CONCERNING HOMOGENEITY IN TOTALLY ORDERED, CONNECTED TOPOLOGICAL SPACE

L. Bruce Treybig
Throughout this paper suppose that $L$ denotes a connected, totally ordered topological space in which there is no first or last point, and whose topology is that induced by the order.

A topological space $S$ is said to be homogeneous provided it is true that if $(x, y)\in S \times S$, there is a homeomorphism $f$ from $S$ onto $S$ such that $f(x) = y$. Let $H$ denote the set of all homeomorphisms from $L$ onto $L$, and let $I$ denote the set of all homeomorphisms which map a closed interval of $L$ onto a closed interval of $L$. Let $H_0(I_0)$ denote the set of all elements of $H(I)$ which preserve order.

**Theorem 1.** If $L$ is homogeneous, then $L$ satisfies the first axiom of countability.

*Proof.* It suffices to show that for some point $z$ of $L$ there exists an increasing sequence $x_1, x_2, \ldots$ and a decreasing sequence $y_1, y_2, \ldots$ such that each of these sequences converges to $z$. Suppose there is no such point. Let $P_1, P_2, \ldots$ denote an increasing sequence which converges to a point $P$ and $Q_1, Q_2, \ldots$ a decreasing sequence which converges to a point $Q$. There is an element $g$ in $H$ such that $g(P) = Q$. In view of the preceding supposition, $g$ is order reversing. There is a point $R$ such that $g(R) = R$, and $R$ is the limit of a sequence $R_1, R_2, \ldots$ which is either increasing or decreasing. Suppose the sequence is decreasing. The sequence $g(R_1), g(R_2), \ldots$ is increasing and converges to $R$. This yields a contradiction. The case where $R_1, R_2, \ldots$ is increasing is similar.

**Theorem 2.** The space $L$ is homogeneous if and only if each pair of closed subintervals of $L$ are topologically equivalent.

*Proof.* Part 1. Suppose each pair of closed subintervals of $L$ are topologically equivalent and $(x, y)\in L \times L$. There exist elements $z$ and $w$ of $L$ such that $z < x < w$ and $z < y < w$, and an element $g$ of $I$ from $[z, x]$ onto $[z, y]$. If $g$ is order reversing there is an element $g'$ of $I_0$ from $[z, x]$ onto $[z, y]$ which may be constructed as follows: Let $t$ denote the point of $[z, x]$ such that $g(t) = t$. $g'$ is defined by
\( g'(u) = \begin{cases} 
  u, & z \leq u \leq t \\
  g(g(u)), & t < u \leq x 
\end{cases} \). In any event, let \( g' \) and \( h' \) denote elements of \( I_o \) which map \([z, x]\) and \([x, w]\), respectively, onto \([z, y]\) and \([y, w]\), respectively. The function \( f \) defined by

\[
\begin{align*}
  f(u) &= \begin{cases}
    u, & u < z \text{ or } u > w \\
    g'(u), & z \leq u \leq x \\
    h'(u), & x < u \leq w 
  \end{cases} 
\end{align*}
\]

is an element of \( H_o \) such that \( f(x) = y \).

**Part 2.** Suppose \( L \) is homogeneous.

**Lemma 1.** If \( (x, y) \in L \times L \), there is an element \( f \) of \( H_o \) such that \( f(x) = y \). Furthermore, if \( f \in I \) there is an element \( g \) of \( I_o \) having the same domain and range, respectively, as \( f \).

*Proof.* Suppose \( g \in H \) and \( g(x) = y \), but \( g \) is not in \( H_o \). There is a point \( b \) such that \( b = g(b) \) and an element \( h \) of \( H \) such that \( h(x) = b \). The function \( f = gh^{-1}g^{-1}h \) is in \( H_o \) and \( f(x) = y \). The proof of the second part of Lemma 1 follows easily from the first part and the proof of Part 1 of Theorem 2.

**Lemma 2.** Suppose \([a, b]\) is a closed interval and \( f \) and \( g \) are elements of \( I_o \) defined on \([a, b]\) such that \( f(a) = g(a) \) (\( f(b) = g(b) \)), but that \( f(x) < g(x) \) for \( a < x \leq b \) (\( a \leq x < b \)). If \( f(a) < x_0 < f(b) \) (\( g(a) < x_0 < g(b) \)) and \( x_1, x_2, \ldots \) is a sequence such that \( x_n = fg^{-1}(x_{n-1}) \) (\( x_n = gf^{-1}(x_{n-1}) \)) for \( n \geq 1 \), then \( x_0, x_1, x_2, \ldots \) is a decreasing (increasing) sequence which converges to \( f(a) \) (\( f(b) \)).

*Proof of first part.* The inequality \( a < g^{-1}(x_0) < f^{-1}(x_0) < b \) implies that \( f(a) < x_1 = fg^{-1}(x_0) < x_0 < f(b) \). Suppose it has been established that \( f(a) < x_n < x_{n-1} < f(b) \). The preceding implies that \( a < g^{-1}(x_n) < f^{-1}(x_n) < b \), which implies that \( f(a) < x_{n+1} = fg^{-1}(x_n) < x_n < f(b) \). Therefore, \( x_0, x_1, x_2, \ldots \) is a decreasing sequence bounded below by \( f(a) \), and thus converges to a point \( x \geq f(a) \). Suppose \( x > f(a) \). Since \( gf^{-1}(x) > x \), there is a positive integer \( n \) such that \( gf^{-1}(x) > x_n > x \), which implies that \( x > gf^{-1}(x_n) = x_{n+1} \). This yields a contradiction, so \( x = f(a) \).

**Lemma 3.** If \( c \in L \) there exist an interval \([a, b]\) and elements \( f \) and \( g \) of \( I_o \) with domain \([a, b]\) such that \( f(a) = g(a) = c \) and \( f(x) < g(x) \), for \( a < x \leq b \); or if \( c \in L \) there exists an interval \([a, b]\) and elements \( f \) and \( g \) of \( I_o \) with domain \([a, b]\) such that \( f(b) = g(b) = c \) and \( f(x) < g(x) \), for \( a \leq x < b \).
Proof. Suppose that for each element \((x, y)\) of \(L \times L\) there is a unique element \(f\) of \(H_0\) such that \(f(x) = y\). Let \(u_1, u_2, \ldots\) denote an increasing sequence converging to a point \(u\), and for each \(n\), let \(f_n\) denote the element of \(H_0\) such that \(f_n(u) = u_n\). If \(x\) is an element of \(L\) and \(n\) a positive integer, then \(f_n(x) < f_{n+1}(x) < x\); for if this is not the case, the graph of \(f_n\) intersects the graph of \(f_{n+1}\), or the graph of \(f_{n+1}\) intersects the graph of the identity homeomorphism, and in either event there is a contradiction to the unique homeomorphism hypothesis. If for some \(x\), the sequence \(f_n(x), f_{n+1}(x), \ldots\) converges to a point \(y < x\), the element \(g\) of \(H_0\) such that \(g(x) = y\) has the property that its graph either intersects the graph of the identity function or the graph of \(f_n\), for some \(n\). Therefore, for any \(x\) in \(L\), the sequence \(f_1(x), f_2(x), \ldots\) is increasing and converges to \(x\).

For each positive integer \(j\), let \(a_{j1}, a_{j2}, \ldots\) and \(b_{j1}, b_{j2}, \ldots\) denote sequences such that (1) \(a_{j1} = f_{j}^{-1}(u)\) and \(b_{j1} = f_{j}(u)\), and (2) \(a_{jn} = f_{j}^{-1}(a_{j, n-1})\) and \(b_{jn} = f_{j}(b_{j, n-1})\), for \(n > 1\). Suppose \(u < x\) and \((r, s)\) is an open interval containing \(x\). Let \(n\) denote an integer such that \(r < f_n(x)\) and \(x < f_n(s)\). Since \(u < x < f_n(s)\), it follows that \(a_{n1} = f_{n}^{-1}(u) < s\). If \(a_{n1}\) is not in \((r, s)\), let \(K\) denote the set of all \(a_{nj}\) such that \(a_{nj} < x\) and let \(z = \text{l.u.b. } K\). If \(z \leq r\), there is an element \(a_{nj}\) of \(K\) such that \(f_n(z) < a_{nj} \leq z < f_n(x)\), which implies that \(z < f_n^{-1}(a_{nj}) = a_{nj+1} < x\), which is a contradiction. In any event, some \(a_{nj}\) is an element of \((r, s)\). The preceding argument clearly indicates that \(\sum (a_{ij} + b_{ij})\) is a countable set dense in \(L\), so \(L\) is a real line and the unique homeomorphism hypothesis is contradicted.

There exist elements \(h\) and \(k\) of \(H_0\) and points \(s\) and \(t\) of \(L\) such that \(h(s) = k(s)\), but \(h(t) < k(t)\). Suppose \(s < t\). Let \(u\) denote the largest element \(x\) of \(L\) such that \(h(x) = k(x)\) and \(x < t\). There is an element \(p\) of \(I_0\) with domain \([k(a), k(t)]\) such that \(p(k(a)) = c\). The functions \(f = p(h)\) and \(g = p(k)\) and the interval \([a, t]\) satisfy the first conclusion of the lemma. The case \(t < s\) yields the second conclusion.

**Lemma 4.** Suppose \([a, b]\) is a closed interval and \(c\) is a point. If \(x > c\), there is a point \(y\) in \((c, x)\) and an element \(f\) of \(I_0\) mapping \([a, b]\) onto \([c, y]\).

Proof. Let \(U\) denote the set of all \(x > c\) such that there is a homeomorphism from \([a, b]\) onto \([c, x]\), and let \(V\) denote the set of all \(x < c\) such that there is a homeomorphism from \([a, b]\) onto \([x, c]\). The sets \(U\) and \(V\) exist because of the existence of elements \(h_1\) and \(h_2\) of \(H_0\) such that \(h_1(a) = c\) and \(h_2(b) = c\). Let \(u = \text{g.1.b. } U\), \(v = \text{1.u.b. } V\) and suppose that \(c < u\).
Case 1. Suppose the first conclusion of Lemma 3 holds. There exists a point \( u \), an interval \([p, q]\), and elements \( f \) and \( g \) of \( I_o \) having domain \([p, q]\), and such that (1) \( c < u < u \), (2) \( f(p) = g(p) = u \), and (3) \( f(x) < g(x) \), for \( p < x \leq q \). There is a point \( r \) such that \( p < r < q \), \( g(r) < u \), and \( g(r) < f(q) \), and an element \( k \) of \( I_o \) having domain \([p, q]\) such that (1) \( k(r) = u \), and (2) \( k(x) \geq g(x) \) for \( x \in [p, q] \). The function \( k \) defined on \([p, q]\) by \( k(x) = kg^{-1}(x) \) is an element of \( I_o \) such that (1) \( k(q) > u \), (2) \( k(p) = k(p) \), and (3) \( k(x) < k(x) \), for \( p < x \leq q \). There is a point \( s \), such that \( u \leq s < h(q) \) and an element \( f' \) of \( I_o \) mapping \([a, b]\) onto \([c, x_0]\). Let \( x_1, x_2, \ldots \) denote a sequence such that \( x_n = h^{-1}(x_{n-1}) \) for \( n \geq 1 \), and let \( f_1, f_2, \ldots \) denote a sequence of functions defined on \([a, b]\) such that for \( n \geq 1 \) (1) \( f_n(x) = f_n(x) \), for \( a \leq x \leq f^{-1}_n(u) \), and (2) \( f_n(x) = h^{-1}f_n(x) \), for \( f^{-1}_n(u) < x \leq b \). For each \( n, f_n \) is a homeomorphism from \([a, b]\) onto \([c, x_0]\), but, according to Lemma 2, \( x_n < u \) for some \( n \). This yields a contradiction, so \( u = c \).

Case 2. If the second conclusion of Lemma 3 holds, then it follows, by an argument similar to the one in Case 1, that \( v = c \). Let \( u_1 \) denote a point between \( c \) and \( u \), and \( g \) an element of \( H_o \) such that \( g(c) = u_1 \). There is a point \( u_2 \) such that \( c < u_2 < u \) and an element \( h \) of \( I_o \) mapping \([a, b]\) onto \([g^{-1}(u_2), c]\). The function \( g(h) \) is an element of \( I_o \) mapping \([a, b]\) onto \([c, d]\), so in this case also, the assumption \( c < u \) leads to a contradiction.

The proof of the main result now follows easily. Suppose \([a, b]\) and \([c, d]\) are closed intervals and \( g \) an element of \( H_o \) such that \( g(b) = d \).

Case 1. \( g(a) \leq c \). There is a point \( e \) such that \( c < e < d \) and an element \( h \) of \( I_o \) mapping \([a, b]\) onto \([c, e]\). As in case 2 of Lemma 4, a homeomorphism from \([a, b]\) onto \([c, d]\) may be constructed from \( g \) and \( h \).

Case 2. \( g(a) > c \). There is a point \( e \) such that \( a < e < b \) and an element \( h \) of \( I_o \) mapping \([c, d]\) onto \([a, e]\). However, \( h^{-1} \) is an element of \( I_o \) mapping \([a, e]\) onto \([c, d]\), and a homeomorphism from \([a, b]\) onto \([c, d]\) may be easily constructed from \( g \) and \( h^{-1} \).

In order to establish the next theorem it is helpful to use a result
of Richard Arens'. A linear homogeneous continuum (LHC) has been defined by G. D. Birkhoff as any set of elements which 1. is simply ordered 2. provides a limit for any monotonely increasing (or decreasing) sequence 3. is isomorphic to every nondegenerate closed subinterval of itself. In [1] Arens shows, among other results, the following (reworded by the author).

**Theorem A.** If I is an LHC and for each positive integer p, \( I_p \) denotes I, then the space \( I' = I_1 \times I_2 \times \cdots \) with the lexicographic order is also an LHC.

**Theorem 3.** If \( I \) is homogeneous, \([a, b]\) is a closed interval, and for each positive integer p, \( I_p \) denotes \([a, b]\), then the space \( x = L \times I_1 \times I_2 \times \cdots \) with the topology induced by the lexicographic order is also homogeneous.

**Proof.** Let \([u_1, u_2, \ldots; v_1, v_2, \ldots]\) and \([x_1, x_2, \ldots; y_1, y_2, \ldots]\) denote closed subintervals of \( X \). Let \( u \) and \( v \) denote elements of \( L \) such that \( u < \min \{u_i, x_i\} \) and \( v > \max \{v_i, y_i\} \) for \( i = 1, 2, 3, \ldots \), and let \( g \) denote an element of \( I_0 \) which maps \([u, v]\) onto \([a, b]\). The function \( F' \) defined by \( F(t_0, t_1, t_2, \cdots) = [g(t_0), t_1, t_2, \cdots] \) is an order preserving homeomorphism from \([u, v] \times I_1 \times I_2 \times \cdots\) onto \([a, b] \times I_1 \times I_2 \times \cdots\). Theorem A shows that any two subintervals of the latter are homeomorphic, so it follows that \([x_1, x_2, \ldots; y_1, y_2, \ldots]\) and \([u_1, u_2, \ldots; v_1, v_2, \ldots]\) are homeomorphic. Therefore, by theorem 2, \( X \) is homogeneous.

Suppose \( L_1, L_2, L_3, \cdots \) denotes a sequence of spaces such that (1) \( L_1 \) is the real line, and (2) for each \( n \), \( L_{n+1} \) is constructed from \( L_n \) by a Theorem 3 type construction. The main theorem of Arens' paper [2] yields the result that if \( i \neq j \), then \( L_i \) is not homeomorphic to \( L_j \). Is it true that if a homogeneous space \( L' \) satisfies the axioms stated on the first page and also has the property that it can be covered by a countable collection of closed intervals, then \( L' \) is one of the spaces \( L_1, L_2, L_3, \cdots \)?

In part 2 of Theorem 2 the construction indicated gives an order preserving homeomorphism from \([a, b]\) onto \([c, d]\). This leads naturally to the following question: If \( L' \) satisfies the axioms of \( L \), is homogeneous, and \([a, b]\) is a closed subinterval of \( L' \), then is there an order reversing homeomorphism from \([a, b]\) onto \([a, b]\) ?

**References**


Tulane University
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Dallas O. Banks, *Bounds for eigenvalues and generalized convexity* ................................................ 1031
Woodrow Wilson Bledsoe and A. P. Morse, *A topological measure construction* .................................. 1067
George Clements, *Entropies of several sets of real valued functions* ..................................................... 1085
Sandra Barkdull Cleveland, *Homomorphisms of non-commutative ∗-algebras* ............................. 1097
William John Andrew Culmer and William Ashton Harris, *Convergent solutions of ordinary linear homogeneous difference equations* ........................................... 1111
Ralph DeMarr, *Common fixed points for commuting contraction mappings* ....................................... 1139
James Robert Dorroh, *Integral equations in normed abelian groups* .................................................. 1143
Adriano Mario Garsia, *Entropy and singularity of infinite convolutions* ......................................... 1159
J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., *Convergence of extended Bernstein polynomials in the complex plane* ........................................... 1171
Irving Leonard Glicksberg, *A remark on analyticity of function algebras* ....................................... 1181
Charles John August Halberg, Jr., *Semigroups of matrices defining linked operators with different spectra* ........................................................................................................ 1187
Isidore Heller, *On a class of equivalent systems of linear inequalities* .............................................. 1209
Joseph Hersch, *The method of interior parallels applied to polygonal or multiply connected membranes* ........................................................................................................ 1229
Hans F. Weinberger, *An effectless cutting of a vibrating membrane* .................................................. 1239
Melvin F. Janowitz, *Quantifiers and orthomodular lattices* ............................................................... 1241
Tilla Weinstein, *Another conformal structure on immersed surfaces of negative curvature* ............. 1281
Gregers Louis Krabbe, *Spectral permanence of scalar operators* ......................................................... 1289
Shige Toshi Kuroda, *Finite-dimensional perturbation and a representation of scattering operator* .......... 1305
Marvin David Marcus and Afton Herbert Cayford, *Equality in certain inequalities* ......................... 1319
Joseph Martin, *A note on uncountably many disks* ............................................................................. 1331
Eugene Kay McLachlan, *Extremal elements of the convex cone of semi-norms* ............................ 1335
John W. Moon, *An extension of Landau’s theorem on tournaments* ............................................... 1343
Louis Joel Mordell, *On the integer solutions of y(y + 1) = x(x + 1)(x + 2)* .................................... 1347
Kenneth Roy Mount, *Some remarks on Fitting’s invariants* ............................................................. 1353
Miroslav Novotný, *Über Abbildungen von Mengen* ........................................................................... 1359
Robert Dean Ryan, *Conjugate functions in Orlicz spaces* ............................................................... 1371
John Vincent Ryff, *On the representation of doubly stochastic operators* ........................................ 1379
Donald Ray Sherbert, *Banach algebras of Lipschitz functions* ......................................................... 1387
James McLean Sloss, *Reflection of biharmonic functions across analytic boundary conditions with examples* ........................................................................................................ 1401
L. Bruce Treybig, *Concerning homogeneity in totally ordered, connected topological space* .......... 1417
John Wermer, *The space of real parts of a function algebra* ............................................................. 1423
James Juei-Chin Yeh, *Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables* ................................................. 1427
William P. Ziemer, *On the compactness of integral classes* ............................................................. 1437