ON FINITE SUMS OF RECIPROCALS OF DISTINCT \( n \)TH POWERS

ROBERT LEWIS GRAHAM
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Introduction. It has long been known that every positive rational number can be represented as a finite sum of reciprocals of distinct positive integers (the first proof having been given by Leonardo Pisano [6] in 1202). It is the purpose of this paper to characterize (cf. Theorem 4) those rational numbers which can be written as finite sums of reciprocals of distinct n-th powers of integers, where n is an arbitrary (fixed) positive integer and "finite sum" denotes a sum with a finite number of summands. It will follow, for example, that \( p/q \) is the finite sum of reciprocals of distinct squares if and only if

\[
\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left(1, \frac{\pi^2}{6}\right).
\]

Our starting point will be the following result:

THEOREM A. Let n be a positive integer and let \( H_n \) denote the sequence \((1^{-n}, 2^{-n}, 3^{-n}, \ldots)\). Then the rational number \( p/q \) is the finite sum of distinct terms taken from \( H_n \) if and only if for all \( \varepsilon > 0 \), there is a finite sum \( s \) of distinct terms taken from \( H_n \) such that \( 0 \leq s - p/q < \varepsilon \).

Theorem A is an immediate consequence of a result of the author [2, Theorem 4] together with the fact that every sufficiently large integer is the sum of distinct n-th powers of positive integers (cf., [8], [7] or [3]).

The main results. We begin with several definitions. Let \( S = (s_1, s_2, \ldots) \) denote a (possibly finite) sequence of real numbers.

DEFINITION 1. \( P(S) \) is defined to be the set of all sums of the form \( \sum_{k=1}^{\infty} \varepsilon_k s_k \) where \( \varepsilon_k = 0 \) or 1 and all but a finite number of the \( \varepsilon_k \) are 0.

DEFINITION 2. \( Ac(S) \) is defined to be the set of all real numbers \( x \) such that for all \( \varepsilon > 0 \), there is an \( s \in P(S) \) such that \( 0 \leq s - x < \varepsilon \).

Note that in this terminology Theorem A becomes:

\[
(1) \quad P(H^n) = Ac(H^n) \cap Q
\]

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1 This result has also been obtained by P. Erdős (not published).
where $Q$ denotes the set of rational numbers.

**DEFINITION 3.** A term $s_n$ of $S$ is said to be smoothly replaceable in $S$ (abbreviated s.r. in $S$) if $s_n \leq \sum_{k=1}^{\infty} s_{n+k}$.

**THEOREM 1.** Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers such that:
1. $s_n \downarrow 0$.
2. There exists an integer $r$ such that $n \geq r$ implies that $s_n$ is smoothly replaceable in $S$.

Then

$$Ac(S) = \bigcup_{x \in P_{r-1}} [\pi, \pi + \sigma]$$

where $P_{r-1} = P((s_1, \cdots, s_{r-1}))$ (note that $P_0 = \{0\}$) and $\sigma = \sum_{k=1}^{\infty} s_k$ (where possibly $\sigma$ is infinite).

**Proof.** Let $x \in \bigcup_{x \in P_{r-1}} [\pi, \pi + \sigma]$ and assume that $x \notin Ac(S)$. Then $x \in [\pi, \pi + \sigma]$ for some $\pi \in P_{r-1}$. A sum of the form $\pi + \sum_{i=1}^{k} s_{i_1}$ where $r \leq i_1 < i_2 < \cdots < i_k$ will be called "minimal" if

$$\pi + \sum_{i=1}^{k-1} s_{i_1} < x < \pi + \sum_{i=1}^{k} s_{i_1}$$

(where a sum of the form $\sum_{i=a}^{b}$ is taken to be 0 for $b < a$). Note that since $x \notin Ac(S)$ then we never get equality in (2). Let $M$ denote the set of minimal sums. Then $M$ must contain infinitely many elements. For suppose $M$ is a finite set. Let $m$ denote the largest index of any $s_j$ which is used in any element of $M$ and let $p = \pi + \sum_{i=1}^{r} s_{j_1} + s_m$ be an element of $M$ which uses $s_m$ (where $r \leq j_1 < j_2 < \cdots < j_n < m$ and possibly $n$ is zero). Thus we have

$$\pi + \sum_{k=1}^{n} s_{j_k} < x < \pi + \sum_{k=1}^{n} s_{j_k} + \sum_{t=1}^{\infty} s_{m+t}$$

since $s_m$ is s.r. in $S$. Therefore there is a least $d \geq 1$ such that $x < p' = \pi + \sum_{k=1}^{n} s_{j_k} + \sum_{i=1}^{d} s_{m+t}$. Hence $p'$ is a "minimal" sum which uses $s_{m+d}$ and $m + d > m$. This is a contradiction to the definition of $m$ and consequently $M$ must be infinite. Now, let $\delta = \inf\{p - x : p \in M\}$. Since $x \notin Ac(S)$ then $\delta > 0$. There exist $p_n, p_{n+1}, \cdots \in M$ such that $p_n - x < \delta + \delta/2^n$. Since $s_n \downarrow 0$ then there exists $c$ such that $n \geq c$ implies that $s_n < \delta/2$. Also, there exists $w$ such that $n \geq w$ implies that $p_n$ uses an $s_k$ for some $k \geq c$ (since only a finite number of $p_i$ can be formed from the $s_k$ with $k < c$). Therefore we can write $p_w = \pi + \sum_{j=1}^{n} s_{k_j}$ where $k_n \geq c$. Hence
which is a contradiction to the assumption that \( p_w \) is "minimal." Thus, we must have \( x \in Ac(S) \) and consequently

\[
\bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \subset Ac(S).
\]

To show inclusion in the other direction let \( x \in Ac(S) \) and suppose that \( x \in \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \). Thus, either \( x < 0 \), \( x \geq \sum_{k=1}^{n} s_k \), or there exist \( \pi \) and \( \pi' \) in \( P_{r-1} \) such that \( \pi + \sigma \leq x < \pi' \) where no element of \( P_{r-1} \) is contained in the interval \([\pi + \sigma, \pi']\). Since the first two possibilities imply that \( x \notin Ac(S) \) (contradicting the hypothesis) then we may assume that the third possibility holds. Therefore there exists \( \delta > 0 \) such that

\[
x \leq \pi' - \delta.
\]

Let \( p \) be any element of \( P(S) \). Then \( p = \sum_{i=1}^{m} s_{i_1} + \sum_{k=1}^{n} s_{j_k} \) for some \( m \) and \( n \) where

\[
1 \leq i_1 < i_2 < \cdots < i_m \leq r - 1 < j_1 < j_2 < \cdots < j_n.
\]

Thus for \( \pi^* = \sum_{i=1}^{m} s_{i_1} \) we have \( p \in [\pi^*, \pi^* + \sigma) \). Consequently any element \( p \) of \( P(S) \) must fall into an interval \([\pi^*, \pi^* + \sigma) \) for some \( \pi^* \in P_{r-1} \) and therefore, if \( p \) exceeds \( x \) then it must exceed \( x \) by at least \( \delta \) (since \( p \in [\pi + \sigma, \pi'] \) and thus by (4) \( p > x \in [\pi + \sigma, \pi'] \) implies \( p \geq \pi' \geq x + \delta \)). This contradicts the hypothesis that \( x \in Ac(S) \) and hence we conclude that \( Ac(S) \supset \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \). Thus, by (3) we have \( Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \) and the theorem is proved.

**Theorem 2.** Let \( S = (s_1, s_2, \cdots) \) be a sequence of real numbers such that:

1. \( s_n \downarrow 0 \).
2. There exists an integer \( r \) such that \( n < r \) implies that \( s_n \) is not s.r. in \( S \) while \( n \geq r \) implies that \( s_n \) is s.r. in \( S \).

Then \( Ac(S) \) is the disjoint union of exactly \( 2^{r-1} \) half-open intervals each of length \( \sum_{k=r}^{\infty} s_k \).

**Proof.** By Theorem 1 we have \( Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \) where \( \sigma = \sum_{k=r}^{\infty} s_k \) and \( P_{r-1} = P(\{s_1, \cdots, s_{r-1}\}) \). Let \( \pi = \sum_{k=1}^{n} s_{i_k} \) and \( \pi' = \sum_{k=1}^{m} s_{j_k} \) be any two formally distinct sums of the \( s_n \) where \( 1 \leq i_1 < \cdots < i_m \leq r - 1 \) and \( 1 \leq j_1 < \cdots < j_n \leq r - 1 \) and we can assume without loss of generality that \( \pi \geq \pi' \). Then either there is a least \( m \geq 1 \) such that \( i_m \neq j_m \) or we have \( i_k = j_k \) for \( k = 1, 2, \cdots, v \) and
\( u > v \). In the first case we have

\[
\pi = \sum_{k=1}^{n} s_{ik} = \sum_{k=1}^{m-1} s_{jk} + \sum_{k=m}^{n} s_{ik} \\
> \sum_{k=1}^{m-1} s_{jk} + \sum_{k=1}^{\infty} s_{i_{m+k}} \text{ (since } s_{i_{m}} \text{ is not s.r. in } S) \\
\geq \pi' + \sigma \text{ (since } j_m \geq i_m + 1). 
\]

In the second case we have

\[
\pi = \sum_{k=1}^{n} s_{ik} = \sum_{k=1}^{n} s_{jk} + \sum_{k=1}^{n} s_{ik} \\
> \sum_{k=1}^{n} s_{jk} + \sum_{k=1}^{\infty} s_{i_{k+1+k}} \text{ (since } s_{i_{k+1}} \text{ is not s.r. in } S) \\
\geq \pi' + \sigma \text{ (since } i_{k+1} + 1 \leq i_n + 1 \leq r). 
\]

Thus, in either case we see that \( \pi > \pi' + \sigma \). Consequently, any two formally distinct sums in \( P_{r-1} \) are separated by a distance of more than \( \sigma \) and hence, each element \( \pi \) of \( P_{r-1} \) gives rise to a half-open interval \([\pi, \pi + \sigma)\) which is disjoint from any other interval \([\pi', \pi' + \sigma)\) for \( \pi \neq \pi' \in P_{r-1} \). Therefore \( Ac(S) = \bigcup_{x \in P_{r-1}} [\pi, \pi + \sigma) \) is the disjoint union of exactly \( 2^{r-1} \) half-open intervals \([\pi, \pi + \sigma), \pi \in P_{r-1} \), (since there are exactly \( 2^{r-1} \) formally distinct sums of the form \( \sum_{k=1}^{r} s_k \cdot \varepsilon_k, \varepsilon_k = 0 \) or \( 1 \)) where each interval is of length \( \sigma \). This proves the theorem.

We now need three additional lemmas in order to prove the main theorems.

**Lemma 1.** Let \( S = (s_1, s_2, \ldots) \) be a sequence of nonnegative real numbers and suppose that there exists an \( m \) such that \( n \geq m \) implies that \( s_n \leq 2s_{n+1} \). Then \( n \geq m \) implies that \( s_n \) is s.r. in \( S \) (i.e., \( s_n \leq \sum_{k=1}^{n} s_{n+k} \)).

**Proof.** If \( \sum_{k=1}^{\infty} s_k = \infty \) then the lemma is immediate. Assume that \( \sum_{k=1}^{\infty} s_k < \infty \). Then

\[
n \geq m \implies s_{n+k} \geq \frac{1}{2} s_{n+k-1}, \text{ \hspace{1cm} } k = 1, 2, 3, \ldots
\]

\[
\implies \sum_{k=1}^{n} s_{n+k} \geq \frac{1}{2} \sum_{k=1}^{n} s_{n+k-1} = \frac{1}{2} s_n + \frac{1}{2} \sum_{k=1}^{n} s_{n+k}. 
\]

Therefore, \( s_n \leq \sum_{k=1}^{\infty} s_{n+k} \), i.e., \( s_n \) is s.r. in \( S \).

**Lemma 2.** Suppose that \( k \leq (2^{1/n} - 1)^{-1} \) and \( k^{-n} \) is s.r. in \( H^n \) (where \( H^n \) was defined to be the sequence \( (1^{-n}, 2^{-n}, \ldots) \)). Then \( (k + 1)^{-n} \) is also s.r. in \( H^n \).
Proof.

\[ k \leq (2^{1/n} - 1)^{-1} \implies \frac{1}{k} \leq 2^{1/n} - 1 \]

(5)

\[ \implies (1 + \frac{1}{k})^n \geq 2 \]

\[ \implies k^{-n} \geq 2(k + 1)^{-n} \]

Since by hypothesis, \( \sum_{j=k+1}^{\infty} j^{-n} \geq k^{-n} \), then by (5)

\[ \sum_{j=k+1}^{\infty} j^{-n} \geq (k + 1)^{-n} \geq 2(k + 1)^{-n} \]

Hence, \( (k + 1)^{-n} \) is s.r. in \( H^n \) and the lemma is proved.

**LEMMA 3.** Suppose that \( k \geq (2^{1/n} - 1)^{-1} \). Then \( k^{-n} \) is s.r. in \( H^n \).

*Proof.*

\[ r \geq k \implies r \geq (2^{1/n} - 1)^{-1} \]

\[ \implies \frac{1}{r} \leq 2^{1/n} - 1 \]

\[ \implies (1 + \frac{1}{r})^n \leq 2 \]

\[ \implies r^{-n} \leq 2(r + 1)^{-n} \]

Therefore, by Lemma 1, \( r^{-n} \) is s.r. in \( H^n \).

**THEOREM 3.** Let \( t_n \) denote the largest integer \( k \) such that \( k^{-n} \) is not s.r. in \( H^n \) and let \( P \) denote \( P((1^{-n}, 2^{-n}, \ldots, k^{-n})) \). Then

\[ Ac(H^n) = \bigcup_{x \in P} [\pi, \pi + \sum_{k=1}^{\infty} (t_n + k)^{-n}] \]

is the disjoint union of exactly \( 2^n \) intervals. Moreover, \( t_n < (2^{1/n} - 1)^{-1} \) and \( t_n \sim n/\ln 2 \) (where \( \ln 2 \) denotes \( \log_2 2 \)).

*Proof.* With the exception of \( t_n \sim n/\ln 2 \), the theorem follows directly from the preceding results. The following argument, due to L. Shepp, shows that \( t_n \sim n/\ln 2 \).

Consider the function \( f_n(x) \) defined by

(6)

\[ f_n(x) = x^n \left( \sum_{k=1}^{\infty} \frac{1}{(x + k)^n} - \frac{1}{x^n} \right) \]

for \( n = 2, 3, \ldots \) and \( x > 0 \). Since

\[ f_n(x) = \sum_{k=1}^{\infty} \left( 1 + \frac{k}{x} \right)^{-n} - 1 \]

then \( f_n(x) < 0 \) for sufficiently small \( x > 0 \), \( f_n(x) > 0 \) for sufficiently
large $x$, and $f_n(x)$ is continuous and monotone increasing for $x > 0$. Hence the equation $f_n(x) = 0$ has a unique positive root $x_n$ and from the definition of $t_n$ it follows by (6) that $0 < x_n - t_n \leq 1$. Thus, to show that $t_n \sim n/\ln 2$, it suffices to show that $x_n \sim n/\ln 2$. Now it is easily shown (cf., [4], p. 18) that for $a > 0$, $(1 + \alpha/n)^{-n}$ is a decreasing function of $n$. Thus, $f_n(\alpha n)$ is a decreasing function of $n$ and since $f_\lambda(2\alpha) < \infty$ for $\alpha > 0$ then

$$
\lim_{n \to \infty} f_n(\alpha n) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \left( 1 + \frac{k}{\alpha n} \right)^{-n} - 1
= \sum_{k=1}^{\infty} \lim_{n \to \infty} \left( 1 + \frac{k}{\alpha n} \right)^{-n} - 1
= -1 + \sum_{k=1}^{\infty} e^{-k/\alpha} = (e^{1/\alpha} - 1)^{-1} - 1
$$

since the monotone convergence theorem (cf., [5]) allows us to interchange the sum and limit. Suppose now that for some $\varepsilon > 0$, there exist $n_1 < n_2 < \cdots$ such that $x_{n_k} > n_1(1/\ln 2 + \varepsilon)$. Then

$$
0 = \lim_{k \to \infty} f_{n_k}(x_{n_k}) \geq \lim_{k \to \infty} f_{n_k}(n_k \left( \frac{1}{\ln 2} + \varepsilon \right))
= (e^{(1/\ln 2) + \varepsilon} - 1) - 1
= (2^{(1/\ln 2) + \varepsilon} - 1) - 1 > 0
$$

which is a contradiction. Similarly, if for some $\varepsilon$, $0 < \varepsilon < 1/\ln 2$, there exist $n_1 < n_2 < \cdots$ such that

$$
x_{n_k} < n_k \left( \frac{1}{\ln 2} - \varepsilon \right),
$$

then

$$
0 = \lim_{k \to \infty} f_{n_k}(x_{n_k}) \leq \lim_{k \to \infty} f_{n_k}(n_k \left( \frac{1}{\ln 2} - \varepsilon \right))
= (e^{(1/\ln 2) - \varepsilon} - 1) - 1
= (2^{(1/\ln 2) - \varepsilon} - 1) - 1 < 0
$$

which is again impossible. Hence we have shown that for all $\varepsilon > 0$, there exists an $n_0$ such that $n > n_0$ implies that

$$
n \left( \frac{1}{\ln 2} - \varepsilon \right) \leq x_n \leq n \left( \frac{1}{\ln 2} + \varepsilon \right)
$$

or equivalently

$$
-\varepsilon \leq \frac{x_n}{n} - \frac{1}{\ln 2} \leq \varepsilon.
$$
Therefore, \( \lim_{n \to \infty} x_n/n = 1/\ln 2 \) and the theorem is proved.\(^2\)

The following table gives the values of \( t_n \) for some small values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n )</th>
<th>( \left( 2^{k/n} - 1 \right)^{-1} )</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
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</tr>
<tr>
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<td>?</td>
<td>143</td>
</tr>
<tr>
<td>1000</td>
<td>?</td>
<td>1442</td>
</tr>
</tbody>
</table>

We may now combine Theorem 3 and Theorem A (cf. Eq. (1)) and express the result in ordinary terminology to give:

**Theorem 4.** Let \( n \) be a positive integer, let \( t_n \) be the largest integer \( k \) such that \( k^{-n} > \sum_{j=1}^{t_n} (k + j)^{-n} \) and let \( P \) denote the set \( \{ \sum_{j=1}^{t_n} \varepsilon_j j^{-n} : \varepsilon_j = 0 \text{ or } 1 \} \). Then the rational number \( p/q \) can be written as a finite sum of reciprocals of distinct \( n \)th powers of integers if and only if

\[
\frac{p}{q} \in \bigcup_{t \in P} \left[ \pi, \pi + \sum_{k=1}^{t} (t_n + k)^{-n} \right].
\]

**Corollary 1.** \( p/q \) can expressed as the finite sum of reciprocals of distinct squares if and only if

\[
\frac{p}{q} \in \left[ 0, \frac{\pi^2}{6} - 1 \right] \cup \left[ 1, \frac{\pi^2}{6} \right] .
\]

**Corollary 2.** \( p/q \) can be expressed as the finite sum of reciprocals of distinct cubes if and only if

\[
\frac{p}{q} \in \left[ 0, \zeta(3) - \frac{9}{8} \right] \cup \left[ \frac{1}{8}, \zeta(3) - 1 \right] \cup \left[ 1, \zeta(3) - \frac{1}{8} \right] \cup \left[ \frac{9}{8}, \zeta(3) \right]
\]

where \( \zeta(3) = \sum_{k=1}^{\infty} k^{-3} = 1.2020569\ldots \)

**Remarks.** In theory it should be possible to calculate directly from the relevant theorems (cf., [2], [3]) an explicit bound for the number of terms of \( H^n \) needed to represent \( p/q \) as an element of \( P(H^n) \). However, since the theorems were not designed to minimize such a bound, but rather merely to show its existence, then understandably, this calculated bound would probably be many orders of

\(^2\) In fact, it can be shown that \( x_n \) has the expansion \( n/\ln 2 - 1/2 + c_1 n^{-1} + \cdots + c_k n^{-k} + O(n^{-k-1}) \) for any \( k \).
magnitude too large. Erdős and Stein [1] and, independently, van Albada and van Lint [9] have shown that if \( f(n) \) denotes the least number of terms of \( H^1 = (1^{-1}, 2^{-1}, \cdots) \) needed to represent the integer \( n \) as an element of \( P(H^1) \) then \( f(n) \sim e^{\gamma n} \) where \( \gamma \) is Euler’s constant.

It should be pointed out that a more general form of Theorem A may be derived from [2] which can be used to prove results of the following type:

**Corollary A.** The rational \( p/q \) with \( (p, q) = 1 \) can be expressed as a finite sum of reciprocals of distinct odd squares if and only if \( q \) is odd and \( p/q \in [0, (\pi^2/8) - 1) \cup [1, \pi^2/8) \).

**Corollary B.** The rational \( p/q \) with \( (p, q) = 1 \) can be expressed as a finite sum of reciprocals of distinct squares which are congruent to 4 modulo 5 if and only if \( (q, 5) = 1 \) and

\[
\frac{p}{q} \in \left[0, \alpha - \frac{13}{36}\right) \cup \left[\frac{1}{9}, \alpha - \frac{1}{4}\right) \cup \left[\frac{1}{4}, \alpha - \frac{1}{9}\right) \cup \left[\frac{13}{36}, \alpha\right)
\]

where

\[
\alpha = 2(5 - \sqrt{5})\pi^2/125 = \sum_{k=0}^\infty ((5k + 2)^{-2} + (5k + 3)^{-2}) = 0.43648\cdots
\]

It is not difficult to obtain representations of specific rationals as elements of \( P(H^n) \) (for small \( n \), e.g.,

\[
\frac{1}{2} = 2^{-3} + 3^{-3} + 4^{-3} + 5^{-3} + 6^{-3} + 15^{-3} + 18^{-3} + 36^{-3} + 60^{-3} + 180^{-3},
\]

\[
\frac{1}{3} = 2^{-3} + 4^{-3} + 10^{-3} + 12^{-3} + 20^{-3} + 30^{-3} + 60^{-3},
\]

\[
\frac{5}{87} = 2^{-3} + 5^{-3} + 10^{-3} + 15^{-3} + 16^{-3} + 74^{-3} + 111^{-3} + 185^{-3} + 240^{-3}
\]

\[
+ 296^{-3} + 444^{-3} + 1480^{-3}, \text{ etc.}!
\]

**References**


**Bell Telephone Laboratories, Inc.**
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