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**NON-LINEAR DIFFERENTIAL EQUATIONS ON CONES IN  
BANACH SPACES**

CHARLES VERNON COFFMAN

# NON-LINEAR DIFFERENTIAL EQUATIONS ON CONES IN BANACH SPACES

CHARLES V. COFFMAN

**1. Introduction.** Let  $y$  denote a vector in  $n$ -dimensional Euclidean space  $E^n$ ,  $y \geq 0$  means that each component of  $y$  is nonnegative. Let  $f(t, y)$  be a function with values in  $E^n$  defined on the domain  $D = \{(t, y), 0 \leq t < \infty, y \in E^n, y \geq 0\}$ . Assume that  $f(t, y)$  is continuous on  $D$  and that for each  $(t, y) \in D$ ,  $-f(t, y) \geq 0$ , and that  $f(t, 0) \equiv 0$ . Let  $C$  be a positive constant. It is known, [\*, 4], that the differential equation  $dy/dt = f(t, y)$  possesses at least one solution  $y = y(t)$ , defined for  $0 \leq t < \infty$ , with  $y(t) \geq 0$  for  $0 \leq t < \infty$ , and satisfying  $|y(0)| = C$ , where  $|\dots|$  denotes the Euclidean norm in  $E^n$ . (Actually the hypothesis of the theorem referred to involves an additional condition on  $f(t, y)$ , however the proof given in [4] can be modified so as to avoid the need for this condition.)

It was shown in [2] that the linear case of the result just quoted has a generalization to an infinite dimensional case. Before describing this result it will be necessary to introduce the following terminology and notation. Throughout the remainder of the paper  $Y$  will denote a (real) Banach space with norm  $\|\dots\|$  and  $K$  some fixed, closed convex cone in  $Y$ . For  $y \in Y$ ,  $y \geq 0$  will mean  $y \in K$ . A linear operator  $A$  on  $Y$  will be said to be nonnegative if  $Ay \geq 0$  whenever  $y \geq 0$ . A nonempty set of the form  $H = H(y^*) = \{y : y \in K, y^*(y) = 1\}$ , where  $y^*$  is in the dual space  $Y^*$  of  $Y$  will be called a cross-section of the cone  $K$ . Theorem 1 of [2] concerns the linear differential equation on the Banach space  $Y$ ,

$$(1.1) \quad dy/dt = -A(t)y,$$

where for each  $t$ ,  $0 \leq t < \infty$ ,  $A(t)$  is a nonnegative bounded linear operator on  $Y$ , and  $A(t)$  is strongly continuous on  $0 \leq t < \infty$ . The result states that if the cone  $K$  has a weakly compact cross-section then the differential equation (1.1) has a nontrivial solution  $y = y(t)$  satisfying  $y(t) \geq 0$  for  $0 \leq t < \infty$ .

Theorem 3 of [2] is the dual of Theorem 1 and concerns the differential equation on  $Y^*$  adjoint to (1.1)

$$(1.2) \quad dy^*/dt = -A^*(t)y^*,$$

where for each  $t$ ,  $0 \leq t < \infty$ ,  $A^*(t)$  is the adjoint of  $A(t)$ . Let  $A(t)$  be as above, this theorem states that if  $K$  has an interior point then (1.2) has a nontrivial solution  $y^* = y^*(t)$  satisfying  $y^*(t) \in K^*$  for  $0 \leq t < \infty$ .  $K^*$  in  $Y^*$  is the dual cone of the cone  $K$  in  $Y$ , consisting of those

elements  $y^*$  such that  $y^*(y) \geq 0$  for each  $y \in K$ .

The main result of this note, Theorem 4.1, generalizes the result of [4] mentioned above to an infinite dimensional case and contains both of the results just quoted.

The following standard terminology will be used. A closed linear manifold  $\Gamma$  in the dual space  $Y^*$  of  $Y$  is called *determining* for  $Y$  if, when  $y \in Y$ ,  $\gamma(y) = 0$  for each  $\gamma \in \Gamma$  implies  $y = 0$ . Such a manifold defines a topology, referred to as the weak  $\Gamma$ -topology, on  $Y$ . The generalized sequence  $\{y_\alpha\}$  of elements of  $Y$  has limit 0 in the weak  $\Gamma$ -topology if and only if  $\lim_\alpha \gamma(y_\alpha) = 0$  for each  $\gamma \in \Gamma$ .

**2. Differential equations on a Banach space.** Let  $\Gamma$  be a closed linear manifold in  $Y^*$  which is determining for  $Y$ . A function  $y(t)$  on some interval  $I$  on the real line, with values in  $Y$ , will be said to have a weak  $\Gamma$ -derivative, or to be weakly  $\Gamma$ -differentiable at  $t \in I$  if the limit as  $h \rightarrow 0$  of the difference quotient  $(y(t+h) - y(t))/h$  exists in the weak  $\Gamma$ -topology. If  $y(t)$  is weakly  $\Gamma$ -differentiable on  $I$ , its weak  $\Gamma$ -derivative will be denoted by  $D_\Gamma y(t)$ . Let  $C$  be a subset of  $Y$  and let  $I$  be, as above, an interval on the real line. Consider the differential equation

$$(2.1) \quad dy/dt = f(t, y)$$

where  $f(t, y)$  is a function from  $I \times C \rightarrow Y$ , continuous in the weak  $\Gamma$ -topology. Let  $y(t)$  be a function on  $I$  with values in  $C$ . If  $y(t)$  is strongly continuous and strongly differentiable on  $I$  with a strongly continuous derivative  $dy/dt$ , and if (2.1) holds,  $y(t)$  will be said to be a strong solution of (2.1) on  $I$ . If  $y(t)$  is strongly continuous, weakly absolutely continuous and strongly differentiable a.e. on  $I$ , and if (2.1) holds a.e. on  $I$ ,  $y(t)$  will be said to be a strong solution in the extended sense of (2.1) on  $I$ . Finally  $y(t)$  will be said to be a weak  $\Gamma$ -solution of (2.1) on  $I$  if it is weakly  $\Gamma$ -differentiable on  $I$  and if  $D_\Gamma y(t) = f(t, y(t))$  on  $I$ . Obviously if  $y(t)$  is a weak  $\Gamma$ -solution of (2.1) then  $D_\Gamma y(t)$  is weakly  $\Gamma$ -continuous.

**LEMMA 2.1.** *If  $y(t)$  is a strong solution in the extended sense of (2.1) on  $I$ , then it can be expressed as the Bochner integral of its derivative.*

*Proof.* Since  $y(t)$  is strongly continuous, its range lies in a separable subspace of  $Y$ . Clearly the values of  $dy/dt$ , where it exists will be in the same subspace. Put

$$(2.2) \quad u(t) = f(t, y(t)) ,$$

then  $u(t)$  is weakly  $\Gamma$ -continuous and coincides with  $dy/dt$  a.e. where the latter is defined. Since  $u(t)$  is almost separably valued and weakly  $\Gamma$ -continuous, it follows from Theorem 1.1.7, [3, p. 330], that  $u(t)$ , hence  $dy/dt$  is locally Bochner integrable on  $I$ . Since  $y(t)$  is weakly absolutely continuous it may be expressed as the indefinite integral of  $dy/dt$ , Theorem 3.8.6, [5].

**LEMMA 2.2.** *Let  $y(t)$  be a function defined on  $I$  with values in  $C$ . In order that  $y(t)$  be a strong solution in the extended sense of (2.1), it is necessary and sufficient that  $y(t)$  be a weak  $\Gamma$ -solution of (2.1) and that  $D_\Gamma y(t)$  be almost separably valued.*

**REMARK.** If  $\Gamma = Y^*$  and if  $y(t)$  is a weak  $\Gamma$ -solution of (2.1), then  $D_\Gamma y(t)$  is necessarily separably valued, since it is weakly  $\Gamma$ -continuous; see [5], p. 59.

*Proof of Lemma 2.2. Necessity.* Let  $y(t)$  be a strong solution in the extended sense of (2.1). Let  $u(t)$  be defined by (2.2). From the proof of Lemma 2.1,  $u(t)$  is almost separably valued and for  $t, t' \in I$ ,  $y(t) - y(t') = \int_{t'}^t u(t) dt$ . For each  $\gamma \in \Gamma$ ,  $\gamma(u(t))$  is continuous, consequently  $\gamma(y(t)) \in C^1$  and  $d[\gamma(y(t))]/dt = \gamma(u(t))$ , since one has  $\gamma\left[\int_{t'}^t u(t) dt\right] = \int_{t'}^t \gamma(u(t)) dt$ . It follows that  $D_\Gamma y(t)$  exists everywhere on  $I$  and that  $D_\Gamma y(t) = u(t)$ .

*Sufficiency.* Let  $y(t)$  be a weak  $\Gamma$ -solution of (2.1) and let  $D_\Gamma y(t)$  be almost separably valued. By Theorem 1.1.7, [3],  $D_\Gamma y(t)$  is locally Bochner integrable on  $I$ . For each  $\gamma \in \Gamma$ ,  $\gamma(D_\Gamma y(t))$  is continuous on  $I$ , so for  $t, t' \in I$ ,  $\gamma(y(t)) - \gamma(y(t')) = \int_{t'}^t \gamma(D_\Gamma y(t)) dt = \gamma\left[\int_{t'}^t D_\Gamma y(t) dt\right]$ . Since  $\Gamma$  is determining for  $Y$  it follows that  $y(t)$  can be expressed as the indefinite Bochner integral of  $D_\Gamma y(t)$ . Hence  $y(t)$  is in fact strongly absolutely continuous, and, by Corollary 2, p. 88, [5],  $y(t)$  is strongly differentiable a.e., and  $dy/dt$  coincides a.e., where it is defined with  $D_\Gamma y(t)$ .

**3. Convergence of approximate solutions.** Let  $Y, \Gamma, C, I$ , and  $f(t, y)$  be as in § 2. The following generalization of the notion of an  $\varepsilon$ -approximate solution of a differential equation will be used. Let  $V$  be a neighborhood of 0 in the weak  $\Gamma$ -topology. A function  $y(t)$  will be called a  $V$ -approximate weak  $\Gamma$ -solution of (2.1) on  $I$  if

- (i)  $y(t)$  is defined on  $I$ , has values in  $C$  and is weakly  $\Gamma$ -continuous;
- (ii) for some finite set of points  $S$  on  $I$ ,  $D_\Gamma y(t)$  exists and is weakly  $\Gamma$ -continuous on  $I - S$ , further  $D_\Gamma y(t)$  has only simple discontinuities, in the weak  $\Gamma$ -topology, at points of  $S$ ;
- (iii)  $(D_\Gamma y(t) - f(t, y)) \in V$  for  $t \in I - S$ .

**REMARK.** In view of the principle of uniform boundedness (ii) implies that  $\|D_r y(t)\|$  is locally bounded on  $I$ .

**LEMMA 3.1.** *Let  $\{y_n(t)\}$  be a sequence of weakly  $\Gamma$ -continuous functions defined on  $I$  with values in  $C$ . Let  $C$  be compact in the weak  $\Gamma$ -topology. For each neighborhood  $V$  of  $0$ , in the weak  $\Gamma$ -topology, and for each compact subinterval  $I'$  of  $I$ , let there exist an  $N = N(V, I')$  such that for all  $n \geq N$ ,  $y_n(t)$  is a  $V$ -approximate weak  $\Gamma$ -solution of (2.1) on  $I'$ . Then there exists a subsequence  $\{y_{n_k}(t)\}$  which converges uniformly on every compact subinterval of  $I$  to a weak  $\Gamma$ -solution  $y(t)$  of (2.1) on  $I$ .*

*Proof.* The existence of a subsequence convergent on each compact subinterval of  $I$  will follow from Ascoli's theorem, [1, p. 43], if it is shown that the sequence is equicontinuous in the weak  $\Gamma$ -topology. Let  $t_0 \in I$  be arbitrary, it will suffice to show that for each  $\gamma \in \Gamma$ , and for each  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon, \gamma, t_0)$  such that for  $n = 1, 2, \dots$

$$(3.1) \quad |\gamma(y_n(t) - y_n(t_0))| < \varepsilon$$

whenever  $t \in I$ , and  $|t - t_0| < \delta$ . For some  $\delta_0 > 0$ , there exists, by the principle of uniform boundedness, an  $M$  such that  $\|f(t, y)\| \leq M$  for  $t \in I$ ,  $|t - t_0| \leq \delta_0$  and  $y \in C$ . For  $n = 1, 2, \dots$ , one has

$$(3.2) \quad \begin{aligned} \gamma(y_n(t)) - \gamma(y_n(t_0)) &= \int_{t_0}^t \gamma(f(t, y_n(t))) dt \\ &+ \int_{t_0}^t \gamma(D_r y_n(t) - f(t, y_n(t))) dt. \end{aligned}$$

For all sufficiently large  $n$ , say  $n \geq n_0$ ,  $|\gamma(D_r y_n(t) - f(t, y_n(t)))| < \varepsilon/2 \|\gamma\| \delta_0$ , for  $t \in I$ ,  $|t - t_0| < \delta_0$ . Thus, if  $\delta_1 < \min(\delta_0, \varepsilon/2M \|\gamma\| \delta_0)$  then (3.1) holds for  $t \in I$ ,  $|t - t_0| < \delta_1$  and  $n \geq n_0$ . Clearly it is possible to choose a  $\delta$ ,  $0 < \delta \leq \delta_1$ , such that (3.1) holds for all  $n$  when  $t \in I$ ,  $|t - t_0| < \delta$ .

Let the subsequence  $\{y_{n_k}(t)\}$  of the original sequence be uniformly convergent on each compact subinterval of  $I$  to  $y(t)$ . For each  $\gamma \in \Gamma$ ,  $\gamma(f(t, y_{n_k}(t))) \rightarrow \gamma(f(t, y(t)))$  and  $\gamma(D_r y_{n_k}(t) - f(t, y_{n_k}(t))) \rightarrow 0$  uniformly on each compact subinterval of  $I$ . It follows that  $\gamma(y(t) - y(t_0)) = \int_{t_0}^t \gamma(f(t, y(t))) dt$ , for  $t, t_0 \in I$ , for each  $\gamma \in \Gamma$ . Consequently  $D_r y(t)$  exists on  $I$  and  $D_r y(t) = f(t, y(t))$  there.

**4. An existence theorem.** As a consequence of the results of the last section, one has the following local existence theorem for the problem (2.1).

**THEOREM 4.1.** *Let the unit sphere in  $Y$  be compact in the weak*

*$\Gamma$ -topology.* For some real interval  $I$ , some point  $y_0 \in Y$  and some positive constant  $k$  let  $f(t, y)$  be defined and weakly  $\Gamma$ -continuous for  $t \in I, \|y - y_0\| \leq k$ . Let  $t_0$  be in the interior of  $I$ , then on some open subinterval  $I'$  of  $I$ , with  $t_0 \in I'$ , there exists a weak  $\Gamma$ -solution  $y(t)$  of (2.1) satisfying the initial condition

$$(4.1) \quad y(t_0) = y_0.$$

*Proof.* It can be assumed that  $I$  is compact, so that  $\|f(t, y)\| \leq M < \infty$  for  $t \in I, \|y - y_0\| \leq k$ . Let  $\delta$  be some number on the range  $0 < \delta < k/M$ , such that the interval  $I' = \{t : |t - t_0| < \delta\} \subset I$ . For a positive integer  $n$ , subdivide  $I'$  by the points  $t_l = t_0 + l\delta/n, l = 0, \pm 1, \dots, \pm n - 1$ . Define a function  $y_n(t)$  on  $I'$  as follows: let  $y_n(t_0) = y_0$ , and let  $y_n(t) = y_0 + (t - t_0)f(t_0, y_0)$  for  $t_0 \leq t \leq t_1$ , clearly then  $\|y_n(t) - y_0\| \leq M(t_1 - t_0) \leq k/n$  for  $t_0 \leq t \leq t_1$ . Assume that  $y_n(t)$  is defined for  $t_0 \leq t \leq t_l, l \leq n - 1$ , with  $\|y_n(t) - y_0\| \leq lk/n$  for  $t_0 \leq t \leq t_l$ , and put  $y_n(t) = y_n(t_l) + (t - t_l)f(t_l, y_n(t_l))$  for  $t_l \leq t \leq t_{l+1}$ . Thus  $\|y_n(t) - y_0\| \leq (l + 1)k/n$ , and the process can be continued to define  $y_n(t)$  on  $t_0 \leq t < \delta$ , with  $\|y_n(t) - y_0\| < k$  there.  $y_n(t)$  is defined to the left of  $t_0$  in a similar fashion.

Let  $y_n(t)$  be so defined for  $n = 1, 2, \dots$ . The interval  $I$  being compact,  $f(t, y)$  is uniformly weakly  $\Gamma$ -continuous for  $t \in I, \|y - y_0\| \leq k$ . It readily follows that, given  $V \subset Y$ , a neighborhood of zero in the weak  $\Gamma$ -topology, then for all sufficiently large  $n$ ,  $y_n(t)$  is a  $V$ -approximate weak  $\Gamma$ -solution of (2.1) on  $I'$ .

The existence of a weak  $\Gamma$ -solution of (2.1) on  $I'$  satisfying the initial condition (4.1) now follows from Lemma 3.1.

**5. Differential equations on cones.** The main result of this note is the following.

**THEOREM 5.1.** *Let  $K$  be a closed convex cone in the Banach space  $Y$ . Let  $\Gamma$  be as above, and let there exist a  $\gamma_0 \in K^* \cap \Gamma$  such that the cross-section  $H(\gamma_0)$  of  $K$  is compact in the weak  $\Gamma$ -topology on  $Y$ . Let  $I = [0, \infty)$  and let  $f(t, y)$  be defined on  $I \times K$ , and continuous in the weak  $\Gamma$ -topology, and suppose that*

$$(5.1) \quad -f(t, y) \in K \quad \text{for } (t, y) \in K.$$

*assume also that*

$$(5.2) \quad f(t, 0) \equiv 0.$$

*Then there exists a weak  $\Gamma$ -solution  $y(t)$  of (2.1) on  $I$  with  $y(t) \in K$  for all  $t \in I$ , and with  $y(0) \in H$ .*

*Proof.* First it will be shown that for each  $T > 0$  there exists a weak  $\Gamma$ -solution  $y_T(t)$  of (2.1) on  $[0, T]$  with  $y_T(t) \in K$  for  $0 \leq t \leq T$ , and having  $y_T(0) \in H$ . Let  $T > 0$  be given, let  $n$  be a positive integer and let  $\eta$  be an arbitrary element of  $K$ . Subdivide  $[0, T]$  into  $n$  subintervals of equal length, and construct a polygonal approximate solution  $y_n(t) = y_n(t, \eta)$  as in § 4, beginning at  $t = T$  with  $y_n(T, \eta) = \eta$  and proceeding to the left. That is,  $t_k = kT/n$ ,  $k = 0, \dots, n$ , being the endpoints of the intervals of the subdivision, if  $y_n(t, \eta)$  is assumed to be defined on  $[t_k, T]$ ,  $k > 0$ , the definition is extended to  $[t_{k-1}, T]$  by putting

$$(5.3) \quad y_n(t, \eta) = y_n(t_k, \eta) + (t - t_k)f(t_k, y_n(t_k, \eta))$$

on  $[t_{k-1}, t_k]$ . Notice that because of (5.1) at no point can the polygon leave  $K$ .

From the fact that  $f(t, y)$  is weakly  $\Gamma$ -continuous it readily follows, using (5.3) and an induction argument that the mapping of  $K \rightarrow K$  given by  $\eta \rightarrow y_n(0, \eta)$  is continuous in the weak  $\Gamma$ -topology. From (5.2), it follows that  $y_n(0, 0) = 0$ . Because of (5.1) one has, in view of (5.3) that  $y_n(t, \eta) \geq y_n(t', \eta)$  for  $0 \leq t < t' \leq T$ , and consequently that

$$(5.4) \quad \gamma_0(y_n(t, \eta)) \geq \gamma_0(y_n(t', \eta)) \quad \text{for } 0 \leq t < t' \leq T.$$

The weak  $\Gamma$ -continuity of the mapping  $\eta \rightarrow y_n(0, \eta)$  implies therefore, the existence of values  $\eta \in K$  for which  $y_n(0, \eta) \in H$ . For each  $n = 1, 2, \dots$ , let  $\eta_n$  be chosen so that  $y_n(0, \eta_n) \in H$ . By (5.4) one has that for  $n = 1, 2, \dots$   $y_n(t, \eta_n) \in K_1$  for  $0 \leq t \leq T$ , where  $K_1 = \{y : y \in K, \gamma_0(y) \leq 1\}$ . Since  $K_1$  is weak  $\Gamma$ -compact,  $f(t, y)$  is uniformly weakly  $\Gamma$ -continuous on  $[0, T] \times K_1$ , thus it follows, as in the proof of Theorem 4.1, that the sequence  $y_n(t, \eta_n)$  satisfies the hypotheses of Lemma 3.1. Consequently that lemma implies that there exists a weak  $\Gamma$ -solution of (2.1) on  $[0, T]$  such that  $y_T(t) \in K_1$  on  $[0, T]$  and  $y_T(0) \in H$ .

For each  $n = 1, 2, \dots$ , let  $y_n(t)$  be a weakly  $\Gamma$ -continuous function from  $I$  to  $K_1$  with  $y_n(0) \in H$  and such that on  $[0, n]$ ,  $y_n(t)$  is a weak  $\Gamma$ -solution of (2.1). The sequence  $\{y_n(t)\}$  satisfies the hypotheses of Lemma 3.1, and the conclusion of the theorem follows by an application of that lemma.

Theorem 3 of [2] follows immediately from Theorem 4.1 above in view of the observation (ii) of [2], namely that the dual cone  $K^*$  of a cone  $K$  with interior has a weak\* compact cross-section. If  $\Gamma = Y^*$  then the solution of (2.1) whose existence is asserted in Theorem 4.1 is, by Lemma 2.2 and the remark following the statement of that lemma, a strong solution, in the extended sense, of (2.1) on  $I$ . If the mapping  $f(t, y)$  of  $I \times K$  to  $-K$  is continuous with respect to the strong topology on  $Y$  it is not difficult to see that a strong solution in the

extended sense, of (2.1) is in fact a strong solution of (2.1). Thus Theorem 1 of [2] also follows from Theorem 5.1.

If  $Y$  is finite dimensional, Theorem 5.1 implies a result similar to Theorem (\*) of [4] referred to in § 1. In the latter result the initial value  $y(0)$  is asserted to lie on the surface of a sphere  $\|y\| = C$ ,  $C > 0$ , rather than on some hyperplane. For the finite dimensional case it is clear that the proof given above can be modified to give this different normalization of the initial value.

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