

# Pacific Journal of Mathematics

**INFINITE PRODUCTS OF ISOLS**

ERIK MAURICE ELLENTUCK

# INFINITE PRODUCTS OF ISOLS

ERIK ELLENTUCK

**Introduction.** In [1] the notion of the sum of an infinite number of isols is introduced. In this paper we shall similarly attack the problem of the product of an infinite number of isols. Before proceeding to this it is necessary to review the concept of exponentiation. Let  $\varepsilon = \{0, 1, \dots\}$  be the set of nonnegative integers.  $f$  is a finite function if  $\delta f = \varepsilon$  ( $\delta f, \rho f$  are domain, range of  $f$  respectively) and  $\{x : f(x) \neq 0\}$  is finite. The set  $\{x : f(x) \neq 0\}$  is called the *essential domain* of  $f$  (denoted  $\delta_e f$ ) and the set  $\{f(x) : f(x) \neq 0\}$  the *essential range* of  $f$  (denoted  $\rho_e f$ ).  $f$  is a finite function from the set  $\beta$  into the set  $\alpha$  if  $\delta_e f \subseteq \beta$  and  $\rho_e f \subseteq \alpha$ . It can be shown (cf. [3], 181) that there exists a recursive function  $r_n(x)$  to two variables such that

- (1) All finite functions are generated without repetitions in the sequence  $\{r_n(x)\}$ .
- (2) From  $n$ , one can effectively find  $r_n(x)$ .
- (3) From  $r_n(x)$ , one can effectively find  $n$ . Then for any subsets  $\alpha$  and  $\beta$  of  $\varepsilon$  we define

$$\alpha^\beta = \{n : r_n(x) \text{ is a finite function from } \beta \text{ into } \alpha\}.$$

In case  $\alpha$  and  $\beta$  are finite it is necessary that  $0 \in \alpha$  in order to make  $\alpha^\beta$  have  $m^n$  elements where  $\alpha$  has  $m$  elements and  $\beta$  has  $n$  elements. If  $A \neq 0$  we let  $A^\beta = \text{Req}(\alpha^\beta)$  where  $0 \in \alpha \in A, \beta \in B$ . Otherwise  $0^\beta = 1$  if  $B = 0, 0^\beta = 0$  if  $B > 0$ .

Let  $R = \text{Req}(\varepsilon)$ . It is known (cf. [3], 189) that  $2^R = R$ . Since we would like an infinite product of identical factors to reduce to an exponentiation, we see that an infinite product of isols may not be an isol. On the other hand, if  $X$  is an infinite isol, then so is  $2^X$  (cf. [3], 182). Thus depending on which exponent we use to formalize the concept of an infinite product of repeated factors we may or may not obtain an isol.

A one-to-one function  $t_n$  from  $\varepsilon$  into  $\varepsilon$  is *regressive* (cf. [1]) if there is a partial recursive function  $p(x)$  such that  $\rho t \subseteq \delta p$  and  $p(t_0) = t_0, (\forall n)(p(t_{n+1}) = t_n)$ . A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. There is no loss of generality by also supposing that  $p$  has the following additional properties:  $\rho p \subseteq \delta p$  and  $(\forall x)(x \in \delta p \rightarrow (\exists n)(p^{n+1}(x) = p^n(x)))$  (superscript denotes iterate). Define  $p^*$  by  $\delta p^* = \delta p$  and  $p^*(x) = (\mu n)(p^{n+1}(x) = p^n(x))$ . Define  $\bar{p}$  by  $\delta \bar{p} = \delta p$  and  $\rho_{\bar{p}(x)} = \{p(x), \dots, p^n(x)\}$  where  $n = p^*(x)$ . Two one-to-one

---

Received March 13, 1963.

functions  $t_n$  and  $t'_n$  from  $\varepsilon$  into  $\varepsilon$  are *recursively equivalent* (denoted  $t_n \simeq t'_n$ ) if there is a partial isomorphism  $f$  such that  $\rho t \subseteq \delta f$  and  $(\forall n)(f(t_n) = t'_n)$ . The following propositions are proven in [1]. Let  $\tau = \rho t$  and  $\tau' = \rho t'$  where  $t_n$  and  $t'_n$  are regressive functions. Then  $\tau \simeq \tau'$  if and only if  $t_n \simeq t'_n$ . Let  $t_n \simeq t'_n$ . Then  $t_n$  is a regressive function if and only if  $t'_n$  is a regressive function. Let  $\tau \simeq \tau'$ . Then  $\tau$  is a regressive set if and only if  $\tau'$  is a regressive set. Every regressive function is recursively equivalent to a strictly increasing regressive function. Every regressive set is recursively equivalent to a retraceable set. An RET is *regressive* if it consists of regressive sets. Let  $\mathcal{A}_R$  be the collection of all regressive isols. It is not difficult to show that there are at least  $\mathfrak{c}$  of them, and that each contains a retraceable set.

Let  $V$  be the class of all subsets of  $\varepsilon$ , let  $Q$  be the class of all finite subsets of  $\varepsilon$ . A mapping  $\Phi: V \rightarrow V$  is called a *combinatorial operator* if (1)  $\alpha \in Q$  implies  $\Phi(\alpha) \in Q$ , (2) the cardinality of  $\Phi(\alpha)$  is determined by that of  $\alpha$ . (3)  $\Phi$  possesses a quasi inverse  $\Phi^{-1}$  such that for any  $x \in \bigcup \{\Phi(\alpha); \alpha \in V\}$ ,  $\Phi^{-1}(x) \in Q$  and  $x \in \Phi(\beta)$  if and only if  $\Phi^{-1}(x) \subseteq \beta$ . Let  $\{\rho_n\}$  be a one-to-one effective enumeration of  $Q$  with  $\rho_0 = \phi$ .  $\Phi$  is called a *recursive combinatorial operator* if there is a recursive function  $g(x)$  such that  $\Phi(\rho_n) = \rho_{g(n)}$ . It is well known that a function  $F$  is recursive combinatorial if and only if it is induced by a recursive combinatorial operator  $\Phi$  in the sense that  $F(\text{Req } \alpha) = \text{Req } \Phi(\alpha)$  for every  $\alpha \in V$ .

**Infinite products.** In this paper we only consider products of an infinite sequence of finite positive isols (positive integers). Let  $\{a_n\}$  be an infinite sequence of positive integers. For any regressive function  $t_n$  with immune range let  $\pi(t_n) = \{n: \delta_e r_n \subseteq \rho t \wedge (\forall x)(r_n(t_x) < a_x)\}$ .

**THEOREM 1.**  $\pi(t_n)$  is an isolated set.

*Proof.* Suppose that  $\gamma$  is a recursively enumerable set,  $\gamma \subseteq \pi(t_n)$ . Let  $\delta = \bigcup \{\delta_e r_n: n \in \gamma\}$ . Then  $\delta$  is recursively enumerable and  $\delta \subseteq \rho t$ . Since  $\rho t$  is immune,  $\delta$  is finite. But this implies that  $\gamma$  is finite as well. Hence  $\pi(t_n)$  is isolated.

**THEOREM 2.** Let  $t_n, t'_n$  be regressive functions with immune ranges. If  $t_n \simeq t'_n$ , then  $\pi(t_n) \simeq \pi(t'_n)$ .

*Proof.* There is a partial isomorphism  $f$  such that  $\rho t \subseteq \delta f$  and  $f(t_n) = t'_n$  for all  $n$ . Let  $\delta = \{n: \delta_e r_n \subseteq \delta f\}$ .  $\delta$  is recursively enumerable. We may define a function  $g$  as follows:  $\delta g = \delta$  and for every  $n \in \delta$ ,  $r_{g(n)}$  is a finite function with  $\delta_e r_{g(n)} = f(\delta_e r_n)$  and such that for each  $x \in \delta_e r_n$ ,

$r_{g(n)}(f(x)) = r_n(x)$ .  $g$  is clearly a partial recursive function and it is not difficult to show that  $g$  is one-to-one,  $\pi(t_n) \subseteq \delta g$ , and  $g(\pi(t_n)) = \pi(t'_n)$ .

Thus the recursive equivalence type of  $\pi(t_n)$  depends only on the recursive equivalence type of  $\rho t$  since for regressive functions  $t_n, t'_n$  we have  $\rho t \simeq \rho t'$  if and only if  $t_n \simeq t'_n$ . This justifies the

**DEFINITION.**  $\prod_T a_n = \text{Req } \pi(t_n)$  where  $T \in A_R$  and  $t_n$  is any regressive function with  $\rho t \in T$ .

By Theorem 1 we know that  $\prod_T a_n$  is an isol and that therefore the product operation (for fixed  $\{a_n\}$ ) maps  $A_R$  into  $A$ . Let  $f$  be a recursive combinatorial function. In [5] it is shown that the partial product  $g(m) = \prod_{n=0}^{m-1} (1 + f(n))$  is also a recursive combinatorial function. It is possible to evaluate  $\prod_T (1 + f(n))$  by using the

**THEOREM 3.** *If  $f$  is a recursive combinatorial function and  $g(m) = \prod_{n=0}^{m-1} (1 + f(n))$  is its partial product, then for every  $T \in A_R$ ,  $\prod_T (1 + f(n)) = G(T)$  (where  $G$  canonically extends  $g$ ).*

*Proof.* For every set  $\alpha \subseteq \varepsilon$  and integer,  $n \in \varepsilon$  let  $1 \oplus \alpha = \{x + 1 : x \in \alpha\}$  and  $\alpha_{<n} = \{x : x \in \alpha \wedge x < n\}$ . Since  $T \in A_R$  there is a retraceable set  $\tau \in T$  which is enumerated by a strictly increasing regressive function  $t_n$ .

$$\prod_T (1 + f(n)) = \text{Req } \pi(t_n),$$

$$\pi(t_n) = \{n : \delta_e r_n \subseteq \tau \wedge (\forall x) (r_n(t_x) < 1 + f(x))\}.$$

Let  $\Phi$  be a recursive combinatorial operator inducing  $f$ , and let

$$\bar{\pi}(\tau) = \{n : \delta_e r_n \subseteq \tau \wedge (\forall x) (x \in \delta_e r_n \rightarrow r_n(x) \in 1 \oplus \Phi(\tau_{<x}))\}.$$

In order to complete our proof we shall show that  $\bar{\pi}$  is a recursive combinatorial operator inducing  $g$  and that  $\bar{\pi}(\tau) \simeq \pi(t_n)$ .

Let  $s$  be an integer and  $\alpha = \{a_1, \dots, a_s\}$ ,  $a_1 < \dots < a_s$  a set having exactly  $s$  elements. If  $n \in \bar{\pi}(\alpha)$ , then  $\delta_e r_n \subseteq \alpha$  and for each  $i$ ,  $1 \leq i \leq s$ ,  $r_n(a_i) = 0$  (if  $a_i \notin \delta_e r_n$ ) or  $r_n(a_i) \in 1 \oplus \Phi(\alpha_{<a_i})$ . Since  $\alpha_{<a_i}$  contains  $i - 1$  elements,  $r_n(a_i)$  may assume any one of  $1 + f(i - 1)$  values. Thus  $\bar{\pi}(\alpha)$  contains  $g(s)$  elements, i.e.  $\bar{\pi}$  induces  $g$ . Now let  $\alpha$  be any set of integers and let  $n \in \bar{\pi}(\alpha)$ . We define

$$\bar{\pi}^{-1}(n) = \delta_e r_n + \mathbf{U} \{\Phi^{-1}(r_n(x) - 1) : r_n(x) \neq 0\}.$$

It is clear that  $\bar{\pi}^{-1}(n) \subseteq \alpha$  and that if  $r_n(x) \neq 0$ , then  $\Phi^{-1}(r_n(x) - 1) \subseteq \varepsilon_{<x}$ . Conversely, suppose that  $\bar{\pi}^{-1}(n) \subseteq \beta$ . Then  $\delta_e r_n \subseteq \beta$  and  $\Phi^{-1}(r_n(x) - 1) \subseteq \beta$  for  $r_n(x) \neq 0$ . But  $\Phi^{-1}(r_n(x) - 1) \subseteq \varepsilon_{<x}$  and therefore  $\Phi^{-1}(r_n(x) - 1) \subseteq \beta_{<x}$  for  $r_n(x) \neq 0$ . Hence  $n \in \bar{\pi}(\beta)$ . Thus  $\bar{\pi}^{-1}$  satisfies the condition of being a quasi inverse function of  $\bar{\pi}$ . Since  $\bar{\pi}$  is clearly effective we see that it is a recursive combinatorial operator inducing  $g$ . Thus in

particular  $G(T) = \text{Req } \bar{\pi}(\tau)$ . Since  $t_n$  is regressive there is a partial recursive function  $p$  such that  $\tau \subseteq \delta p$  and  $p(t_{n+1}) = t_n$ ,  $p(t_0) = t_0$  for all  $n$  (satisfying all those conditions given in the introduction). Let  $\delta = \{n : \delta_e r_n \subseteq \delta p \wedge (\forall x) (x \in \delta_e r_n \rightarrow r_n(x) \in 1 \oplus \Phi(\rho_{\bar{p}(x)}))\}$ . Since  $\rho_{\bar{p}(x)} = \tau_{<x}$  for  $x \in \tau$ ,  $\bar{\pi}(\tau) \subseteq \delta$ . Then for any integer  $n \in \delta$  we may effectively calculate  $\delta_e r_n$ , and for each  $x \in \delta_e r_n$  we may effectively calculate  $\bar{p}(x)$ ,  $\rho_{\bar{p}(x)}$ ,  $1 \oplus \Phi(\rho_{\bar{p}(x)})$  and finally  $s_n(x) = \text{card} [(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_n(x)}]$ . If we let  $s_n(x) = 0$  for  $x \notin \delta_e r_n$ , then there is a partial recursive function  $h(n)$  such that  $\delta h = \delta$  and  $r_{h(n)} = s_n$ . It is clear that  $h$  is defined on  $\delta$  and that it is a partial recursive function. Let  $m, n \in \delta$  and suppose that  $h(m) = h(n)$ . Since  $\delta_e r_{h(n)} = \delta_e r_n$  it follows that  $\delta_e r_m = \delta_e r_n$ . For  $x \in \delta_e r_m$

$$r_{h(m)}(x) = \text{card} [(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_m(x)}]$$

and

$$r_{h(n)}(x) = \text{card} [(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_n(x)}].$$

Since  $r_{h(m)}(x) = r_{h(n)}(x)$  it is clear that  $r_m(x) = r_n(x)$ . Thus  $r_m$  and  $r_n$  are the same function and therefore  $m = n$ .  $h$  is a one-to-one function. If  $n \in \bar{\pi}(\tau)$ , then  $\delta_e r_n \subseteq \tau$  and  $x \in \delta_e r_n$  implies that  $r_n(x) \in 1 \oplus \Phi(\rho_{\bar{p}(x)}) = 1 \oplus \Phi(\tau_{<x})$ . If  $x = t_m$ , then  $r_n(t_m) \in 1 \oplus \Phi(\{t_0, \dots, t_{m-1}\})$ ,  $r_{h(n)}(t_m) < 1 + f(m)$  and therefore  $h(n) \in \pi(t_n)$ . Thus  $h$  maps  $\bar{\pi}(\tau)$  into  $\pi(t_n)$ . Finally if  $s \in \pi(t_n)$ , and  $t_m \in \delta_e r_s$  choose  $n$  such that  $\delta_e r_n = \delta_e r_s$  and such that  $r_n(t_m)$  is  $r_s(t_m)$ th element of  $1 \oplus \Phi(\{t_0, \dots, t_{m-1}\})$ . Since  $1 \leq r_s(t_m) \leq f(m)$  the function  $r_n$  is defined. Thus  $h$  maps  $\bar{\pi}(\tau)$  onto  $\pi(t_n)$ .  $\bar{\pi}(\tau) \simeq \pi(t_n)$ .

If we denote the product  $\prod_T a_n$  by  $T: a_0 \cdot a_1 \cdot a_2 \cdots$ , then the following two formulas hold as a consequence of Theorem 3:  $T: 1 \cdot 2 \cdot 3 \cdots = T!$  and  $T: a \cdot a \cdot a \cdots = a^T$  where  $a > 1$ . Thus the infinite product operation as defined is consistent with the previously defined exponentiation and factorial operations.

## REFERENCES

1. J. C. E. Dekker, *Infinite Series of Isols*, Proc. Sympos. Pure Math., **V** (1962), 77-96., Amer. Math. Soc., Providence, R. I.
2. ——— and J. Myhill, *Retraceable Sets*, Canad. J. of Math., **10** (1958), 357-373.
3. ———, *Recursive Equivalence Types*, University of California Publ. in Math., **3** (1960), 67-214.
4. J. Myhill, *Recursive Equivalence Types and Combinatorial Functions*, Bull. Amer. Math. Soc., **64** (1958), 373-376.
5. A. Nerode, *Extensions to Isols*, Ann. of Math., **73** (1961), 362-403.

SHELL DEVELOPMENT COMPANY  
EMERYVILLE, CALIFORNIA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

ROBERT OSSERMAN  
Stanford University  
Stanford, California

M. G. ARSOVE  
University of Washington  
Seattle 5, Washington

J. DUGUNDJI  
University of Southern California  
Los Angeles 7, California

LOWELL J. PAIGE  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

# Pacific Journal of Mathematics

Vol. 14, No. 1

May, 1964

Richard Arens, <i>Normal form for a Pfaffian</i> .....	1
Charles Vernon Coffman, <i>Non-linear differential equations on cones in Banach spaces</i> .....	9
Ralph DeMarr, <i>Order convergence in linear topological spaces</i> .....	17
Peter Larkin Duren, <i>On the spectrum of a Toeplitz operator</i> .....	21
Robert E. Edwards, <i>Endomorphisms of function-spaces which leave stable all translation-invariant manifolds</i> .....	31
Erik Maurice Ellentuck, <i>Infinite products of isols</i> .....	49
William James Firey, <i>Some applications of means of convex bodies</i> .....	53
Haim Gaifman, <i>Concerning measures on Boolean algebras</i> .....	61
Richard Carl Gilbert, <i>Extremal spectral functions of a symmetric operator</i> .....	75
Ronald Lewis Graham, <i>On finite sums of reciprocals of distinct <math>n</math>th powers</i> .....	85
Hwa Suk Hahn, <i>On the relative growth of differences of partition functions</i> .....	93
Isidore Isaac Hirschman, Jr., <i>Extreme eigen values of Toeplitz forms associated with Jacobi polynomials</i> .....	107
Chen-jung Hsu, <i>Remarks on certain almost product spaces</i> .....	163
George Seth Innis, Jr., <i>Some reproducing kernels for the unit disk</i> .....	177
Ronald Jacobowitz, <i>Multiplicativity of the local Hilbert symbol</i> .....	187
Paul Joseph Kelly, <i>On some mappings related to graphs</i> .....	191
William A. Kirk, <i>On curvature of a metric space at a point</i> .....	195
G. J. Kurowski, <i>On the convergence of semi-discrete analytic functions</i> .....	199
Richard George Laatsch, <i>Extensions of subadditive functions</i> .....	209
V. Marić, <i>On some properties of solutions of <math>\Delta\psi + A(r^2)X\nabla\psi + C(r^2)\psi = 0</math></i> ...	217
William H. Mills, <i>Polynomials with minimal value sets</i> .....	225
George James Minty, Jr., <i>On the monotonicity of the gradient of a convex function</i> .....	243
George James Minty, Jr., <i>On the solvability of nonlinear functional equations of 'monotonic' type</i> .....	249
J. B. Muskat, <i>On the solvability of <math>x^e \equiv e \pmod{p}</math></i> .....	257
Zeev Nehari, <i>On an inequality of P. R. Bessack</i> .....	261
Raymond Moos Redheffer and Ernst Gabor Straus, <i>Degenerate elliptic equations</i> .....	265
Abraham Robinson, <i>On generalized limits and linear functionals</i> .....	269
Bernard W. Roos, <i>On a class of singular second order differential equations with a non linear parameter</i> .....	285
Tôru Saitô, <i>Ordered completely regular semigroups</i> .....	295
Edward Silverman, <i>A problem of least area</i> .....	309
Robert C. Sine, <i>Spectral decomposition of a class of operators</i> .....	333
Jonathan Dean Swift, <i>Chains and graphs of Ostrom planes</i> .....	353
John Griggs Thompson, <i>2-signalizers of finite groups</i> .....	363
Harold Widom, <i>On the spectrum of a Toeplitz operator</i> .....	365