Pacific Journal of Mathematics

EXTREMAL SPECTRAL FUNCTIONS OF A SYMMETRIC OPERATOR

RICHARD CARL GILBERT

Vol. 14, No. 1 May 1964

EXTREMAL SPECTRAL FUNCTIONS OF A SYMMETRIC OPERATOR

RICHARD C. GILBERT

1. Introduction. Let H_1 be a symmetric operator in a Hilbert space $\mathfrak{H}_1$. If H is a self-adjoint operator in a Hilbert space $\mathfrak{H}$ such that $\mathfrak{H}_1 \subset \mathfrak{H}$ and $H_1 \subset H$, then H is called a *self-adjoint extension* of H_1. If $\mathfrak{H} \ominus \mathfrak{H}_1$ is finite-dimensional, then H is called a *finite-dimensional* self-adjoint extension of H_1. H is called a *minimal* self-adjoint extension if neither $\mathfrak{H} \ominus \mathfrak{H}_1$ nor any of its subspaces different from $\{0\}$ reduces H.

Suppose H is a self-adjoint extension of H_1. If $E(\lambda)$ is the spectral function of H and if P_1 is the operator in $\mathfrak{H}$ of orthogonal projection on $\mathfrak{H}_1$, then the operator function $E_1(\lambda) = P_1 E(\lambda)$ restricted to $\mathfrak{H}_1$ is called a *spectral function* of H_1. We shall say that the spectral function $E_1(\lambda)$ is *defined* by the self-adjoint extension H.

The family of spectral functions of H_1 is a *convex set*, i.e., if $E_1'(\lambda)$ and $E_1''(\lambda)$ are spectral functions of H_1 and if a and b are non-negative real numbers such that $a + b = 1$, then $aE_1'(\lambda) + bE_1''(\lambda)$ is also a spectral function of H_1. A spectral function $E_1(\lambda)$ of H_1 is said to be *extremal* if it is impossible to find two different spectral functions $E_1'(\lambda)$, $E_1''(\lambda)$ and positive real numbers a and b, $a + b = 1$, such that $E_1(\lambda) = aE_1'(\lambda) + bE_1''(\lambda)$.

For further information we refer the reader to Achieser and Glasmann [1].

M. A. Naimark [6] has shown that the finite-dimensional extensions of a symmetric operator define extremal spectral functions of the operator. Finite-dimensional extensions exist, however, only for symmetric operators with equal deficiency indices. In § 4 of this paper it is shown that self-adjoint extensions defined by the addition of maximal symmetric operators determine extremal spectral functions for a symmetric operator with unequal deficiency indices. The proof uses the proposition of M. A. Naimark [6] that if $E_1(\lambda)$ is defined by the minimal self-adjoint extension H, then $E_1(\lambda)$ is extremal if and only if every bounded self-adjoint operator A which commutes with H and satisfies the condition $(Af, g) = (f, g)$ for all $f, g \in \mathfrak{H}_1$ is reduced by $\mathfrak{H}_1$. Section 2 is devoted to a description of the self-adjoint extensions of a symmetric operator, and section 3 identifies some extremal spectral functions of a symmetric operator with infinite equal deficiency indices other than the ones defined by finite-dimensional extensions.

Received May 15, 1963. This work was supported by the Mathematics Research Center, U.S. Army, Madison, Wisconsin, under Contract No.: DA-11-022-ORD-2059.

The proof is based on the proposition of M. A. Naimark mentioned above.

2. Self-adjoint extensions of a symmetric operator. The linear operator H in the Hilbert space $\mathfrak{H}$ is said to be *Hermitian* if $(Hf, g) = (f, Hg)$ for all $f, g \in \mathfrak{D}(H)$. H is *symmetric* if it is Hermitian and $\overline{\mathfrak{D}(H)} = \mathfrak{H}$. If H is a closed Hermitian operator and λ is a nonreal number, we define the subspaces $\mathfrak{M}(\lambda)$ and $\mathfrak{L}(\lambda)$ by the equations $\mathfrak{L}(\lambda) = \mathfrak{R}(H - \bar{\lambda}E)$ and $\mathfrak{M}(\lambda) = \mathfrak{H} \ominus \mathfrak{L}(\lambda)$. ($E$ stands for the identity operator.) $\mathfrak{M}(\lambda)$ is called a *deficiency subspace* of H and has the same dimensions for all λ in the same half-plane (upper or lower.) If $m = \dim \mathfrak{M}(\bar{\lambda})$, $n = \dim \mathfrak{M}(\lambda)$, then (m, n) are called the *deficiency indices* of H (with respect to λ). (We add "with respect to λ" because the ordered pair (m, n) depends on the half-plane λ is in.) The operator $U(\lambda) = (H - \bar{\lambda}E)(H - \lambda E)^{-1}$ is an isometry mapping $\mathfrak{L}(\bar{\lambda})$ onto $\mathfrak{L}(\lambda)$. It is called the *Cayley transform* of H. We have that $H = (\lambda U(\lambda) - \bar{\lambda}E)(U(\lambda) - E)^{-1}$. Since λ is a fixed non-real number in the following, we shall write U in place of $U(\lambda)$. For fixed λ the correspondence between a Hermitian operator and its Cayley transform is a one-to-one inclusion-preserving correspondence between the set of closed Hermitian operators H and the set of closed isometric operators U for which $(U - E)^{-1}$ exists. We note, finally, that a subspace $\mathfrak{H}_1$ reduces H if and only if $\mathfrak{H}_1$ reduces U. In this circumstance, if $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, and if H_i and U_i are H and U respectively restricted to $\mathfrak{H}_i$, then U_i is the Cayley transform of H_i and $H = H_1 \oplus H_2$, $U = U_1 \oplus U_2$.

M. A. Naimark [5] has proved the following theorem which describes all self-adjoint extensions of a symmetric operator.

THEOREM 1. *Let λ be any fixed nonreal number. Let H_1 be a closed symmetric operator with deficiency indices (m_1, n_1) (with respect to λ). Then every self-adjoint extension H of H_1 is obtained as follows:*

(1) *Let H_2 be a closed Hermitian operator in $\mathfrak{H}_2$ with deficiency indices (m_2, n_2) (with respect to λ) satisfying $m_1 + m_2 = n_1 + n_2$, $m_2 \leqq n_1$.*

(2) *Let $H_0 = H_1 \oplus H_2$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. ($H_0$ is therefore a closed Hermitian operator with equal deficiency indices $(m_1 + m_2, n_1 + n_2)$, and if U_i is the Cayley transform of H_i, $i = 0, 1, 2$, then $U_0 = U_1 \oplus U_2$. Further, $\mathfrak{M}_0(\bar{\lambda}) = \mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$, $\mathfrak{M}_0(\lambda) = \mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$).*

(3) *Let V be an arbitrary isometric operator mapping $\mathfrak{M}_0(\bar{\lambda})$ onto $\mathfrak{M}_0(\lambda)$ satisfying the condition $\varphi \in \mathfrak{M}_2(\bar{\lambda})$, $V\varphi \in \mathfrak{M}_2(\lambda)$ implies $\varphi = 0$.*

(4) *Let $\mathfrak{D}(H)$ be defined as all $g = f + V\varphi - \varphi$, where $f \in \mathfrak{D}(H_0)$, $\varphi \in \mathfrak{M}_0(\bar{\lambda})$.*

(5) *If* $g \in \mathfrak{D}(H)$, *let* $Hg = H_0 f + \lambda V\varphi - \bar{\lambda}\varphi$.

Then, H is self-adjoint extension in $\mathfrak{H}$ *of* H_1, *and every self-adjoint extension of* H_1 *is obtained in this way. We have that* $\mathfrak{D}(H_2) = \mathfrak{D}(H) \cap \mathfrak{H}_2$.

We say that H_2 and V of Theorem 1 *define* the self-adjoint extension H of H_1.

We can put the operator V into correspondence with a matrix (V_{ik}) of operators such that V_{11}: $\mathfrak{M}_1(\bar{\lambda}) \to \mathfrak{M}_1(\lambda)$, V_{12}: $\mathfrak{M}_2(\bar{\lambda}) \to \mathfrak{M}_1(\lambda)$, V_{21}: $\mathfrak{M}_1(\bar{\lambda}) \to \mathfrak{M}_2(\lambda)$, V_{22}: $\mathfrak{M}_2(\bar{\lambda}) \to \mathfrak{M}_2(\lambda)$. Then condition on V in (3) of theorem 1 then becomes $V_{12}\varphi = 0$ implies $\varphi = 0$.

We now give a theorem which gives a more detailed analysis of the structure of V.

THEOREM 2. *Suppose that* $\mathfrak{M}_1(\lambda)$, $\mathfrak{M}_1(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$, $\mathfrak{M}_2(\bar{\lambda})$ *are Hilbert spaces and that V is an isometry which maps* $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ *onto* $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$. ($\lambda$ *here has nothing to do with the theorem and is retained only as a notational convenience.) If* $V = (V_{ik})$ *in matrix form, suppose that* $V_{12}\varphi = 0$ *implies that* $\varphi = 0$. *Then the following conclusions are true*:

(1) *If* $\mathfrak{M}_1^-(\lambda)$ *is defined by the equation* $\mathfrak{M}_1^-(\lambda) = [V_{12}\mathfrak{M}_2(\bar{\lambda})]^c$ (c *indicates closure of a set) and if* $\mathfrak{N}_1(\lambda)$ *is defined by* $\mathfrak{N}_1(\lambda) = \mathfrak{M}_1(\lambda) \ominus \mathfrak{M}_1^-(\lambda)$, *then* $\mathfrak{N}_1(\lambda)$ *is the null space of* V_{12}^*. *Thus,* V_{12}^* *is one-to-one on* $\mathfrak{M}_1^-(\lambda)$. *Further,* $\mathfrak{M}_2(\bar{\lambda}) = [V_{12}^*\mathfrak{M}_1^-(\lambda)]^c$.

(2) $V^* = V^{-1}$ *maps* $\mathfrak{N}_1(\lambda)$ *onto a subspaces of* $\mathfrak{M}_1(\bar{\lambda})$, *which we denote by* $\mathfrak{N}_1(\bar{\lambda})$. *Thus,* $\mathfrak{N}_1(\bar{\lambda}) = V^*\mathfrak{N}_1(\lambda)$, $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar{\lambda})$.

(3) *If* $\mathfrak{M}_1^-(\bar{\lambda})$ *is defined by the equation* $\mathfrak{M}_1^-(\bar{\lambda}) = \mathfrak{M}_1(\bar{\lambda}) \ominus \mathfrak{N}_1(\bar{\lambda})$, *then V maps* $\mathfrak{M}_1^-(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ *isometrically onto* $\mathfrak{M}_1^-(\lambda) \oplus \mathfrak{M}_2(\lambda)$.

Thus, $V_{11}\mathfrak{M}_1^-(\bar{\lambda}) \subset \mathfrak{M}_1^-(\lambda)$.

(4) V_{21} *is one-to-one on* $\mathfrak{M}_1^-(\bar{\lambda})$, *and* $\mathfrak{N}_1(\bar{\lambda})$ *is the null space of* V_{21}. $\mathfrak{M}_2(\lambda) = [V_{21}\mathfrak{M}_1^-(\bar{\lambda})]^c$.

(5) V_{21}^* *is one-to-one on* $\mathfrak{M}_2(\lambda)$ *and* $\mathfrak{M}_1^-(\bar{\lambda}) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$.

(6) *If* $m_1 = \dim \mathfrak{M}_1(\bar{\lambda})$, $n_1 = \dim \mathfrak{M}_1(\lambda)$, $m_2 = \dim \mathfrak{M}_2(\bar{\lambda})$, $n_2 = \dim \mathfrak{M}_2(\lambda)$, *then* $m_1 + m_2 = n_1 + n_2$, $m_2 = \dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_1^-(\lambda) \leqq n_1$, $n_2 = \dim \mathfrak{M}_2(\lambda) = \dim \mathfrak{M}_1^-(\bar{\lambda}) \leqq m_1$.

(7) *If* $m_2 = n_2$, $m_1 = n_1$.

Proof. (1) Since $\mathfrak{N}_1(\lambda)$ is the orthogonal complement of the closure of the range of V_{12}, $\mathfrak{N}_1(\lambda)$ is the null space of V_{12}^*, and V_{12}^* is one-to-one on $\mathfrak{M}_1^-(\lambda)$.

Suppose $g \in \mathfrak{M}_2(\bar{\lambda})$ and g is perpendicular to $V_{12}^*\mathfrak{M}_1^-(\lambda)$. Then $0 = (g, V_{12}^* f) = (V_{12}g, f)$ for all $f \in \mathfrak{M}_1^-(\lambda)$. Therefore, $V_{12}g = 0$, and, since V_{12} is one-to-one, $g = 0$. Thus, $M_2(\bar{\lambda}) = [V_{12}^*\mathfrak{M}_1^-(\lambda)]^c$.

(2) Since

$$V^* = \begin{pmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & V_{22}^* \end{pmatrix},$$

$V^*\mathfrak{N}_1(\lambda) = V_{11}^*\mathfrak{N}_1(\lambda) \subset \mathfrak{M}_1(\bar{\lambda})$. Thus, $V^* = V^{-1}$ maps $\mathfrak{N}_1(\lambda)$ onto a subspace of $\mathfrak{M}_1(\bar{\lambda})$.

(3) Clear, since $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar{\lambda})$.

(4) We first show that V_{21} is one-to-one on $\mathfrak{M}_1^-(\bar{\lambda})$. Suppose $f \in \mathfrak{M}_1^-(\bar{\lambda})$, $V_{21}f = 0$. Then, $Vf = V_{11}f + V_{21}f = V_{11}f \in \mathfrak{M}_1^-(\lambda)$. Let $g = V_{11}f = Vf$, so that $f = V^*g = V_{11}^*g + V_{12}^*g$. Since $f \in \mathfrak{M}_1^-(\bar{\lambda})$, $V_{11}^*g \in \mathfrak{M}_1^-(\bar{\lambda})$, $V_{12}^*g \in \mathfrak{M}_2(\bar{\lambda})$, we have that $V_{12}^*g = 0$. By (1) and the fact that $g \in \mathfrak{M}_1^-(\lambda)$, $g = 0$. Thus, $f = V^*g = 0$, and our contention is proved.

Since $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar{\lambda})$, $V_{21}f = 0$ for all $f \in \mathfrak{N}_1(\bar{\lambda})$. On the other hand, we have just shown that V_{21} is one-to-one on $\mathfrak{M}_1^-(\bar{\lambda})$. It follows that $\mathfrak{N}_1(\bar{\lambda})$ is the null space of V_{21}.

Because $(V_{21}^*)^* = V_{21}$ and the null space of $(V_{21}^*)^*$ is the orthogonal complement of the closure of the range of V_{21}^*, we see that $\mathfrak{M}_1^-(\bar{\lambda}) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$.

We claim finally that $\mathfrak{M}_2(\lambda) = [V_{21}M_1^-(\bar{\lambda})]^c$. Suppose $g \in \mathfrak{M}_2(\lambda)$ and that g is perpendicular to $V_{21}\mathfrak{M}_1^-(\bar{\lambda})$. Therefore, $0 = (V_{21}f, g) = (f, V_{21}^*g)$ for all $f \in \mathfrak{M}_1^-(\bar{\lambda})$. Since $V_{21}^*g \in \mathfrak{M}_1^-(\bar{\lambda})$, it follows that $V_{21}^*g = 0$. Thus, $V^*g = V_{22}^*g \in \mathfrak{M}_2(\bar{\lambda})$. Let $f = V^*g$. Then, $g = Vf = V_{12}f + V_{22}f$, where $g \in \mathfrak{M}_2(\lambda)$, $V_{12}f \in \mathfrak{M}_1^-(\lambda)$, $V_{22}f \in \mathfrak{M}_2(\lambda)$. Hence, $V_{12}f = 0$ and $f = 0$. Whence, $g = Vf = 0$. This proves our claim and completes the proof of (4).

(5) We have already shown in (4) that $\mathfrak{M}_1^-(\bar{\lambda}) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$. Since we also showed in (4) that $\mathfrak{M}_2(\lambda) = [V_{21}\mathfrak{M}_1^-(\bar{\lambda})]^c$, it follows that the null space of V_{21}^* is empty and therefore V_{21}^* is one-to-one on $\mathfrak{M}_2(\lambda)$.

(6) $m_1 + m_2 = n_1 + n_2$ follows from the fact that V maps $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ isometrically onto $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$.

We claim now that $\dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_1^-(\lambda)$. Let $\{\varphi_\alpha\}$ be a complete orthonormal system in $\mathfrak{M}_2(\bar{\lambda})$. Then $\{V_{12}\varphi_\alpha\}$ is a fundamental set in $\mathfrak{M}_1^-(\lambda)$. (See Nagy [4] for definitions.) Therefore $\dim \mathfrak{M}_2(\bar{\lambda}) = P\{\varphi_\alpha\} = P\{V_{12}\varphi_\alpha\} \geqq \dim \mathfrak{M}_1^-(\lambda)$, where P stands for cardinality. Using V_{12}^* and an analogous argument, we obtain that $\dim \mathfrak{M}_1^-(\lambda) \geqq \dim \mathfrak{M}_2(\bar{\lambda})$. Thus, $\dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_1^-(\lambda)$, and $m_2 = \dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_1^-(\lambda) \leqq n_1$. Similarly, $n_2 = \dim \mathfrak{M}_2(\lambda) = \dim \mathfrak{M}_1^-(\bar{\lambda}) \leqq m_1$.

(7) The proof is clear from the inequalities in (6).

Theorem 2 is therefore completely proved.

THEOREM 3. (M. A. Naimark [5]). *For each self-adjoint extension H in $\mathfrak{H}$ of a symmetric operator H_1 in $\mathfrak{H}_1$ there exists a minimal self-adjoint extension H_0 in $\mathfrak{H}_0$ such that*

(1) $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$;

(2) $H_1 \subset H_0 \subset H$;

(3) *H_0 and H define the same spectral function of H_1.*

THEOREM 4. *Suppose that H_1 is a closed symmetric operator and that H_2 and V define a self-adjoint extension H of H_1. Let H_0 be a self-adjoint extension of H_1 having the properties that $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$ and $H_1 \subset H_0 \subset H$. Then the following statements are true:*

(1) *If we write $\mathfrak{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_3$, $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_4 = \mathfrak{H}_1 \oplus \mathfrak{H}_3 \oplus \mathfrak{H}_4$, $\mathfrak{H}_2 = \mathfrak{H}_3 \oplus \mathfrak{H}_4$, then H is reduced by $\mathfrak{H}_4$ and $H = H_0 \oplus H_4$, where H_4 is a self-adjoint operator in $\mathfrak{H}_4$.*

(2) $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda}) \cap \mathfrak{L}_2(\lambda)$, $\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{H}_3$, $\mathfrak{M}_2(\lambda) \subset \mathfrak{H}_3$.

(3) *H_2 is reduced by $\mathfrak{H}_4$ and $H_2 = H_3 \oplus H_4$, where H_3 is a closed Hermitian operator in $\mathfrak{H}_3$ with the same deficiency subspaces $M_2(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$ as H_2.*

(4) *H_0 is defined by H_3 and V.*

(5) *H and H_0 define the same spectral function of H_1.*

Proof. (1) Since $H_1 \subset H_0 \subset H$, we have that $U_1 \subset U_0 \subset U$. Because U_0 maps $\mathfrak{H}_0$ isometrically onto $\mathfrak{H}_0$ and U maps $\mathfrak{H}$ isometrically onto $\mathfrak{H}$, we have that U maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$. Thus, $\mathfrak{H}_4$ reduces U, and hence $U = U_0 \oplus U_4$, $H = H_0 \oplus H_4$, where H_4 is a self-adjoint operator in $\mathfrak{H}_4$ with Cayley transform U_4. This proves (1).

(2) We claim first that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda})$. Let $f \in \mathfrak{H}_4$. Since $H_4 \subset \mathfrak{H}_2 = \mathfrak{M}_2(\bar{\lambda}) \oplus \mathfrak{L}_2(\bar{\lambda})$, $f = f' + f''$, where $f' \in \mathfrak{M}_2(\bar{\lambda})$, $f'' \in \mathfrak{L}_2(\bar{\lambda})$. Hence, $Uf = Uf' + Uf'' = Vf' + U_2 f'' = V_{12} f' + V_{22} f' + U_2 f''$, where $Uf \in \mathfrak{H}_4 \subset \mathfrak{H}_2$, $V_{12} f' \in \mathfrak{M}_1(\lambda) \subset \mathfrak{H}_1$, $V_{22} f' \in \mathfrak{M}_2(\lambda) \subset \mathfrak{H}_2$, $U_2 f'' \in \mathfrak{L}_2(\lambda) \subset \mathfrak{H}_2$. Thus, $V_{12} f' = 0$, and therefore $f' = 0$. It follows that $f = f'' \in \mathfrak{L}_2(\bar{\lambda})$ and that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda})$.

Since $\mathfrak{H}_4 \subset \mathfrak{H}_2(\bar{\lambda})$, and since U maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$ and $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$, we conclude that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\lambda)$. Hence, $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda}) \cap \mathfrak{L}_2(\lambda)$. It follows immediately that $\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{H}_3$, $\mathfrak{M}_2(\lambda) \subset \mathfrak{H}_3$. (2) is therefore completely proved.

(3) Because $U_2 = U$ on $\mathfrak{L}_2(\bar{\lambda})$, we see that U_2 maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$. We know, however, that U_2 maps $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$. It follows that $\mathfrak{H}_4$ reduces U_2. Thus, $U_2 = U_3 \oplus U_4$, where U_3 maps $\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4$ isometrically onto $\mathfrak{L}_2(\lambda) \ominus \mathfrak{H}_4$, and $H_2 = H_3 \oplus H_4$, where H_3 is a closed Hermitian operator in $\mathfrak{H}_3$ with Cayley transform U_3. Noting that $\mathfrak{H}_3 = \mathfrak{M}_2(\lambda) \oplus [\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4] = \mathfrak{M}_2(\lambda) \oplus [\mathfrak{L}_2(\lambda) \ominus \mathfrak{H}_4]$, we see that H_3 has deficiency subspaces $\mathfrak{M}_2(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$. This proves (3).

(4) By Theorem 1, H_3 and V define a self-adjoint extension H_0' of H_1 in $\mathfrak{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_3$. If U_0' is the Cayley transform of H_0', then $U_0' = U_1 = U$ on $\mathfrak{L}_1(\bar{\lambda})$, $U_0' = V = U$ on $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$, $U_0' = U_3 = U$ on $\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4$. It follows that $U_0' = U$ on $\mathfrak{H}_1 \oplus \mathfrak{H}_3 = \mathfrak{H}_0$. But since $U_0 \subset U$, $U_0 = U$ on $\mathfrak{H}_0$, hence, $U_0 = U_0'$, and therefore $H_0 = H_0'$. This

proves (4).

(5) As we have shown, $H = H_0 \oplus H_4$. Thus, $E(\lambda) = E_0(\lambda) \oplus E_4(\lambda)$, and therefore $E(\lambda)f = E_0(\lambda)f$ for all $f \in \mathfrak{H}_1$. If P is the operator of orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_1$ and if P_0 is the operator of orthogonal projection of $\mathfrak{H}_0$ onto $\mathfrak{H}_1$, $PE(\lambda)f = PE_0(\lambda)f = P_0E_0(\lambda)f$ for all $f \in \mathfrak{H}_1$, so that H and H_0 define the same spectral function of H_1. This proves (5), and the proof of theorem 4 is completed.

3. Extremal spectral functions of a symmetric operator with equal deficiency indices.

THEOREM 5. *Let H be a self-adjoint extension of the closed symmetric operator H_1. Suppose that H is defined by H_2 and V. Then the following statements are equivalent*:

(1) $\mathfrak{D}(H_2) = \{0\}$.

(2) $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$.

(3) $\mathfrak{D}(H) \cap \mathfrak{H}_2 = \{0\}$.

Proof. That (1) implies (2) is clear from the definition of $\mathfrak{M}_2(\bar{\lambda})$ and $\mathfrak{M}_2(\lambda)$. Suppose, on the other hand, that $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$. Then, $\mathfrak{R}(H_2 - \lambda E) = \mathfrak{R}(H_2 - \bar{\lambda}E) = \{0\}$. If $f \in \mathfrak{D}(H_2)$, $H_2f - \lambda f = 0$ and $H_2f - \bar{\lambda}f = 0$. Subtracting the first equation from the second, $(\lambda - \bar{\lambda})f = 0$, and therefore $f = 0$. Thus, $\mathfrak{D}(H_2) = \{0\}$, and we have proved that (2) implies (1).

By Theorem 1, $\mathfrak{D}(H_2) = \mathfrak{D}(H) \cap \mathfrak{H}_2$, so that (1) and (3) are clearly equivalent.

THEOREM 6. *Let H_1 be a closed symmetric operator. Suppose that H is a self-adjoint extension of H_1 defined by H_2 and V. If $\mathfrak{D}(H_2) = \{0\}$, the following statements are true*:

(1) $m_1 = n_1$, i.e., *the deficiency indices of H_1 are equal.*

(2) *H is minimal.*

(3) *The spectral function $E_1(\lambda)$ of H_1 defined by H is extremal.*

Proof. (1) By Theorem 5, $\mathfrak{D}(H_2) = \{0\}$ implies that $m_2 = n_2$. By theorem 2, (7), $m_1 = n_1$.

(2) By Theorem 5, $\mathfrak{D}(H_2) = \{0\}$ implies that $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$. Hence, $\mathfrak{L}_2(\bar{\lambda}) = \mathfrak{L}_2(\lambda) = \{0\}$. It follows from Theorem 3 and Theorem 4, (2), that H is minimal.

(3) Let A be any bounded operator in $\mathfrak{H}$ having a matrix representation,

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where E is the identity in $\mathfrak{H}_1$, B maps $\mathfrak{H}_2$ into $\mathfrak{H}_1$, C maps $\mathfrak{H}_2$ into $\mathfrak{H}_2$, and C is self-adjoint. Suppose that A commutes with H. We shall show that this implies that $B \equiv 0$. By the proposition of M. A. Naimark [6] mentioned in the introduction, then, it follows that the spectral function $E_1(\lambda)$ defined by H is extremal.

Since A commutes with H, it commutes with the Cayley transform U of H. If we represent U as a matrix, $U \sim (U_{jk})$, where U_{jk} maps $\mathfrak{H}_k$ into $\mathfrak{H}_j$, then the fact that A commutes with U implies that $BU_{21} = U_{12}B^*$. Taking adjoints, we also have that $U_{21}^*B^* = BU_{12}^*$. We observe, further, that $U = V$ on $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ and that $U^* = U^{-1} = V^{-1} = V^*$ on $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$.

Using the equation $BU_{12}^* = U_{21}^*B^*$, the fact that $\mathfrak{M}_2(\lambda) = \mathfrak{H}_2$, and Theorem 2, we obtain that $BV_{12}^*\mathfrak{M}_1^-(\lambda) = BU_{12}^*\mathfrak{M}_1^-(\lambda) = U_{21}^*B^*\mathfrak{M}_1^-(\lambda) \subset U_{21}^*\mathfrak{H}_2 = U_{21}^*\mathfrak{M}_2(\lambda) = V_{21}^*\mathfrak{M}_2(\lambda) \subset \mathfrak{M}_1(\bar{\lambda})$. Since by Theorem 2 $V_{12}^*\mathfrak{M}_1^-(\lambda)$ is dense in $M_2(\bar{\lambda}) = \mathfrak{H}_2$ and since B is bounded, it follows that $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\bar{\lambda})$.

Similarly, using the equation $BU_{21} = U_{12}B^*$, we obtain that $BV_{21}\mathfrak{M}_1^-(\bar{\lambda}) = BU_{21}\mathfrak{M}_1^-(\bar{\lambda}) = U_{12}B^*\mathfrak{M}_1^-(\bar{\lambda}) \subset U_{12}\mathfrak{H}_2 = U_{12}\mathfrak{M}_2(\bar{\lambda}) = V_{12}\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{M}_1(\lambda)$, and therefore $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\lambda)$.

Thus, $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\bar{\lambda}) \cap \mathfrak{M}_1(\lambda)$. But $\mathfrak{M}_1(\bar{\lambda}) \cap \mathfrak{M}_1(\lambda) = \{0\}$, because $\mathfrak{M}_1(\bar{\lambda})$ and $\mathfrak{M}_1(\lambda)$ are the deficiency subspaces of a symmetric operator. Hence, $B \equiv 0$. This complete the proof of Theorem 6.

By use of a somewhat less general form of Theorem 6, M. A. Naimark [6] has shown that every finite-dimensional extension H of a closed symmetric operator H_1 defines an extremal spectral function of H_1.

THEOREM 7. *If H is a finite-dimensional extension of a closed symmetric operator H_1, then H_1 must have equal deficiency indices.*

Proof. Suppose that H is defined by H_2 and V. Then H_2 is a Hermitian operator in the finite-dimensional space $\mathfrak{H}_2$. Since U_2 maps $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$, it follows that $\dim \mathfrak{L}_2(\bar{\lambda}) = \dim \mathfrak{L}_2(\lambda)$. Hence $\dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_2(\lambda)$, i.e., $m_2 = n_2$. By Theorem 2, (7), $m_1 = n_1$. This proves Theorem 7.

4. Extremal spectral functions of a symmetric operator with unequal deficiency indices. We first introduce the notion of a partial *isometry* and some of the properties thereof. (See Murray and von Neumann [3].) A bounded linear operator W in a Hilbert space $\mathfrak{H}$ is called a *partial isometry* if it maps a subspace $\mathfrak{E}$ isometrically onto another subspace $\mathfrak{F}$, while it maps $\mathfrak{H} \ominus \mathfrak{E}$ onto $\{0\}$. $\mathfrak{E}$ is called the *initial set* of W, and $\mathfrak{F}$ is called the *final set* of W. If W is a partial isometry, then the following statements hold:

(1) If $P(\mathfrak{E})$ is the operator of orthogonal projection on $\mathfrak{E}$ and if $P(\mathfrak{F})$ is the operator of orthogonal projection on $\mathfrak{F}$, then $P(\mathfrak{E}) = W^*W$;

$P(\mathfrak{F}) = WW^*$.

(2) U^* is a partial isometry with initial set $\mathfrak{F}$ and final set $\mathfrak{E}$.

(3) As a mapping of $\mathfrak{F}$ onto $\mathfrak{E}$, U^* is the inverse of U as a mapping of $\mathfrak{E}$ onto $\mathfrak{F}$.

THEOREM 8. *Suppose that* W *is a partial isometry with initial set* $\mathfrak{M}$ *and final set* $\mathfrak{H}$. *Let* $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{M}$. *Then,* $\mathfrak{M} = \mathfrak{M}' \oplus \mathfrak{M}''$, *where*

(1) W *maps* $\mathfrak{M}''$ *isometrically onto* $\mathfrak{M}''$;

(2) *if* $f \in \mathfrak{N} \oplus \mathfrak{M}'$, $\lim_{p\to\infty} W^p f = 0$.

Proof. Let $\mathfrak{M}_i = (W^*)^i \mathfrak{N}$, $i = 0, 1, 2, \cdots$. Then each $\mathfrak{M}_i$ is a subspace (i.e., a closed linear manifold), and the following statements are true:

(a) $\mathfrak{M}_i \subset \mathfrak{M}$ for $i = 1, 2, \cdots$. This is clear because W^* is a partial isometry with initial set $\mathfrak{H}$ and final set $\mathfrak{M}$.

(b) If $f \in \mathfrak{M}_n$, where $n \geqq 0$, then $W^p f \in \mathfrak{M}_{n-p}$ for $1 \leqq p \leqq n$, and $W^p f = 0$ for $p > n$. Proof: If $f \in \mathfrak{M}_n$, then $f = (W^*)^n g$ for some $g \in \mathfrak{N}$. Since $WW^* = E$, $W^p f = (W^*)^{n-p} g \in \mathfrak{M}_{n-p}$, $1 \leqq p \leqq n$. If $p > n$, $W^p f = W^{p-n} g = 0$.

(c) If $f \in \mathfrak{M}_i$, $i = 0, 1, 2, \cdots$, and if n is a positive integer, then $(W^*)^n f \in \mathfrak{M}_{i+n}$. Proof: If $f \in \mathfrak{M}_i$, $f = (W^*)^i g$, where $g \in \mathfrak{N}$. Therefore, $(W^*)^n f = (W^*)^{i+n} g \in \mathfrak{M}_{i+n}$.

(d) $\mathfrak{M}_i$ is perpendicular to $\mathfrak{M}_j$ if $i \neq j$. Proof: Suppose $i < j$, and let $f \in \mathfrak{M}_i$, $g \in \mathfrak{M}_j$. Then there exists $f_1 \in \mathfrak{N}$ and $g_1 \in \mathfrak{N}$ such that $f = (W^*)^i f_1$, $g = (W^*)^j g_1$. Hence, $(f, g) = ((W^*)^i f_1, (W^*)^j g_1) = (f_1, (W^*)^{j-i} g_1) = 0$, since $f_1 \in \mathfrak{N}$, $(W^*)^{j-i} g_1 \in \mathfrak{M}_{j-i} \subset \mathfrak{M}$.

Now let $\mathfrak{M}' = \sum_{i=1}^{\infty} \mathfrak{M}_i$. Then $\mathfrak{M}'$ is a subspace of $\mathfrak{M}$. Let $\mathfrak{M}'' = \mathfrak{M} \ominus \mathfrak{M}'$. We shall show that $\mathfrak{M}'$ and $\mathfrak{M}''$ satisfy (1) and (2).

Since $\mathfrak{M} = \mathfrak{M}' \oplus \mathfrak{M}''$ and $\mathfrak{H} = \mathfrak{N} \oplus \mathfrak{M}' \oplus \mathfrak{M}''$, and since W maps $\mathfrak{M}$ isometrically onto $\mathfrak{H}$, in order to prove (1) it is sufficient to show that W maps $\mathfrak{M}'$ onto $\mathfrak{N} \oplus \mathfrak{M}'$. Suppose $f \in \mathfrak{M}'$. Then, $f = \sum_{i=1}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$, and $Wf = \sum_{i=1}^{\infty} Wf_i$. Because by (b) $Wf_i \in \mathfrak{M}_{i-1}$, we see that $Wf \in \mathfrak{N} \oplus \mathfrak{M}'$. Thus, W maps $\mathfrak{M}'$ into $\mathfrak{N} \oplus \mathfrak{M}'$. To show that the map is onto, let $g \in \mathfrak{N} \oplus \mathfrak{M}'$. Then, $g = \sum_{i=0}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$. If $f = W^* f = \sum_{i=0}^{\infty} W^* f_i \in \mathfrak{M}'$, by (c). Further, $Wf = WW^* g = g$. Hence, W maps $\mathfrak{M}'$ onto $\mathfrak{N} \oplus \mathfrak{M}'$.

We now prove (2). Let $f \in \mathfrak{N} \oplus \mathfrak{M}'$. Then, $f = \sum_{i=0}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$. By (b), $W^p f = \sum_{i=0}^{\infty} W^p f_i = \sum_{i=p}^{\infty} W^p f_i$. Hence, $\| W^p f \|^2 = \sum_{i=p}^{\infty} \| W^p f_i \|^2 = \sum_{i=p}^{\infty} \| f_i \|^2$. Thus, $\lim_{p\to\infty} \| W^p f \|^2 = 0$. This proves (2) and completes the proof of the theorem.

THEOREM 9. *Let* λ *be a fixed nonreal number. Suppose that* H_1 *is a closed symmetric operator in* $\mathfrak{H}_1$ *with deficiency indices* (m, n)

(with respect to λ*), and suppose that* $m \neq n$*. Let* H *be a self-adjoint extension of* H_1 *defined by* H_2 *and* V*, where* H_2 *is a closed Hermitian operator with deficiency indices* $(0, s)$*,* $n + s = m$*, if* $m > n$ *and* $(s, 0)$*,* $m + s = n$*, if* $m < n$*. Then the spectral function defined by* H *is extremal.*

Proof. Assume that $m > n$. The case $m < n$ then follows by interchanging the roles of $\bar{\lambda}$ and λ in Theorem 1 and defining H by H_2 and V^*.

By Theorem 3 there exists a minimal self-adjoint extension H_0 of H_1 such that $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$, $H_1 \subset H_0 \subset H$, and H_0 and H define the same spectral function of H_1. By Theorem 4, H_0 is defined by V and a Hermitian operator H_3 with the same deficiency subspaces as H_2. Since we can always consider H_0 instead of H, it follows that without loss of generality we can consider H to be a minimal self-adjoint extension.

Since $\mathfrak{M}_2(\bar{\lambda}) = \{0\}$ and $\mathfrak{L}_2(\bar{\lambda}) = \mathfrak{H}_2$, we have that if $f \in \mathfrak{H}_2$, $Uf \in \mathfrak{L}_2(\lambda) \subset \mathfrak{H}_2$. If we represent U as a matrix, $U \sim (U_{jk})$, where U_{jk} maps $\mathfrak{H}_k$ into $\mathfrak{H}_j$, then it follows that $U_{12} \equiv 0$ on $\mathfrak{H}_2$. Further, $Uf = U_{22}f$ for all $f \in \mathfrak{H}_2$, so that U_{22} maps $\mathfrak{H}_2$ isometrically onto $\mathfrak{L}_2(\lambda)$. U_{22} is thus a partial isometry in $\mathfrak{H}_2$ with initial set $\mathfrak{H}_2$ and final set $\mathfrak{L}_2(\lambda)$, while U_{22}^* is a partial isometry with initial set $\mathfrak{L}_2(\lambda)$ and final set $\mathfrak{H}_2$. We have that $E = P(\mathfrak{H}_2) = U_{22}^* U_{22}$, while $P(\mathfrak{L}_2(\lambda)) = U_{22}U_{22}^*$.

Now let A be any bounded operator in $\mathfrak{H}$ with matrix representation

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where E is the identity in $\mathfrak{H}_1$, B maps $\mathfrak{H}_2$ into $\mathfrak{H}_1$, C maps $\mathfrak{H}_2$ into $\mathfrak{H}_2$, and C is self-adjoint. Suppose that A commutes with H. We shall show that this implies $B \equiv 0$. Then by the proposition of M. A. Naimark [6] mentioned in the introduction, it follows that the spectral function $E_1(\lambda)$ defined by H is extremal.

Since A commutes with H, it commutes with the Cayley transform U of H. This implies that $BU_{21} = U_{12}B^*$ and $U_{12} + BU_{22} = U_{11}B + U_{12}C$. Since $U_{12} \equiv 0$, these equations become $BU_{21} \equiv 0$ and $BU_{22} = U_{11}B$. On $\mathfrak{M}_1(\bar{\lambda})$, $U_{21} = V_{21}$ and therefore $BV_{21}\mathfrak{M}_1(\bar{\lambda}) = BU_{21}\mathfrak{M}_1(\bar{\lambda}) = \{0\}$. Becaese by Theorem 2, $V_{21}\mathfrak{M}_1(\bar{\lambda})$ is dense in $\mathfrak{M}_2(\lambda)$, $B\mathfrak{M}_2(\lambda) = \{0\}$, i.e., $BP(\mathfrak{M}_2(\lambda)) = 0$. From the equation $BU_{22} = U_{11}B$ we have that $BP(\mathfrak{L}_2(\lambda)) = BU_{22}U_{22}^* = U_{11}BU_{22}^*$. Adding $BP(\mathfrak{L}_2(\lambda)) = U_{11}BU_{22}^*$ with $BP(\mathfrak{M}_2(\lambda)) = 0$, we obtain that $B = U_{11}BU_{22}^*$. By iterating this equation we see that $B = U_{11}^pB(U_{22}^*)^p$ for every positive integer p. Since $\| U_{11} \| \leqq 1$, $\| Bf \| \leqq \| B \| \; \| (U_{22}^*)^p f \|$ for each $f \in \mathfrak{H}_2$ and each positive integer p.

By Theorem 8, $\mathfrak{L}_2(\lambda) = \mathfrak{M}' \oplus \mathfrak{M}''$, where U_{22}^* maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$, and if $f \in \mathfrak{M}_2(\lambda) \oplus \mathfrak{M}'$, then $\lim_{p\to\infty} \|(U_{22}^*)^p f\| = 0$. But if U_{22}^* maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$, then U_{22} and therefore U maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$. This means that U and therefore H is reduced by $\mathfrak{M}''$, a subspace of $\mathfrak{H}_2$. Since H is a minimal self-adjoint extension of H_1, $\mathfrak{M}'' = \{0\}$. Hence, $\mathfrak{H}_2 = \mathfrak{M}_2(\lambda) \oplus \mathfrak{M}'$, and thus if $f \in \mathfrak{H}_2$, $\lim_{p\to\infty} \|(U_{22}^*)^p f\| = 0$. Since $\|Bf\| \leqq \|B\| \; \|(U_{22}^*)^p f\|$ for each $f \in \mathfrak{H}_2$ and for every positive integer p, it follows that $B \equiv 0$ on $\mathfrak{H}_2$. This completes the proof of Theorem 9.

Since the operator H_2 in Theorem 9 is a *Hermitian* operator with deficiency indices $(0, s)$ or $(s, 0)$, it may seem that we are dealing with a wider class of operators than the maximal symmetric operators. That this is not so is shown by Theorem 10 below.

THEOREM 10. *If H is a Hermitian operator with deficiency indices* $(0, s)$ *or* $(s, 0)$, *then H is a maximal symmetric operator. If H is a Hermitian operator with deficiency indices* $(0, 0)$, *then H is a self-adjoint operator.*

Proof. If H is a Hermitian operator and $\mathfrak{B} = \mathfrak{H} \ominus [\mathfrak{D}(H)]^c$, then $\mathfrak{B} \cap \mathfrak{L}(\bar{\lambda}) = \{0\}$. (If $h \in \mathfrak{B} \cap \mathfrak{L}(\bar{\lambda})$, then $h = (H - \lambda E)g$, Hence, $0 = (h, g) = (Hg, g) - \lambda(g, g)$. Since (Hg, g) is real while λ is not, $g = 0$. This simple argument is due to M. A. Krasnosel'skii [2, Lemma 2].) If H has deficiency indices $(0, s)$, $\mathfrak{M}(\bar{\lambda}) = \{0\}$ so that $\mathfrak{B} \subset \mathfrak{L}(\bar{\lambda})$. Thus, $\mathfrak{B} = \{0\}$ and H is symmetric. Similarly, H is symmetric if its deficiency indices are $(s, 0)$. It follows immediately that if H has deficiency indices $(0, 0)$, H is self-adjoint. Theorem 10 is proved.

REFERENCES

1. N. I. Achieser and I. M. Glasmann, *Theorie der linearen operatoren im Hilbert-Raum*, Akademie-Verlag, Berlin, 1954.

2. M. A. Krasnosel'skii, *On self-adjoint extensions of Hermitian operators*, Ukrainskii Mat. Zhurnal, No. **1** (1949), 21-38.

3. F. J. Murray and J. v. Neumann, *On rings of operators*, Annals of Math., **37** (1936), 116-229.

4. Bela v. Sz. Nagy, *Spektraldarstellung linearer transformationen des Hilbertschen Raumes*, Springer Verlag, Berlin, 1942.

5. M. A. Naimark, *Spectral functions of a symmetric operator*, Izvest. Akad. Nauk SSSR, Ser. Mat., **4** (1940), 277-318.

6. ———, *Extremal spectral functions of a symmetric operator*, Izvest. Akad. Nauk SSSR, Ser. Mat., **11** (1942), 327-344.

MATHEMATICS RESEARCH CENTER, U.S. ARMY
UNIVERSITY OF WISCONSIN, AND
UNIVERSITY OF CALIFORNIA, RIVERSIDE

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 14, No. 1 May, 1964