EXTREMAL SPECTRAL FUNCTIONS OF A SYMMETRIC OPERATOR

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1. Introduction. Let $H_1$ be a symmetric operator in a Hilbert space $\mathcal{H}_1$. If $H$ is a self-adjoint operator in a Hilbert space $\mathcal{H}$ such that $\mathcal{H}_1 \subset \mathcal{H}$ and $H_1 \subset H$, then $H$ is called a self-adjoint extension of $H_1$. If $\mathcal{H} \supset \mathcal{H}_1$ is finite-dimensional, then $H$ is called a finite-dimensional self-adjoint extension of $H_1$. $H$ is called a minimal self-adjoint extension if neither $\mathcal{H} \supset \mathcal{H}_1$ nor any of its subspaces different from $\{0\}$ reduces $H$.

Suppose $H$ is a self-adjoint extension of $H_1$. If $E(\lambda)$ is the spectral function of $H$ and if $P_1$ is the operator in $\mathcal{H}$ of orthogonal projection on $\mathcal{H}_1$, then the operator function $E_1(\lambda) = P_1E(\lambda)$ restricted to $\mathcal{H}_1$ is called a spectral function of $H_1$. We shall say that the spectral function $E_1(\lambda)$ is defined by the self-adjoint extension $H$.

The family of spectral functions of $H_1$ is a convex set, i.e., if $E'_1(\lambda)$ and $E''_1(\lambda)$ are spectral functions of $H_1$ and if $a$ and $b$ are non-negative real numbers such that $a + b = 1$, then $aE'_1(\lambda) + bE''_1(\lambda)$ is also a spectral function of $H_1$. A spectral function $E_1(\lambda)$ of $H_1$ is said to be extremal if it is impossible to find two different spectral functions $E'_1(\lambda)$, $E''_1(\lambda)$ and positive real numbers $a$ and $b$, $a + b = 1$, such that $E_1(\lambda) = aE'_1(\lambda) + bE''_1(\lambda)$.

For further information we refer the reader to Achieser and Glasmann [1].

M. A. Naimark [6] has shown that the finite-dimensional extensions of a symmetric operator define extremal spectral functions of the operator. Finite-dimensional extensions exist, however, only for symmetric operators with equal deficiency indices. In § 4 of this paper it is shown that self-adjoint extensions defined by the addition of maximal symmetric operators determine extremal spectral functions for a symmetric operator with unequal deficiency indices. The proof uses the proposition of M. A. Naimark [6] that if $E_1(\lambda)$ is defined by the minimal self-adjoint extension $H_1$, then $E_1(\lambda)$ is extremal if and only if every bounded self-adjoint operator $A$ which commutes with $H$ and satisfies the condition $(Af, g) = (f, g)$ for all $f, g \in \mathcal{H}_1$ is reduced by $\mathcal{H}_1$. Section 2 is devoted to a description of the self-adjoint extensions of a symmetric operator, and section 3 identifies some extremal spectral functions of a symmetric operator with infinite equal deficiency indices other than the ones defined by finite-dimensional extensions.
The proof is based on the proposition of M. A. Naimark mentioned above.

2. Self-adjoint extensions of a symmetric operator. The linear operator $H$ in the Hilbert space $\mathcal{H}$ is said to be Hermitian if $(Hf, g) = (f, Hg)$ for all $f, g \in \mathcal{D}(H)$. $H$ is symmetric if it is Hermitian and $\mathcal{D}(H) = \mathcal{H}$. If $H$ is a closed Hermitian operator and $\lambda$ is a nonreal number, we define the subspaces $\mathcal{M}(\lambda)$ and $\mathcal{S}(\lambda)$ by the equations $\mathcal{S}(\lambda) = \mathcal{H} - \lambda \mathcal{E}$ and $\mathcal{M}(\lambda) = \mathcal{S} \ominus \mathcal{S}(\lambda)$. ($E$ stands for the identity operator.) $\mathcal{M}(\lambda)$ is called a deficiency subspace of $H$ and has the same dimensions for all $\lambda$ in the same half-plane (upper or lower.) If $m = \dim \mathcal{M}(\lambda)$, $n = \dim \mathcal{M}(\lambda)$, then $(m, n)$ are called the deficiency indices of $H$ (with respect to $\lambda$). (We add "with respect to $\lambda$" because the ordered pair $(m, n)$ depends on the half-plane $\lambda$ is in.) The operator $U(\lambda) = (H - \lambda \mathcal{E})(H - \lambda \mathcal{E})^{-1}$ is an isometry mapping $\mathcal{S}(\lambda)$ onto $\mathcal{S}(\lambda)$. It is called the Cayley transform of $H$. We have that $H = \lambda U(\lambda) - \lambda \mathcal{E})(U(\lambda) - E)^{-1}$. Since $\lambda$ is a fixed non-real number in the following, we shall write $U$ in place of $U(\lambda)$. For fixed $\lambda$ the correspondence between a Hermitian operator and its Cayley transform is a one-to-one inclusion-preserving correspondence between the set of closed Hermitian operators $H$ and the set of closed isometric operators $U$ for which $(U - E)^{-1}$ exists. We note, finally, that a subspace $\mathcal{S}$ reduces $H$ if and only if $\mathcal{S}$ reduces $U$. In this circumstance, if $\mathcal{S} = \mathcal{S} \ominus \mathcal{S}$, and if $H_i$ and $U_i$ are $H$ and $U$ respectively restricted to $\mathcal{S}$, then $U_i$ is the Cayley transform of $H_i$ and $H = H_1 \oplus H_2$, $U = U_1 \oplus U_2$.

M. A. Naimark [5] has proved the following theorem which describes all self-adjoint extensions of a symmetric operator.

THEOREM 1. Let $\lambda$ be any fixed nonreal number. Let $H_1$ be a closed symmetric operator with deficiency indices $(m_1, n_1)$ (with respect to $\lambda$). Then every self-adjoint extension $H$ of $H_1$ is obtained as follows:

1. Let $H_2$ be a closed Hermitian operator in $\mathcal{S}$ with deficiency indices $(m_2, n_2)$ (with respect to $\lambda$) satisfying $m_1 + m_2 = n_1 + n_2$, $m_2 \leq n_1$.

2. Let $H_0 = H_1 \oplus H_2$ in $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$. ($H_0$ is therefore a closed Hermitian operator with equal deficiency indices $(m_1 + m_2, n_1 + n_2)$, and if $U_i$ is the Cayley transform of $H_i$, $i = 0, 1, 2$, then $U_0 = U_1 \oplus U_2$. Further, $\mathcal{M}_a(\lambda) = \mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$, $\mathcal{M}_b(\lambda) = \mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$).

3. Let $V$ be an arbitrary isometric operator mapping $\mathcal{M}_a(\lambda)$ onto $\mathcal{M}_b(\lambda)$ satisfying the condition $\varphi \in \mathcal{M}_a(\lambda)$, $V \varphi \in \mathcal{M}_b(\lambda)$ implies $\varphi = 0$.

4. Let $\mathcal{D}(H)$ be defined as all $g = f + V \varphi - \varphi$, where $f \in \mathcal{D}(H_0)$, $\varphi \in \mathcal{M}_a(\lambda)$. 


If \( g \in \mathcal{D}(H) \), let \( H_g = H_0 + \lambda V\varphi - \overline{\lambda}\varphi \).

Then, \( H \) is self-adjoint extension in \( \mathcal{D} \) of \( H_0 \), and every self-adjoint extension of \( H_0 \) is obtained in this way. We have that \( \mathcal{D}(H) = \mathcal{D}(H) \cap \mathcal{S} \).

We say that \( H_0 \) and \( V \) of Theorem 1 define the self-adjoint extension \( H \) of \( H_0 \).

We can put the operator \( V \) into correspondence with a matrix \((V_{ik})\) of operators such that \( F_{\mathcal{S}u\mathcal{L}a(\lambda)} = F_{\mathcal{S}u\mathcal{L}a(\lambda)} \). Then condition on \( F \) in (3) of theorem 1 then becomes \( V_{12}\varphi = 0 \) implies \( \varphi = 0 \).

We now give a theorem which gives a more detailed analysis of the structure of \( V \).

**Theorem 2.** Suppose that \( \mathcal{M}_1(\lambda), \mathcal{M}_3(\lambda), \mathcal{M}_4(\lambda), \mathcal{M}_5(\lambda) \) are Hilbert spaces and that \( V \) is an isometry which maps \( \mathcal{M}_1(\lambda) \oplus \mathcal{M}_3(\lambda) \) onto \( \mathcal{M}_2(\lambda) \oplus \mathcal{M}_4(\lambda) \). (\( \lambda \) here has nothing to do with the theorem and is retained only as a notational convenience.) If \( V = (V_{ik}) \) in matrix form, suppose that \( V_{12}\varphi = 0 \) implies \( \varphi = 0 \). Then the following conclusions are true:

1. If \( \mathcal{R}_1(\lambda) \) is defined by the equation \( \mathcal{R}_1(\lambda) = \text{closure} \left( \mathcal{R}_1(\lambda) \right) \) (\( \text{c} \) indicates closure of a set) and if \( \mathcal{R}_1(\lambda) = \mathcal{M}_1(\lambda) \oplus \mathcal{M}_5(\lambda) \), then \( \mathcal{R}_1(\lambda) \) is the null space of \( V_{12}^* \). Thus, \( V_{12}^* \) is one-to-one on \( \mathcal{M}_1(\lambda) \). Further, \( \mathcal{M}_1(\lambda) = \text{closure} \left( \mathcal{M}_1(\lambda) \right) \).

2. \( V^* \) maps \( \mathcal{R}_1(\lambda) \) onto a subspace of \( \mathcal{M}_3(\lambda) \), which we denote by \( \mathcal{R}_3(\lambda) \). Thus, \( \mathcal{R}_3(\lambda) = V^*\mathcal{R}_1(\lambda) \).

3. If \( \mathcal{R}_1(\lambda) \) is defined by the equation \( \mathcal{R}_1(\lambda) = \mathcal{M}_1(\lambda) \oplus \overline{\mathcal{R}_1(\lambda)} \), then \( V \) maps \( \mathcal{M}_1(\lambda) \oplus \overline{\mathcal{R}_1(\lambda)} \) isometrically onto \( \mathcal{M}_5(\lambda) \).

Thus, \( V_{12}\mathcal{M}_1(\lambda) \subseteq \mathcal{M}_5(\lambda) \).

4. \( V_{21} \) is one-to-one on \( \mathcal{M}_1(\lambda) \), and \( \mathcal{R}_1(\lambda) \) is the null space of \( V_{21}^* \).

5. \( V_{34}^* \) is one-to-one on \( \mathcal{M}_3(\lambda) \) and \( \mathcal{R}_1(\lambda) = \text{closure} \left( \mathcal{R}_1(\lambda) \right) \).

6. If \( m_1 = \dim \mathcal{M}_1(\lambda), \ n_1 = \dim \mathcal{M}_3(\lambda), \ m_2 = \dim \mathcal{M}_4(\lambda), \ n_2 = \dim \mathcal{M}_5(\lambda) \), then \( m_1 + m_2 = n_1 + n_2 \), \( m_3 = \dim \mathcal{M}_3(\lambda) = \dim \mathcal{R}_1(\lambda) \leq n_1, \ n_3 = \dim \mathcal{M}_5(\lambda) = \dim \mathcal{M}_1(\lambda) \leq m_1 \).

7. If \( m_2 = n_2 \), \( m_1 = n_1 \).

**Proof.**

1. Since \( \mathcal{R}_1(\lambda) \) is the orthogonal complement of the closure of the range of \( V_{12} \), \( \mathcal{R}_1(\lambda) \) is the null space of \( V_{12}^* \), and \( V_{12}^* \) is one-to-one on \( \mathcal{M}_1(\lambda) \).

Suppose \( g \in \mathcal{M}_5(\lambda) \) and \( g \) is perpendicular to \( V_{12}^*\mathcal{M}_1(\lambda) \). Then \( 0 = (g, V_{12}^*f) = (V_{12}g, f) \) for all \( f \in \mathcal{M}_1(\lambda) \). Therefore, \( V_{12}g = 0 \), and, since \( V_{12} \) is one-to-one, \( g = 0 \). Thus, \( \mathcal{M}_5(\lambda) = \text{closure} \left( \mathcal{M}_5(\lambda) \right) \).
(2) Since
\[ V^* = \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{21}^* & V_{22}^* \end{pmatrix}, \]
\[ V^*\mathcal{R}_i(\lambda) = V_{11}^*\mathcal{R}_i(\lambda) \subseteq \mathcal{M}_i(\lambda). \]
Thus, \( V^* = V^{-1} \) maps \( \mathcal{R}_i(\lambda) \) onto a subspace of \( \mathcal{M}_i(\lambda) \).

(3) Clear, since \( \mathcal{R}_i(\lambda) = V\mathcal{R}_i(\lambda) \).

(4) We first show that \( V_{31} \) is one-to-one on \( \mathcal{M}_i(\lambda) \). Suppose \( f \in \mathcal{M}_i(\lambda), \ V_{31} f = 0 \). Then, \( Vf = V_{11} f + V_{31} f = V_{11} f + 0 = V_{11} f \in \mathcal{M}_i(\lambda) \). Let \( g = V_{11} f = Vf \), so that \( f = V^* g = V_{11}^* g + V_{12}^* g \). Since \( f \in \mathcal{M}_i(\lambda) \), \( V_{11}^* g \in \mathcal{M}_i(\lambda), \ V_{12}^* g \in \mathcal{M}_i(\lambda) \), we have that \( V_{11}^* g = 0 \). By (1) and the fact that \( g \in \mathcal{M}_i(\lambda) \), \( g = 0 \). Thus, \( f = V^* g = 0 \), and our contention is proved.

Since \( \mathcal{R}_i(\lambda) = V\mathcal{R}_i(\lambda) \), \( V_{31} f = 0 \) for all \( f \in \mathcal{R}_i(\lambda) \). On the other hand, we have just shown that \( V_{31} \) is one-to-one on \( \mathcal{M}_i(\lambda) \). It follows that \( \mathcal{R}_i(\lambda) \) is the null space of \( V_{31} \).

Because \( (V_{31}^*)^* = V_{31} \) and the null space of \( (V_{31}^*)^* \) is the orthogonal complement of the closure of the range of \( V_{31}^* \), we see that \( \mathcal{M}_i(\lambda) = [V_{31} \mathcal{M}_i(\lambda)]^\perp \).

We claim finally that \( \mathcal{M}_3(\lambda) = [V_{31} \mathcal{M}_i(\lambda)]^\perp \). Suppose \( g \in \mathcal{M}_3(\lambda) \) and that \( g \) is perpendicular to \( V_{31} \mathcal{M}_i(\lambda) \). Therefore, \( 0 = (V_{31} f, g) = (f, V_{31}^* g) \) for all \( f \in \mathcal{M}_i(\lambda) \). Since \( V_{31}^* g \in \mathcal{M}_i(\lambda) \), it follows that \( V_{31}^* g = 0 \). Thus, \( V^* g = V_{31}^* g \in \mathcal{M}_i(\lambda) \). Let \( f = V^* g \). Then, \( g = Vf = V_{11} f + V_{31} f \), where \( g \in \mathcal{M}_2(\lambda), \ V_{11} f \in \mathcal{M}_i(\lambda), \ V_{31} f \in \mathcal{M}_i(\lambda) \). Hence, \( V_{11} f = 0 \) and \( f = 0 \). Whence, \( g = Vf = 0 \). This proves our claim and completes the proof of (4).

(5) We have already shown in (4) that \( \mathcal{M}_i(\lambda) = [V_{31} \mathcal{M}_i(\lambda)]^\perp \). Since we also showed in (4) that \( \mathcal{M}_2(\lambda) = [V_{31} \mathcal{M}_i(\lambda)]^\perp \), it follows that the null space of \( V_{31}^* \) is empty and therefore \( V_{31}^* \) is one-to-one on \( \mathcal{M}_i(\lambda) \).

(6) \( m_1 + m_2 = n_1 + n_2 \) follows from the fact that \( V \) maps \( \mathcal{M}_i(\lambda) \) isometrically onto \( \mathcal{M}_i(\lambda) \).

We claim now that \( \dim \mathcal{M}_4(\lambda) = \dim \mathcal{M}_i(\lambda) \). Let \( \{\varphi_a\} \) be a complete orthonormal system in \( \mathcal{M}_i(\lambda) \). Then \( \{V_{12} \varphi_a\} \) is a fundamental set in \( \mathcal{M}_i(\lambda) \). (See Nagy [4] for definitions.) Therefore \( \dim \mathcal{M}_4(\lambda) = P\{\varphi_a\} = P\{V_{12} \varphi_a\} \geq \dim \mathcal{M}_i(\lambda) \), where \( P \) stands for cardinality. Using \( V_{12}^* \) and an analogous argument, we obtain that \( \dim \mathcal{M}_4(\lambda) \geq \dim \mathcal{M}_3(\lambda) \). Thus, \( \dim \mathcal{M}_4(\lambda) = \dim \mathcal{M}_i(\lambda) \), and \( m_4 = \dim \mathcal{M}_4(\lambda) = \dim \mathcal{M}_i(\lambda) \leq n_1 \). Similarly, \( n_4 = \dim \mathcal{M}_4(\lambda) = \dim \mathcal{M}_i(\lambda) \leq m_1 \).

(7) The proof is clear from the inequalities in (6).

Theorem 2 is therefore completely proved.

THEOREM 3. (M. A. Naimark [5]). For each self-adjoint extension \( H \) in \( \mathcal{S} \) of a symmetric operator \( H_1 \) in \( \mathcal{S} \) there exists a minimal self-adjoint extension \( H_0 \) in \( \mathcal{S} \) such that

(1) \( \mathcal{S}_1 \subset \mathcal{S}_0 \subset \mathcal{S} \);
(2) \( H_1 \subset H_0 \subset H \);
(3) \( H_0 \) and \( H \) define the same spectral function of \( H \).

**Theorem 4.** Suppose that \( H_1 \) is a closed symmetric operator and that \( H_0 \) and \( V \) define a self-adjoint extension \( H \) of \( H_1 \). Let \( H_0 \) be a self-adjoint extension of \( H_1 \) having the properties that \( \mathcal{D}_1 \subset \mathcal{D}_0 \subset \mathcal{D} \) and \( H_1 \subset H_0 \subset H \). Then the following statements are true:

(1) If we write \( \mathcal{D}_0 = \mathcal{D}_1 \oplus \mathcal{D}_2 \), \( \mathcal{D}_0 = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 \), \( \mathcal{D}_2 = \mathcal{D}_1 \oplus \mathcal{D}_4 \), then \( H \) is reduced by \( \mathcal{D}_1 \) and \( H = H_0 \oplus H_1 \), where \( H_1 \) is a self-adjoint operator in \( \mathcal{D}_1 \).

(2) \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda') \), \( \mathcal{M}(\lambda) \subset \mathcal{D}_1 \), \( \mathcal{M}(\lambda) \subset \mathcal{D}_2 \).

(3) \( H_1 \) is reduced by \( \mathcal{D}_4 \) and \( H = H_0 \oplus H_1 \), where \( H_1 \) is a closed Hermitian operator in \( \mathcal{D}_4 \) with the same deficiency subspaces \( \mathcal{M}(\lambda) \), \( \mathcal{M}(\lambda) \) as \( H \).

(4) \( H_0 \) is defined by \( H_2 \) and \( V \).

(5) \( H \) and \( H_0 \) define the same spectral function of \( H \).

**Proof.** (1) Since \( H_1 \subset H_0 \subset H \), we have that \( U_1 \subset U_0 \subset U \). Because \( U_0 \) maps \( \mathcal{D}_0 \) isometrically onto \( \mathcal{D}_0 \), and \( U \) maps \( \mathcal{D}_0 \) isometrically onto \( \mathcal{D}_0 \), we have that \( U \) maps \( \mathcal{D}_0 \) isometrically onto \( \mathcal{D}_0 \). Thus, \( \mathcal{D}_1 \) reduces \( U \), and hence \( U = U_0 \oplus U_0 = H_0 \oplus H_0 \), where \( H_1 \) is a self-adjoint operator in \( \mathcal{D}_0 \) with Cayley transform \( U_0 \). This proves (1).

(2) We claim first that \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \). Let \( f \in \mathcal{D}_1 \). Since \( H_1 \subset \mathcal{D}_2 = \mathcal{L}(\lambda) + \mathcal{L}(\lambda) \), \( f = f' + f'' \), where \( f' \in \mathcal{L}(\lambda) \), \( f'' \in \mathcal{L}(\lambda) \). Hence, \( Uf = Uf' + Uf'' = Vf' + U_0f'' = V_0f' + V_0f'' + U_0f'' \), where \( Uf \in \mathcal{D}_4 \subset \mathcal{D}_0 \), \( V_0f' \in \mathcal{M}(\lambda) \subset \mathcal{D}_1 \), \( V_0f'' \in \mathcal{M}(\lambda) \subset \mathcal{D}_1 \), \( U_0f'' \in \mathcal{L}(\lambda) \subset \mathcal{D}_2 \). Thus, \( V_0f' = 0 \), and therefore \( f' = 0 \). It follows that \( f = f'' \in \mathcal{L}(\lambda) \) and that \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \).

Since \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \), and since \( U \) maps \( \mathcal{D}_1 \) isometrically onto \( \mathcal{D}_1 \) and \( \mathcal{L}(\lambda) \) isometrically onto \( \mathcal{L}(\lambda) \), we conclude that \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \). Hence, \( \mathcal{D}_1 \subset \mathcal{L}(\lambda) \cap \mathcal{L}(\lambda) \). It follows immediately that \( \mathcal{M}(\lambda) \subset \mathcal{D}_0 \), \( \mathcal{M}(\lambda) \subset \mathcal{D}_1 \).

(2) is therefore completely proved.

(3) Because \( U_2 = U \) on \( \mathcal{L}(\lambda) \), we see that \( U \) maps \( \mathcal{D}_1 \) isometrically onto \( \mathcal{D}_2 \). We know, however, that \( U_2 \) maps \( \mathcal{L}(\lambda) \) isometrically onto \( \mathcal{L}(\lambda) \). It follows that \( \mathcal{D}_1 \) reduces \( U_2 \). Thus, \( U_2 = U_0 \oplus U_0 \), where \( U \) maps \( \mathcal{L}(\lambda) \subset \mathcal{D}_1 \) isometrically onto \( \mathcal{L}(\lambda) \subset \mathcal{D}_1 \), and \( H = H_0 \oplus H_1 \), where \( H_1 \) is a closed Hermitian operator in \( \mathcal{D}_1 \) with Cayley transform \( U_0 \). Noting that \( \mathcal{D}_1 = \mathcal{M}(\lambda) \oplus [\mathcal{L}(\lambda) \oplus \mathcal{D}_1] = \mathcal{M}(\lambda) \oplus [\mathcal{L}(\lambda) \oplus \mathcal{D}_1] \), we see that \( H_1 \) has deficiency subspaces \( \mathcal{M}(\lambda) \), \( \mathcal{M}(\lambda) \). This proves (3).

(4) By Theorem 1, \( H_1 \) and \( V \) define a self-adjoint extension \( H'_1 \) of \( H_1 \) in \( \mathcal{D}_0 = \mathcal{D}_1 \oplus \mathcal{D}_2 \). If \( U_0 \) is the Cayley transform of \( H_0 \), then \( U' = U_1 = U \) on \( \mathcal{L}(\lambda) \), \( U' = V = U \) on \( \mathcal{L}(\lambda) \oplus \mathcal{M}(\lambda) \), \( U' = U_0 = U \) on \( \mathcal{L}(\lambda) \oplus \mathcal{D}_1 \). It follows that \( U' = U \) on \( \mathcal{D}_1 \oplus \mathcal{D}_2 = \mathcal{D}_0 \). But since \( U_0 \subset U \), \( U_0 = U \) on \( \mathcal{D}_0 \), hence, \( U_0 = U'_0 \), and therefore \( H_0 = H'_0 \). This
proves (4).

(5) As we have shown, \( H = H_0 \oplus H_4 \). Thus, \( E(\lambda) = E_0(\lambda) \oplus E_4(\lambda) \), and therefore \( E(\lambda)f = E_0(\lambda)f \) for all \( f \in \mathcal{D} \). If \( P \) is the operator of orthogonal projection of \( \mathcal{D} \) onto \( \mathcal{D}_1 \), and if \( P_0 \) is the operator of orthogonal projection of \( \mathcal{D}_0 \) onto \( \mathcal{D}_2 \), \( PE(\lambda)f = P_0E_0(\lambda)f \) for all \( f \in \mathcal{D}_1 \), so that \( H \) and \( H_0 \) define the same spectral function of \( H_4 \). This proves (5), and the proof of theorem 4 is completed.

3. Extremal spectral functions of a symmetric operator with equal deficiency indices.

**Theorem 5.** Let \( H \) be a self-adjoint extension of the closed symmetric operator \( H_2 \). Suppose that \( H \) is defined by \( H_2 \) and \( V \). Then the following statements are equivalent:

1. \( \mathcal{D}(H_2) = \{0\} \).
2. \( \mathcal{M}_4(\lambda) = \mathcal{M}_4(\lambda) = \mathcal{D}_2 \).
3. \( \mathcal{D}(H) \cap \mathcal{D}_2 = \{0\} \).

**Proof.** That (1) implies (2) is clear from the definition of \( \mathcal{M}_4(\lambda) \) and \( \mathcal{M}_4(\lambda) \). Suppose, on the other hand, that \( \mathcal{M}_4(\lambda) = \mathcal{M}_4(\lambda) = \mathcal{D}_2 \). Then, \( \mathcal{R}(H_2 - \lambda E) = \mathcal{R}(H_2 - \mathcal{R}_2 E) = \{0\} \). If \( f \in \mathcal{D}(H_2) \), \( H_2 f - \lambda f = 0 \) and \( H_2 f - \mathcal{R}_2 f = 0 \). Subtracting the first equation from the second, \( (\lambda - \mathcal{R}_2)f = 0 \), and therefore \( f = 0 \). Thus, \( \mathcal{D}(H_2) = \{0\} \), and we have proved that (2) implies (1).

By Theorem 1, \( \mathcal{D}(H_2) = \mathcal{D}(H) \cap \mathcal{D}_2 \), so that (1) and (3) are clearly equivalent.

**Theorem 6.** Let \( H_1 \) be a closed symmetric operator. Suppose that \( H \) is a self-adjoint extension of \( H_1 \) defined by \( H_2 \) and \( V \). If \( \mathcal{D}(H_2) = \{0\} \), the following statements are true:

1. \( m_1 = n_1 \), i.e., the deficiency indices of \( H_1 \) are equal.
2. \( H \) is minimal.
3. The spectral function \( E_i(\lambda) \) of \( H_1 \) defined by \( H \) is extremal.

**Proof.** (1) By Theorem 5, \( \mathcal{D}(H_2) = \{0\} \) implies that \( m_2 = n_2 \). By theorem 2, (7), \( m_1 = n_1 \).

(2) By Theorem 5, \( \mathcal{D}(H_2) = \{0\} \) implies that \( \mathcal{M}_4(\lambda) = \mathcal{M}_4(\lambda) = \mathcal{D}_2 \). Hence, \( \mathcal{L}_4(\lambda) = \mathcal{L}_4(\lambda) = \{0\} \). It follows from Theorem 3 and Theorem 4, (2), that \( H \) is minimal.

(3) Let \( A \) be any bounded operator in \( \mathcal{D} \) having a matrix representation,

\[
A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},
\]
where \( E \) is the identity in \( \mathcal{H} \), \( B \) maps \( \mathcal{D}_2 \) into \( \mathcal{D}_3 \), \( C \) maps \( \mathcal{D}_2 \) into \( \mathcal{D}_3 \), and \( C \) is self-adjoint. Suppose that \( A \) commutes with \( H \). We shall show that this implies that \( B = 0 \). By the proposition of M. A. Naimark [6] mentioned in the introduction, then, it follows that the spectral function \( E_2(\lambda) \) defined by \( H \) is extremal.

Since \( A \) commutes with \( H \), it commutes with the Cayley transform \( U \) of \( H \). If we represent \( Z \) as a matrix, \( U = (U_{jk}) \), where \( U_{jk} \) maps \( \mathcal{H} \) into \( \mathcal{H} \), then the fact that \( A \) commutes with \( U \) implies that \( BU_{31} = U_{13}B^* \). Taking adjoints, we also have that \( U_{31}^*B^* = BU_{13}^* \). We observe, further, that \( U = V \) on \( \mathcal{M}_2(\lambda) \oplus \mathcal{M}_3(\lambda) \) and that \( U^* = U^{-1} = V^{-1} = V^* \) on \( \mathcal{M}_2(\lambda) \oplus \mathcal{M}_3(\lambda) \).

Using the equation \( BU_{31} = U_{13}B^* \), the fact that \( \mathcal{M}_2(\lambda) = \mathcal{D}_2 \), and Theorem 2, we obtain that \( BV_{13} = BU_{31}^* = BU_{13}^* \). Hence \( BV_{13} = BU_{31}^* \). This proves Theorem 7.

**Theorem 7.** If \( H \) is a finite-dimensional extension of a closed symmetric operator \( H \), then \( H \) must have equal deficiency indices.

**Proof.** Suppose that \( H \) is defined by \( H = V \). Then \( H \) is a Hermitian operator in the finite-dimensional space \( \mathcal{D}_2 \). Since \( U_{13} \) maps \( \mathcal{D}_2(\lambda) \) isometrically onto \( \mathcal{D}_2(\lambda) \), it follows that \( \dim \mathcal{D}_2(\lambda) = \dim \mathcal{D}_2(\lambda) \). Hence \( \dim \mathcal{M}_2(\lambda) = \dim \mathcal{M}_3(\lambda) \), i.e., \( m_1 = m_2 \). By Theorem 2, (7), \( m_2 = n_2 \). This proves Theorem 7.

4. Extremal spectral functions of a symmetric operator with unequal deficiency indices. We first introduce the notion of a partial isometry and some of the properties thereof. (See Murray and von Neumann [3].) A bounded linear operator \( W \) in a Hilbert space \( \mathcal{H} \) is called a partial isometry if it maps a subspace \( \mathbb{E} \) isometrically onto another subspace \( \mathcal{F} \), while it maps \( \mathcal{H} \) onto \( \mathbb{E} \). \( \mathcal{F} \) is called the initial set of \( W \), and \( \mathcal{F} \) is called the final set of \( W \). If \( W \) is a partial isometry, then the following statements hold:

1) If \( P(\mathbb{E}) \) is the operator of orthogonal projection on \( \mathbb{E} \) and if \( P(\mathcal{F}) \) is the operator of orthogonal projection on \( \mathcal{F} \), then \( P(\mathbb{E}) = W^*W \);
\[ P(\mathfrak{F}) = WW^*. \]

(2) \( U^* \) is a partial isometry with initial set \( \mathfrak{F} \) and final set \( \mathfrak{E} \).

(3) As a mapping of \( \mathfrak{F} \) onto \( \mathfrak{E} \), \( U^* \) is the inverse of \( U \) as a mapping of \( \mathfrak{E} \) onto \( \mathfrak{F} \).

**Theorem 8.** Suppose that \( W \) is a partial isometry with initial set \( \mathfrak{M} \) and final set \( \mathfrak{N} \). Let \( \mathfrak{R} = \mathfrak{N} \oplus \mathfrak{M} \). Then, \( \mathfrak{R} = \mathfrak{M} \oplus \mathfrak{M}' \), where

1. \( W \) maps \( \mathfrak{M}' \) isometrically onto \( \mathfrak{M} \);
2. if \( f \in \mathfrak{R} \oplus \mathfrak{M}' \), \( \lim_{p \to \infty} W^p f = 0. \)

**Proof.** Let \( \mathfrak{M}_i = (W^*)^i \mathfrak{M}, i = 0, 1, 2, \ldots \). Then each \( \mathfrak{M}_i \) is a subspace (i.e., a closed linear manifold), and the following statements are true:

(a) \( \mathfrak{M}_i \subseteq \mathfrak{M} \) for \( i = 1, 2, \ldots \). This is clear because \( W^* \) is a partial isometry with initial set \( \mathfrak{F} \) and final set \( \mathfrak{M} \).

(b) If \( f \in \mathfrak{M}_n \), where \( n \geq 0 \), then \( W^p f \in \mathfrak{M}_{n-p} \) for \( 1 \leq p \leq n \), and \( W^p f = 0 \) for \( p > n \). Proof: If \( f \in \mathfrak{M}_n \), then \( f = (W^*)^n g \) for some \( g \in \mathfrak{R} \).

Since \( WW^* = \mathbb{1} \), \( W^p f = (W^*)^{n-p} g \in \mathfrak{M}_{n-p} \), \( 1 \leq p \leq n \). If \( p > n \), \( W^p f = W^{p-n} g = 0. \)

(c) If \( f \in \mathfrak{M}_i \), \( i = 0, 1, 2, \ldots \), and if \( n \) is a positive integer, then \( (W^*)^n f \in \mathfrak{M}_{i+n} \). Proof: If \( f \in \mathfrak{M}_i \), \( f = (W^*)^i g \), where \( g \in \mathfrak{R} \). Therefore, \( (W^*)^n f = (W^*)^{i+n} g \in \mathfrak{M}_{i+n} \).

(d) \( \mathfrak{M}_i \) is perpendicular to \( \mathfrak{M}_j \) if \( i \neq j \). Proof: Suppose \( i < j \), and let \( f \in \mathfrak{M}_i \), \( g \in \mathfrak{M}_j \). Then there exists \( f_1 \in \mathfrak{R} \) and \( g_1 \in \mathfrak{R} \) such that \( f = (W^*)^i f_1 \), \( g = (W^*)^j g_1 \). Hence, \( (f, g) = ((W^*)^i f_1, (W^*)^j g_1) = (f_i, (W^*)^{i-j} g_j) = 0 \), since \( f_i \in \mathfrak{R}, (W^*)^{i-j} g_j \in \mathfrak{M}_{i-j} \subseteq \mathfrak{N} \).

Now let \( \mathfrak{W} = \sum_{i=1}^\infty \mathfrak{M}_i \). Then \( \mathfrak{W} \) is a subspace of \( \mathfrak{R} \). Let \( \mathfrak{W}' = \mathfrak{R} \oplus \mathfrak{W} \). We shall show that \( \mathfrak{W} \) and \( \mathfrak{W}' \) satisfy (1) and (2).

Since \( \mathfrak{R} = \mathfrak{W} \oplus \mathfrak{W}' \) and \( \mathfrak{F} = \mathfrak{R} \oplus \mathfrak{W} \), \( \mathfrak{N} = \mathfrak{M} \oplus \mathfrak{W} \), and since \( W \) maps \( \mathfrak{W} \) isometrically onto \( \mathfrak{F} \), in order to prove (1) it is sufficient to show that \( W \) maps \( \mathfrak{W}' \) onto \( \mathfrak{R} \oplus \mathfrak{W}' \). Suppose \( f \in \mathfrak{W}' \). Then, \( f = \sum_{i=1}^\infty f_i \), where \( f_i \in \mathfrak{M}_i \), and \( Wf = \sum_{i=1}^\infty Wf_i \). Because by (b) \( Wf_i \in \mathfrak{M}_{i-1} \), we see that \( Wf \in \mathfrak{R} \oplus \mathfrak{W}' \). Thus, \( W \) maps \( \mathfrak{W}' \) into \( \mathfrak{R} \oplus \mathfrak{W}' \). To show that the map is onto, let \( g \in \mathfrak{R} \oplus \mathfrak{W}' \). Then, \( g = \sum_{i=0}^\infty f_i \), where \( f_i \in \mathfrak{M}_i \).

If \( f = W^* f = \sum_{i=0}^\infty W^* f_i \in \mathfrak{W}' \), by (c). Further, \( Wf = WW^* g = g \). Hence, \( W \) maps \( \mathfrak{W}' \) onto \( \mathfrak{R} \oplus \mathfrak{W}' \).

We now prove (2). Let \( f \in \mathfrak{R} \oplus \mathfrak{W}' \). Then, \( f = \sum_{i=0}^\infty f_i \), where \( f_i \in \mathfrak{M}_i \). By (b), \( W^2 f = \sum_{i=0}^\infty W^2 f_i = \sum_{i=p}^\infty W^2 f_i \). Hence, \( \| W^2 f \|^2 = \sum_{i=p}^\infty \| W^2 f_i \|^2 = \sum_{i=p}^\infty \| f_i \|^2 \). Thus, \( \lim_{p \to \infty} \| W^2 f \|^2 = 0 \). This proves (2) and completes the proof of the theorem.

**Theorem 9.** Let \( \lambda \) be a fixed nonreal number. Suppose that \( H \) is a closed symmetric operator in \( \mathfrak{F} \), with deficiency indices \( (m, n) \).
(with respect to $\lambda$), and suppose that $m \neq n$. Let $H$ be a self-adjoint extension of $H_\lambda$ defined by $H_\lambda$ and $V$, where $H_\lambda$ is a closed Hermitian operator with deficiency indices $(0, s)$, $n + s = m$, if $m > n$ and $(s, 0)$, $m + s = n$, if $m < n$. Then the spectral function defined by $H$ is extremal.

Proof. Assume that $m > n$. The case $m < n$ then follows by interchanging the roles of $\lambda$ and $\lambda$ in Theorem 1 and defining $H$ by $H_\lambda^2$ and $V^*$. By Theorem 3 there exists a minimal self-adjoint extension $H_0$ of $JH\lambda$ such that $H_0 \subset H_\lambda \subset H$, and $H_0$ and $H$ define the same spectral function of $H\lambda$. By Theorem 4, $H_0$ is defined by $V$ and a Hermitian operator $H_2$ with the same deficiency subspaces as $H_\lambda$. Since we can always consider $H_0$ instead of $H$, it follows that without loss of generality we can consider $H$ to be a minimal self-adjoint extension.

Since $W_\lambda(\lambda) = \{0\}$ and $\mathcal{L}_\lambda(\lambda) = \mathcal{E}_\lambda$, we have that if $f \in \mathcal{E}_\lambda$, $Uf \in \mathcal{L}_\lambda(\lambda) \subset \mathcal{E}_\lambda$. If we represent $U$ as a matrix, $U \sim (U_{jk})$, where $U_{jk}$ maps $\mathcal{E}_\lambda$ into $\mathcal{E}_\lambda$, then it follows that $U_{13} = 0$ on $\mathcal{E}_\lambda$. Further, $Uf = U_{21}f$ for all $f \in \mathcal{E}_\lambda$, so that $U_{22}$ maps $\mathcal{E}_\lambda$ isometrically onto $\mathcal{L}_\lambda(\lambda)$. $U_{22}$ is thus a partial isometry in $\mathcal{E}_\lambda$ with initial set $\mathcal{E}_\lambda$ and final set $\mathcal{L}_\lambda(\lambda)$, while $U_{21}$ is a partial isometry with initial set $\mathcal{L}_\lambda(\lambda)$ and final set $\mathcal{E}_\lambda$. We have that $E = P(\mathcal{E}_\lambda) = U_{21}U_{22}$, while $P(\mathcal{L}_\lambda(\lambda)) = U_{22}U_{21}^*$. Now let $A$ be any bounded operator in $\mathcal{E}_\lambda$ with matrix representation

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where $E$ is the identity in $\mathcal{E}_\lambda$, $B$ maps $\mathcal{E}_\lambda$ into $\mathcal{E}_\lambda$, $C$ maps $\mathcal{E}_\lambda$ into $\mathcal{E}_\lambda$, and $C$ is self-adjoint. Suppose that $A$ commutes with $H$. We shall show that this implies $B = 0$. Then by the proposition of M. A. Naimark [6] mentioned in the introduction, it follows that the spectral function $E(\lambda)$ defined by $H$ is extremal.

Since $A$ commutes with $H$, it commutes with the Cayley transform $U$ of $H$. This implies that $BU_{21} = U_{21}B^*$ and $U_{13} + BU_{21} = U_{13}B + U_{21}C$. Since $U_{13} = 0$, these equations become $BU_{21} = 0$ and $BU_{22} = U_{21}B$. On $\mathcal{E}_\lambda(\lambda)$, $U_{21} = V_{21}$ and therefore $BV_{21}\mathcal{E}_\lambda(\lambda) = BU_{21}\mathcal{E}_\lambda(\lambda) = \{0\}$. Because by Theorem 2, $V_{21}\mathcal{E}_\lambda(\lambda)$ is dense in $\mathcal{E}_\lambda(\lambda)$, $BV_{21}(\lambda) = \{0\}$, i.e., $BP(\mathcal{E}_\lambda(\lambda)) = 0$. From the equation $BU_{22} = U_{21}B$ we have that $BP(\mathcal{L}_\lambda(\lambda)) = BU_{21}U_{22}^* = U_{13}BU_{22}^*$. Adding $BP(\mathcal{E}_\lambda(\lambda)) = U_{21}BU_{22}^*$ with $BP(\mathcal{E}_\lambda(\lambda)) = 0$, we obtain that $B = U_{13}BU_{22}^*$. By iterating this equation we see that $B = U_{13}^p B(U_{22}^*)^p$ for every positive integer $p$. Since $\|U_{13}\| \leq 1$, $\|Bf\| \leq \|B\| \|(U_{22}^*)^p f\|$ for each $f \in \mathcal{E}_\lambda$ and each positive integer $p$. 


By Theorem 8, \( S_2(\lambda) = W_0 \oplus W' \), where \( U_{2*} \) maps \( W'' \) isometrically onto \( W' \), and if \( f \in W_2(\lambda) \oplus W' \), then \( \lim_{\rho \to \infty} \| (U_{2*})^p f \| = 0 \). But if \( U_{2*} \) maps \( W'' \) isometrically onto \( W' \), then \( U_2 \) and therefore \( U \) maps \( W'' \) isometrically onto \( W' \). This means that \( U \) and therefore \( H \) is reduced by \( W'' \), a subspace of \( \mathcal{D}_2 \). Since \( H \) is a minimal self-adjoint extension of \( H_1 \), \( W'' = \{0\} \). Hence, \( \mathcal{S}_2 = W_2(\lambda) \oplus W' \), and thus if \( f \in \mathcal{S}_2 \), \( \lim_{\rho \to \infty} \| (U_{2*})^p f \| = 0 \). Since \( \| B f \| \leq \| B \| \| (U_{2*})^p f \| \) for each \( f \in \mathcal{S}_2 \) and for every positive integer \( p \), it follows that \( B \equiv 0 \) on \( \mathcal{S}_2 \). This completes the proof of Theorem 9.

Since the operator \( H_1 \) in Theorem 9 is a Hermitian operator with deficiency indices \((0, s)\) or \((s, 0)\), it may seem that we are dealing with a wider class of operators than the maximal symmetric operators. That this is not so is shown by Theorem 10 below.

**Theorem 10.** If \( H \) is a Hermitian operator with deficiency indices \((0, s)\) or \((s, 0)\), then \( H \) is a maximal symmetric operator. If \( H \) is a Hermitian operator with deficiency indices \((0, 0)\), then \( H \) is a self-adjoint operator.

**Proof.** If \( H \) is a Hermitian operator and \( \mathcal{B} = \mathcal{S} \ominus [\mathcal{D}(H)]' \), then \( \mathcal{B} \cap \mathcal{S}(\lambda) = \{0\} \). (If \( h \in \mathcal{B} \cap \mathcal{S}(\lambda) \), then \( h = (H - \lambda E) g \), Hence, \( 0 = (h, g) = (Hg, g) - \lambda (g, g) \). Since \( (Hg, g) \) is real while \( \lambda \) is not, \( g = 0 \). This simple argument is due to M. A. Krasnosel'skii [2, Lemma 2].) If \( H \) has deficiency indices \((0, s)\), \( \mathcal{M}(\lambda) = \{0\} \) so that \( \mathcal{B} \subset \mathcal{S}(\lambda) \). Thus, \( \mathcal{B} = \{0\} \) and \( H \) is symmetric. Similarly, \( H \) is symmetric if its deficiency indices are \((s, 0)\). It follows immediately that if \( H \) has deficiency indices \((0, 0)\), \( H \) is self-adjoint. Theorem 10 is proved.

**References**


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