

Pacific Journal of Mathematics

THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA

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1. Introduction. Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X . We consider a closed subalgebra A of $C(X)$ which separates the points of X and contains the constants. We call A "a function algebra on X ".

Let $Re A$ denote the class of functions u real and continuous on X such that for some f in A , $u = Re f$. Then $Re A$ is a real vector space of real continuous functions on X . What more can be said about $Re A$?

In [3] it was shown that $Re A$ cannot be closed under uniform convergence on X unless $A = C(X)$. Here we shall show that $Re A$ cannot be closed under multiplication unless $A = C(X)$. In other words:

Theorem 1: If $Re A$ is a ring, then $A = C(X)$.

This result was conjectured by K. Hoffman. As a corollary one gets the existence of a continuous function u on the unit circle having the following property: u has a continuous conjugate function (in the sense of Fourier theory) whereas u^2 does not. For we may take for A the algebra of continuous functions on the circle which extend analytically to the unit disk. Then $Re A$ is the class of all functions which are continuous and have continuous conjugates. But $A \neq C(X)$. Hence by Theorem 1, $Re A$ is not a ring, hence not closed under squaring, and so the desired u exists.

The existence of such a u had been shown in 1961 by J. P. Kahane (unpublished). It should be noted that if a function u is sufficiently smooth to have an absolutely convergent Fourier series, then u^2 does also, and hence u^2 does have a continuous conjugate.

2. The antisymmetric case. In this section we assume that A is anti-symmetric, i.e. contains no real functions except constants, and prove Theorem 1 under this hypothesis. This amounts to proving:

THEOREM 1'. *Let A be anti-symmetric and let $Re A$ form a ring. Then X consists of a single point.*

Assume X contains a point x_0 and another point x_1 . We must deduce a contradiction. Fix u in $Re A$. Then (because of antisymmetry),

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there exists exactly one f in A with $u = \operatorname{Re} f$ and $\operatorname{Im} f(x_0) = 0$. The map: u into f is now a real-linear map of $\operatorname{Re} A$ into A which is one-to-one. We can then norm $\operatorname{Re} A$ by the norm N :

$$N(u) = \max_x |f| = \|f\|.$$

In this norm $\operatorname{Re} A$ is then evidently a real Banach space. By standard application of the closed graph theorem, we have

LEMMA 1. *There exists a constant K such that for all u, u' in $\operatorname{Re} A$*

$$N(u \cdot u') \leq K \cdot N(u) \cdot N(u').$$

LEMMA 2. *If p lies in $\operatorname{Re} A$ and $p > 0$ on X , then $\log p$ is in $\operatorname{Re} A$.*

Proof. Let S be the class of functions $u + iu'$ with u and u' in $\operatorname{Re} A$. Then S is an algebra of complex-valued functions on X containing A as a subalgebra and closed under complex conjugation. Define $N(u + iu') = N(u) + N(u')$ and $\|f\|' = \sup_{\theta} N(e^{i\theta} f)$ for all f in S . Then S is a (complex) Banach space under $\|\cdot\|'$ as norm and also $\|f \cdot g\|' \leq K \|f\|' \cdot \|g\|'$. Hence S is a Banach algebra under a norm equivalent to $\|\cdot\|'$.

Let M_S denote the space of homomorphisms of S into the complex numbers and M_A be the corresponding space for A . Fix m in M_S . Restricted to A , m is an element σ of M_A . Also the map: f into $\overline{m(f)}$, restricted to A , is an element τ of M_A . Since p lies in $\operatorname{Re} A$, we can find some r in A such that

$$p = \frac{1}{2} (r + \bar{r}) \quad \text{whence} \quad m(p) = \frac{1}{2} (m(r) + \overline{m(r)}),$$

or

$$m(p) = \frac{1}{2} (\sigma(r) + \overline{\tau(r)}).$$

By hypothesis, $\operatorname{Re} r = p > 0$ on X . Hence by a well-known property of function algebras, $\operatorname{Re} \beta(r) > 0$ for all β in M_A . In particular $\operatorname{Re} \sigma(r) > 0$ and $\operatorname{Re} \tau(r) > 0$. Hence $\operatorname{Re} m(p) > 0$.

Since this holds for all m in M_S , we can, by the general theory of Banach algebras, apply to p any function analytic in the right half-plane and still stay in the algebra S . Hence $\log p$ is an element of S , and, being real valued, of $\operatorname{Re} A$.

Let now K^* be any positive number. Choose g in A with $g(x_0) = 0$

and $\|g\| = 1$. Let a be some point in X where $|g(a)| = 1$. Next choose φ analytic in $|z| < 1$, continuous in $|z| \leq 1$, such that $0 < \operatorname{Re} \varphi \leq 1$ in $|z| \leq 1$, $\operatorname{Im} \varphi(0) = 0$ and $\operatorname{Im} \varphi(g(a)) \geq K^*$. Put $f = \varphi(g)$. Then f belongs to A and we have:

$$0 < \operatorname{Re} f \leq 1 \text{ on } X, \quad \operatorname{Im} f(x_0) = 0 \text{ and } \|f\| \geq K^* .$$

Then $\operatorname{Re} f$ is in $\operatorname{Re} A$ and > 0 . By Lemma 2, then, $\log(\operatorname{Re} f)$ also is in $\operatorname{Re} A$, i.e. there is some F in A with $\operatorname{Re} F = \log(\operatorname{Re} f)$. Put now $V = \exp(\frac{1}{2}F)$. Then again V is in A . Also $|V|^2 = \operatorname{Re} f$. Then $\max_x |V| = \|V\| \leq 1$.

We now use the following identity, true for each complex z :

$$(\operatorname{Re} z)^2 = \frac{1}{2} (\operatorname{Re} z^2 + |z|^2) .$$

Applying this to V and using that $|V|^2 = \operatorname{Re} f$, we get

$$(\operatorname{Re} V)^2 = \operatorname{Re} \left(\frac{1}{2} (V^2 + f) \right) .$$

Clearly for each h in A , we have $N(\operatorname{Re} h) \geq \|h\| - |\operatorname{Im} h(x_0)|$. Hence $N((\operatorname{Re} V)^2) \geq \frac{1}{2} (\|V^2 + f\| - |\operatorname{Im} V^2(x_0)|) \geq \frac{1}{2} (K^* - 2)$, since $\|f\| \geq K^*$ while $\|V^2\| \leq 1$.

On the other hand, by Lemma 1,

$$N((\operatorname{Re} V)^2) \leq K \cdot (N(\operatorname{Re} V))^2 \quad \text{and} \quad N((\operatorname{Re} V)) \leq 2 \|V\| \leq 2 .$$

Since K^* is arbitrary while K is fixed, we have a contradiction. Thus Theorem 1' is proved.

3. The general case. To deduce the result in the general case from Theorem 1', we use the following theorem of Bishop [1]. (See also [2].):

THEOREM. *Let A be any function algebra on X . Then there exists a collection Φ of closed, pairwise disjoint sets covering X so that*

- (a) f in $C(X)$ and $f|K$ in $A|K$ for every K in Φ imply f in A ;
- (b) $A|K$ is closed in $C(K)$ for each K in Φ .
- (c) $A|K$ is antisymmetric on K for each K in Φ .

Because of Bishop's theorem, one has the following method of reasoning: let (P) be a property which has meaning for every function algebra A . Assume

(i) Whenever a given A has property (P) , then so does each restriction algebra $A|K$ for K in Φ , and

(ii) Whenever A is antisymmetric on the space X and A has

property (P) , then X consists of a single point.

We then conclude, using the Theorem, that if A is a function algebra on a space X such that A has property (P) , then $A = C(X)$. Thus, if (P) is the property “ A is closed under complex conjugation”, (i) and (ii) clearly hold, and one concludes the Stone-Weierstrass theorem.

If (P) is the property “ $Re A$ is a ring”, then (i) also clearly holds, and that (ii) holds was the content of Theorem 1'. Thus we may conclude Theorem 1.

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