

# Pacific Journal of Mathematics

**THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA**

JOHN WERMER

# THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA

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**1. Introduction.** Let  $X$  be a compact Hausdorff space and  $C(X)$  the algebra of all complex-valued continuous functions on  $X$ . We consider a closed subalgebra  $A$  of  $C(X)$  which separates the points of  $X$  and contains the constants. We call  $A$  "a function algebra on  $X$ ".

Let  $Re A$  denote the class of functions  $u$  real and continuous on  $X$  such that for some  $f$  in  $A$ ,  $u = Re f$ . Then  $Re A$  is a real vector space of real continuous functions on  $X$ . What more can be said about  $Re A$ ?

In [3] it was shown that  $Re A$  cannot be closed under uniform convergence on  $X$  unless  $A = C(X)$ . Here we shall show that  $Re A$  cannot be closed under multiplication unless  $A = C(X)$ . In other words:

*Theorem 1: If  $Re A$  is a ring, then  $A = C(X)$ .*

This result was conjectured by K. Hoffman. As a corollary one gets the existence of a continuous function  $u$  on the unit circle having the following property:  $u$  has a continuous conjugate function (in the sense of Fourier theory) whereas  $u^2$  does not. For we may take for  $A$  the algebra of continuous functions on the circle which extend analytically to the unit disk. Then  $Re A$  is the class of all functions which are continuous and have continuous conjugates. But  $A \neq C(X)$ . Hence by Theorem 1,  $Re A$  is not a ring, hence not closed under squaring, and so the desired  $u$  exists.

The existence of such a  $u$  had been shown in 1961 by J. P. Kahane (unpublished). It should be noted that if a function  $u$  is sufficiently smooth to have an absolutely convergent Fourier series, then  $u^2$  does also, and hence  $u^2$  does have a continuous conjugate.

**2. The antisymmetric case.** In this section we assume that  $A$  is anti-symmetric, i.e. contains no real functions except constants, and prove Theorem 1 under this hypothesis. This amounts to proving:

**THEOREM 1'.** *Let  $A$  be anti-symmetric and let  $Re A$  form a ring. Then  $X$  consists of a single point.*

Assume  $X$  contains a point  $x_0$  and another point  $x_1$ . We must deduce a contradiction. Fix  $u$  in  $Re A$ . Then (because of antisymmetry),

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there exists exactly one  $f$  in  $A$  with  $u = \operatorname{Re} f$  and  $\operatorname{Im} f(x_0) = 0$ . The map:  $u$  into  $f$  is now a real-linear map of  $\operatorname{Re} A$  into  $A$  which is one-to-one. We can then norm  $\operatorname{Re} A$  by the norm  $N$ :

$$N(u) = \max_x |f| = \|f\|.$$

In this norm  $\operatorname{Re} A$  is then evidently a real Banach space. By standard application of the closed graph theorem, we have

**LEMMA 1.** *There exists a constant  $K$  such that for all  $u, u'$  in  $\operatorname{Re} A$*

$$N(u \cdot u') \leq K \cdot N(u) \cdot N(u').$$

**LEMMA 2.** *If  $p$  lies in  $\operatorname{Re} A$  and  $p > 0$  on  $X$ , then  $\log p$  is in  $\operatorname{Re} A$ .*

*Proof.* Let  $S$  be the class of functions  $u + iu'$  with  $u$  and  $u'$  in  $\operatorname{Re} A$ . Then  $S$  is an algebra of complex-valued functions on  $X$  containing  $A$  as a subalgebra and closed under complex conjugation. Define  $N(u + iu') = N(u) + N(u')$  and  $\|f\|' = \sup_{\theta} N(e^{i\theta}f)$  for all  $f$  in  $S$ . Then  $S$  is a (complex) Banach space under  $\|\cdot\|'$  as norm and also  $\|f \cdot g\|' \leq K\|f\|' \cdot \|g\|'$ . Hence  $S$  is a Banach algebra under a norm equivalent to  $\|\cdot\|'$ .

Let  $M_S$  denote the space of homomorphisms of  $S$  into the complex numbers and  $M_A$  be the corresponding space for  $A$ . Fix  $m$  in  $M_S$ . Restricted to  $A$ ,  $m$  is an element  $\sigma$  of  $M_A$ . Also the map:  $f$  into  $\overline{m(\bar{f})}$ , restricted to  $A$ , is an element  $\tau$  of  $M_A$ . Since  $p$  lies in  $\operatorname{Re} A$ , we can find some  $r$  in  $A$  such that

$$p = \frac{1}{2}(r + \bar{r}) \quad \text{whence} \quad m(p) = \frac{1}{2}(m(r) + m(\bar{r})),$$

or

$$m(p) = \frac{1}{2}(\sigma(r) + \overline{\tau(r)}).$$

By hypothesis,  $\operatorname{Re} r = p > 0$  on  $X$ . Hence by a well-known property of function algebras,  $\operatorname{Re} \beta(r) > 0$  for all  $\beta$  in  $M_A$ . In particular  $\operatorname{Re} \sigma(r) > 0$  and  $\operatorname{Re} \tau(r) > 0$ . Hence  $\operatorname{Re} m(p) > 0$ .

Since this holds for all  $m$  in  $M_S$ , we can, by the general theory of Banach algebras, apply to  $p$  any function analytic in the right half-plane and still stay in the algebra  $S$ . Hence  $\log p$  is an element of  $S$ , and, being real valued, of  $\operatorname{Re} A$ .

Let now  $K^*$  be any positive number. Choose  $g$  in  $A$  with  $g(x_0) = 0$

and  $\|g\| = 1$ . Let  $a$  be some point in  $X$  where  $|g(a)| = 1$ . Next choose  $\varphi$  analytic in  $|z| < 1$ , continuous in  $|z| \leq 1$ , such that  $0 < \operatorname{Re} \varphi \leq 1$  in  $|z| \leq 1$ ,  $\operatorname{Im} \varphi(0) = 0$  and  $\operatorname{Im} \varphi(g(a)) \geq K^*$ . Put  $f = \varphi(g)$ . Then  $f$  belongs to  $A$  and we have:

$$0 < \operatorname{Re} f \leq 1 \text{ on } X, \quad \operatorname{Im} f(x_0) = 0 \text{ and } \|f\| \geq K^* .$$

Then  $\operatorname{Re} f$  is in  $\operatorname{Re} A$  and  $> 0$ . By Lemma 2, then,  $\log(\operatorname{Re} f)$  also is in  $\operatorname{Re} A$ , i.e. there is some  $F$  in  $A$  with  $\operatorname{Re} F = \log(\operatorname{Re} f)$ . Put now  $V = \exp(\frac{1}{2}F)$ . Then again  $V$  is in  $A$ . Also  $|V|^2 = \operatorname{Re} f$ . Then  $\max_x |V| = \|V\| \leq 1$ .

We now use the following identity, true for each complex  $z$ :

$$(\operatorname{Re} z)^2 = \frac{1}{2} (\operatorname{Re} z^2 + |z|^2) .$$

Applying this to  $V$  and using that  $|V|^2 = \operatorname{Re} f$ , we get

$$(\operatorname{Re} V)^2 = \operatorname{Re} \left( \frac{1}{2} (V^2 + f) \right) .$$

Clearly for each  $h$  in  $A$ , we have  $N(\operatorname{Re} h) \geq \|h\| - |\operatorname{Im} h(x_0)|$ . Hence  $N((\operatorname{Re} V)^2) \geq \frac{1}{2} (\|V^2 + f\| - |\operatorname{Im} V^2(x_0)|) \geq \frac{1}{2} (K^* - 2)$ , since  $\|f\| \geq K^*$  while  $\|V^2\| \leq 1$ .

On the other hand, by Lemma 1,

$$N((\operatorname{Re} V)^2) \leq K \cdot (N(\operatorname{Re} V))^2 \quad \text{and} \quad N(\operatorname{Re} V) \leq 2 \|V\| \leq 2 .$$

Since  $K^*$  is arbitrary while  $K$  is fixed, we have a contradiction. Thus Theorem 1' is proved.

**3. The general case.** To deduce the result in the general case from Theorem 1', we use the following theorem of Bishop [1]. (See also [2].):

**THEOREM.** *Let  $A$  be any function algebra on  $X$ . Then there exists a collection  $\Phi$  of closed, pairwise disjoint sets covering  $X$  so that*

- (a)  $f$  in  $C(X)$  and  $f|_K$  in  $A|_K$  for every  $K$  in  $\Phi$  imply  $f$  in  $A$ ;
- (b)  $A|_K$  is closed in  $C(K)$  for each  $K$  in  $\Phi$ .
- (c)  $A|_K$  is antisymmetric on  $K$  for each  $K$  in  $\Phi$ .

Because of Bishop's theorem, one has the following method of reasoning: let  $(P)$  be a property which has meaning for every function algebra  $A$ . Assume

- (i) Whenever a given  $A$  has property  $(P)$ , then so does each restriction algebra  $A|_K$  for  $K$  in  $\Phi$ , and
- (ii) Whenever  $A$  is antisymmetric on the space  $X$  and  $A$  has

property  $(P)$ , then  $X$  consists of a single point.

We then conclude, using the Theorem, that if  $A$  is a function algebra on a space  $X$  such that  $A$  has property  $(P)$ , then  $A = C(X)$ . Thus, if  $(P)$  is the property “ $A$  is closed under complex conjugation”, (i) and (ii) clearly hold, and one concludes the Stone-Weierstrass theorem.

If  $(P)$  is the property “ $Re A$  is a ring”, then (i) also clearly holds, and that (ii) holds was the content of Theorem 1'. Thus we may conclude Theorem 1.

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