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# THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA

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**1. Introduction.** Let $X$ be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on $X$. We consider a closed subalgebra $A$ of $C(X)$ which separates the points of $X$ and contains the constants. We call $A$ "a function algebra on $X$".

Let $Re\,A$ denote the class of functions $u$ real and continuous on $X$ such that for some $f$ in $A$, $u = Re\,f$. Then $Re\,A$ is a real vector space of real continuous functions on $X$. What more can be said about $Re\,A$?

In [3] it was shown that $Re\,A$ cannot be closed under uniform convergence on $X$ unless $A = C(X)$. Here we shall show that $Re\,A$ cannot be closed under multiplication unless $A = C(X)$. In other words:

*Theorem* 1: *If $Re\,A$ is a ring, then $A = C(X)$.*

This result was conjectured by K. Hoffman. As a corollary one gets the existence of a continuous function $u$ on the unit circle having the following property: $u$ has a continuous conjugate function (in the sense of Fourier theory) whereas $u^2$ does not. For we may take for $A$ the algebra of continuous functions on the circle which extend analytically to the unit disk. Then $Re\,A$ is the class of all functions which are continuous and have continuous conjugates. But $A \neq C(X)$. Hence by Theorem 1, $Re\,A$ is not a ring, hence not closed under squaring, and so the desired $u$ exists.

The existence of such a $u$ had been shown in 1961 by J. P. Kahane (unpublished). It should be noted that if a function $u$ is sufficiently smooth to have an absolutely convergent Fourier series, then $u^2$ does also, and hence $u^2$ does have a continuous conjugate.

**2. The antisymmetric case.** In this section we assume that $A$ is anti-symmetric, i.e. contains no real functions except constants, and prove Theorem 1 under this hypothesis. This amounts to proving:

THEOREM 1′. *Let $A$ be anti-symmetric and let $Re\,A$ form a ring. Then $X$ consists of a single point.*

Assume $X$ contains a point $x_0$ and another point $x_1$. We must deduce a contradiction. Fix $u$ in $Re\,A$. Then (because of antisymmetry),

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there exists exactly one $f$ in $A$ with $u = Re\, f$ and $Im\, f(x_0) = 0$. The map: $u$ into $f$ is now a real-linear map of $Re\, A$ into $A$ which is one-to-one. We can then norm $Re\, A$ by the norm $N$:

$$N(u) = \max_X |f| = \|f\| .$$

In this norm $Re\, A$ is then evidently a real Banach space. By standard application of the closed graph theorem, we have

LEMMA 1. *There exists a constant $K$ such that for all $u, u'$ in $Re\, A$*

$$N(u \cdot u') \leqq K \cdot N(u) \cdot N(u') .$$

LEMMA 2. *If $p$ lies in $Re\, A$ and $p > 0$ on $X$, then* $\log p$ *is in $Re\, A$.*

*Proof.* Let $S$ be the class of functions $u + iu'$ with $u$ and $u'$ in $Re\, A$. Then $S$ is an algebra of complex-valued functions on $X$ containing $A$ as a subalgebra and closed under complex conjugation. Define $N(u + iu') = N(u) + N(u')$ and $\|f\|' = \sup_\theta N(e^{i\theta} f)$ for all $f$ in $S$. Then $S$ is a (complex) Banach space under $\| \quad \|'$ as norm and also $\|f \cdot g\|' \leqq K \|f\|' \cdot \|g\|'$. Hence $S$ is a Banach algebra under a norm equivalent to $\| \quad \|'$.

Let $M_S$ denote the space of homomorphisms of $S$ into the complex numbers and $M_A$ be the corresponding space for $A$. Fix $m$ in $M_S$. Restricted to $A$, $m$ is an element $\sigma$ of $M_A$. Also the map: $f$ into $\overline{m(\bar{f})}$, restricted to $A$, is an element $\tau$ of $M_A$. Since $p$ lies in $Re\, A$, we can find some $r$ in $A$ such that

$$p = \frac{1}{2}(r + \bar{r}) \quad \text{whence } m(p) = \frac{1}{2}(m(r) + m(\bar{r})) ,$$

or

$$m(p) = \frac{1}{2}(\sigma(r) + \overline{\tau(r)}) .$$

By hypothesis, $Re\, r = p > 0$ on $X$. Hence by a well-known property of function algebras, $Re\, \beta(r) > 0$ for all $\beta$ in $M_A$. In particular $Re\, \sigma(r) > 0$ and $Re\, \tau(r) > 0$. Hence $Re\, m(p) > 0$.

Since this holds for all $m$ in $M_S$, we can, by the general theory of Banach algebras, apply to $p$ any function analytic in the right half-plane and still stay in the algebra $S$. Hence $\log p$ is an element of $S$, and, being real valued, of $Re\, A$.

Let now $K^*$ be any positive number. Choose $g$ in $A$ with $g(x_0) = 0$

and $\|g\| = 1$. Let $a$ be some point in $X$ where $|g(a)| = 1$. Next choose $\varphi$ analytic in $|z| < 1$, continuous in $|z| \leqq 1$, such that $0 < Re\,\varphi \leqq 1$ in $|z| \leqq 1$, $Im\,\varphi(0) = 0$ and $Im\,\varphi(g(a)) \geqq K^*$. Put $f = \varphi(g)$. Then $f$ belongs to $A$ and we have:

$$0 < Re f \leqq 1 \text{ on } X\,, \qquad Im\, f(x_0) = 0 \text{ and } \|f\| \geqq K^*\,.$$

Then $Re f$ is in $Re\,A$ and $>0$. By Lemma 2, then, $\log(Re f)$ also is in $Re\,A$, i.e. there is some $F$ in $A$ with $Re\,F = \log(Re f)$. Put now $V = \exp(\frac{1}{2}F)$. Then again $V$ is in $A$. Also $|V|^2 = Re f$. Then $\max_x |V| = \|V\| \leqq 1$.

We now use the following identity, true for each complex $z$:

$$(Re\,z)^2 = \frac{1}{2}(Re\,z^2 + |z|^2)\,.$$

Applying this to $V$ and using that $|V|^2 = Re f$, we get

$$(Re\,V)^2 = Re\Big(\frac{1}{2}(V^2 + f)\Big)\,.$$

Clearly for each $h$ in $A$, we have $N(Re\,h) \geqq \|h\| - |Im\,h(x_0)|$. Hence $N((Re\,V)^2) \geqq \frac{1}{2}(\|V^2 + f\| - |Im\,V^2(x_0)|) \geqq \frac{1}{2}(K^* - 2)$, since $\|f\| \geqq K^*$ while $\|V^2\| \leqq 1$.

On the other hand, by Lemma 1,

$$N((Re\,V)^2) \leqq K\cdot(N(Re\,V))^2 \quad\text{and}\quad N((Re\,V)) \leqq 2\,\|V\| \leqq 2\,.$$

Since $K^*$ is arbitrary while $K$ is fixed, we have a contradiction. Thus Theorem 1′ is proved.

**3. The general case.** To deduce the result in the general case from Theorem 1′, we use the following theorem of Bishop [1]. (See also [2].):

THEOREM. *Let $A$ be any function algebra on $X$. Then there exists a collection $\Phi$ of closed, pairwise disjoint sets covering $X$ so that*

(a) *$f$ in $C(X)$ and $f\,|\,K$ in $A\,|\,K$ for every $K$ in $\Phi$ imply $f$ in $A$;*

(b) *$A\,|\,K$ is closed in $C(K)$ for each $K$ in $\Phi$.*

(c) *$A\,|\,K$ is antisymmetric on $K$ for each $K$ in $\Phi$.*

Because of Bishop's theorem, one has the following method of reasoning: let $(P)$ be a property which has meaning for every function algebra $A$. Assume

(i) Whenever a given $A$ has property $(P)$, then so does each restriction algebra $A\,|\,K$ for $K$ in $\Phi$, and

(ii) Whenever $A$ is antisymmetric on the space $X$ and $A$ has property $(P)$, then $X$ consists of a single point.

We then conclude, using the Theorem, that if $A$ is a function algebra on a space $X$ such that $A$ has property $(P)$, then $A = C(X)$. Thus, if $(P)$ is the property "$A$ is closed under complex conjugation", (i) and (ii) clearly hold, and one concludes the Stone-Weierstrass theorem.

If $(P)$ is the property "$Re\, A$ is a ring", then (i) also clearly holds, and that (ii) holds was the content of Theorem 1′. Thus we may conclude Theorem 1.

## REFERENCES

1. E. Bishop, *A generalization of the Stone-Weierstrass theorem*, Pacific J. Math., **11**, (1961), 777-783.

2. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc., **105** (1962), 415-435.

3. K. Hoffman and J. Wermer, *A characterisation of $C(X)$* Pacific J. Math., 12 (1962), 941-944.

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