f\) of \(I_0\) mapping \([a, b]\) onto \([c, y]\).

Proof. Let \(U\) denote the set of all \(x > c\) such that there is a homeomorphism from \([a, b]\) onto \([c, x]\), and let \(V\) denote the set of all \(x < c\) such that there is a homeomorphism from \([a, b]\) onto \([x, c]\). The sets \(U\) and \(V\) exist because of the existence of elements \(h_1\) and \(h_2\) of \(H_{\infty}\) such that \(h_1(a) = c\) and \(h_2(b) = c\). Let \(u = \text{g.l.b. } U\), \(v = \text{l.u.b. } V\) and suppose that \(c < u\).
Case 1. Suppose the first conclusion of Lemma 3 holds. There exists a point $u$, an interval $[p, q]$, and elements $f$ and $g$ of $I_0$ having domain $[p, q]$, and such that (1) $c < u < u_1$, (2) $f(p) = g(p) = u_1$, and (3) $f(x) < g(x)$, for $p < x \leq q$. There is a point $r$ such that $p < r < q$, $g(r) < f(q)$, and an element $k$ of $I_0$ having domain $[p, q]$ such that (1) $k(r) = u$, and (2) $k(x) \geq g(x)$ for $x \in [p, q]$. The function $h$ defined on $[p, q]$ by $h(x) = kg^{-1}(x)$ is an element of $I_0$ such that (1) $h(q) > u$, (2) $h(p) = k(p)$, and (3) $h(x) < k(x)$, for $p < x \leq q$. There is a point $x_0$ such that $u \leq x_0 < h(q)$ and an element $f_0$ of $I_0$ mapping $[a, b]$ onto $[c, x_0]$. Let $x_1, x_2, \ldots$ denote a sequence such that $x_n = h^{-1}(x_{n-1})$ for $n \geq 1$, and let $f_1, f_2, \ldots$ denote a sequence of functions defined on $[a, b]$ such that for $n \geq 1$ (1) $f_n(a) = f_0(a)$, for $a \leq x \leq f_{n-1}(u_1)$, and (2) $f_n(x) = h^{-1}(f_{n-1}(x))$, for $f_{n-1}(u_1) < x \leq b$. For each $n$, $f_n$ is a homeomorphism from $[a, b]$ onto $[c, u_2]$. The function $g$ defined by

$$f(x) = \begin{cases} h(x), & a \leq x \leq t \\ gh(x), & t < x \leq b \end{cases}$$

is an element of $I_0$ which maps $[a, b]$ onto $[c, u_1]$, so in this case also, the assumption $c < u$ leads to a contradiction.

The proof of the main result now follows easily. Suppose $[a, b]$ and $[c, d]$ are closed intervals and $g$ an element of $H_0$ such that $g(b) = d$.

Case 2. If the second conclusion of Lemma 3 holds, then it follows, by an argument similar to the one in Case 1, that $v = c$. Let $u_1$ denote a point between $c$ and $u$, and $g$ an element of $H_0$ such that $g(c) = u_1$. There is a point $u_2$ such that $c < u_2 < u_1$ and an element $h$ of $I_0$ mapping $[a, b]$ onto $[c, u_2]$. The function $g(h)$ is an element of $I_0$ mapping $[a, b]$ onto $[u_2, u_1]$. Let $k$ denote an element of $H_0$ such that $k(a) = c$. Since $k(\delta) \geq u$, there is a point $t$ such that $k(t) = gh(t)$. The function $f$ defined by

$$f(x) = \begin{cases} h(x), & a \leq x \leq t \\ gh(x), & t < x \leq b \end{cases}$$

is an element of $I_0$ which maps $[a, b]$ onto $[c, u_1]$, so in this case also, the assumption $c < u$ leads to a contradiction.

In order to establish the next theorem it is helpful to use a result
of Richard Arens'. A linear homogeneous continuum (LHC) has been defined by G. D. Birkhoff as any set of elements which 1. is simply ordered 2. provides a limit for any monotonelly increasing (or decreasing) sequence 3. is isomorphic to every nondegenerate closed sub-interval of itself. In [1] Arens shows, among other results, the following (reworded by the author).

**Theorem A.** If \( I \) is an LHC and for each positive integer \( p \), \( I_p \) denotes \( I \), then the space \( I' = I_1 \times I_2 \times \cdots \) with the lexicographic order is also an LHC.

**Theorem 3.** If \( L \) is homogeneous, \([a, b]\) is a closed interval, and for each positive integer \( p \), \( I_p \) denotes \([a, b]\), then the space \( X = L \times I_1 \times I_2 \times \cdots \) with the topology induced by the lexicographic order is also homogeneous.

**Proof.** Let \([u_1, u_2, \cdots; v_1, v_2, \cdots]\) and \([x_1, x_2, \cdots; y_1, y_2, \cdots]\) denote closed subintervals of \( X \). Let \( u \) and \( v \) denote elements of \( L \) such that \( u < \min \{u_i, x_i\} \) and \( v > \max \{v_i, y_i\} \) for \( i = 1, 2, 3, \ldots \), and let \( g \) denote an element of \( I_0 \) which maps \([u, v]\) onto \([a, b]\). The function \( F \) defined by \( F(t_0, t_1, t_2, \cdots) = [g(t_0), t_1, t_2, \cdots] \) is an order preserving homeomorphism from \([u, v] \times I_1 \times I_2 \times \cdots \) onto \([a, b] \times I_1 \times I_2 \times \cdots \). Theorem A shows that any two subintervals of the latter are homeomorphic, so it follows that \([x_1, x_2, \cdots; y_1, y_2, \cdots]\) and \([u_1, u_2, \cdots; v_1, v_2, \cdots]\) are homeomorphic. Therefore, by theorem 2, \( X \) is homogeneous.

Suppose \( L_1, L_2, L_3, \cdots \) denotes a sequence of spaces such that 1) \( L_1 \) is the real line, and 2) for each \( n \), \( L_{n+1} \) is constructed from \( L_n \) by a Theorem 3 type construction. The main theorem of Arens’ paper [2] yields the result that if \( i \neq j \), then \( L_i \) is not homeomorphic to \( L_j \). Is it true that if a homogeneous space \( L' \) satisfies the axioms stated on the first page and also has the property that it can be covered by a countable collection of closed intervals, then \( L' \) is one of the spaces \( L_1, L_2, L_3, \cdots \)?

In part 2 of Theorem 2 the construction indicated gives an order preserving homeomorphism from \([a, b]\) onto \([c, d]\). This leads naturally to the following question: If \( L' \) satisfies the axioms of \( L \), is homogeneous, and \([a, b]\) is a closed subinterval of \( L' \), then is there an order reversing homeomorphism from \([a, b]\) onto \([a, b]\) ?

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