ON FINITE SUMS OF RECIPROCALS OF DISTINCT $n$TH POWERS

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DISTINCT nth POWERS

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Introduction. It has long been known that every positive rational number can be represented as a finite sum of reciprocals of distinct positive integers (the first proof having been given by Leonardo Pisano [6] in 1202). It is the purpose of this paper to characterize (cf. Theorem 4) those rational numbers which can be written as finite sums of reciprocals of distinct nth powers of integers, where n is an arbitrary (fixed) positive integer and "finite sum" denotes a sum with a finite number of summands. It will follow, for example, that \( \frac{p}{q} \) is the finite sum of reciprocals of distinct squares if and only if

\[
\frac{p}{q} \in \left[ 0, \frac{\pi^2}{6} - 1 \right) \cup \left[ 1, \frac{\pi^2}{6} \right).
\]

Our starting point will be the following result:

Theorem A. Let \( n \) be a positive integer and let \( H^n \) denote the sequence \((1^{-n}, 2^{-n}, 3^{-n}, \ldots)\). Then the rational number \( \frac{p}{q} \) is the finite sum of distinct terms taken from \( H^n \) if and only if for all \( \varepsilon > 0 \), there is a finite sum \( s \) of distinct terms taken from \( H^n \) such that \( 0 \leq s - \frac{p}{q} < \varepsilon \).

Theorem A is an immediate consequence of a result of the author [2, Theorem 4] together with the fact that every sufficiently large integer is the sum of distinct nth powers of positive integers (cf., [8], [7] or [3]).

The main results. We begin with several definitions. Let \( S = (s_1, s_2, \ldots) \) denote a (possibly finite) sequence of real numbers.

Definition 1. \( P(S) \) is defined to be the set of all sums of the form \( \sum_{k=1}^{\infty} \varepsilon_k s_k \) where \( \varepsilon_k = 0 \) or \( 1 \) and all but a finite number of the \( \varepsilon_k \) are 0.

Definition 2. \( Ac(S) \) is defined to be the set of all real numbers \( x \) such that for all \( \varepsilon > 0 \), there is an \( s \in P(S) \) such that \( 0 \leq s - x < \varepsilon \). Note that in this terminology Theorem A becomes:

\[
(1) \quad P(H^n) = Ac(H^n) \cap Q
\]

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1 This result has also been obtained by P. Erdös (not published).
where $Q$ denotes the set of rational numbers.

**Definition 3.** A term $s_n$ of $S$ is said to be smoothly replaceable in $S$ (abbreviated $s.r.$ in $S$) if $s_n \leq \sum_{k=1}^{\infty} s_{n+k}$.

**Theorem 1.** Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers such that:
1. $s_n \downarrow 0$.
2. There exists an integer $r$ such that $n \geq r$ implies that $s_n$ is smoothly replaceable in $S$.

Then

$$Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$$

where $P_{r-1} = P((s_1, \cdots, s_{r-1}))$ (note that $P_0 = \{0\}$) and $\sigma = \sum_{k=r}^{\infty} s_k$ (where possibly $\sigma$ is infinite).

**Proof.** Let $x \in \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$ and assume that $x \notin Ac(S)$. Then $x \in [\pi, \pi + \sigma)$ for some $\pi \in P_{r-1}$. A sum of the form $\pi + \sum_{i=1}^{k} s_{i}$ where $r \leq i_1 < i_2 < \cdots < i_k$ will be called “minimal” if

$$\pi + \sum_{i=1}^{k-1} s_{i} < x < \pi + \sum_{i=1}^{k} s_{i}$$

(where a sum of the form $\sum_{i=a}^{b} s_{i}$ is taken to be 0 for $b < a$). Note that since $x \notin Ac(S) \supset P(S)$ then we never get equality in (2). Let $M$ denote the set of minimal sums. Then $M$ must contain infinitely many elements. For suppose $M$ is a finite set. Let $m$ denote the largest index of any $s_j$ which is used in any element of $M$ and let $p = \pi + \sum_{k=1}^{n} s_{j_k} + s_m$ be an element of $M$ which uses $s_m$ (where $r \leq j_1 < j_2 < \cdots < j_m < m$ and possibly $n$ is zero). Thus we have

$$\pi + \sum_{k=1}^{n} s_{j_k} < x < \pi + \sum_{k=1}^{n} s_{j_k} + \sum_{t=m}^{\infty} s_{m+t}$$

since $s_m$ is s.r. in $S$. Therefore there is a least $d \geq 1$ such that $x < p' = \pi + \sum_{k=1}^{n} s_{j_k} + \sum_{t=m}^{d} s_{m+t}$. Hence $p'$ is a “minimal” sum which uses $s_{m+d}$ and $m + d > m$. This is a contradiction to the definition of $M$ and consequently $M$ must be infinite. Now, let $\delta = \inf\{p - x : p \in M\}$. Since $x \notin Ac(S)$ then $\delta > 0$. There exist $p_1, p_2, \cdots \in M$ such that $p_n - x < \delta + \delta/2^n$. Since $s_n \downarrow 0$ then there exists $c$ such that $n \geq c$ implies that $s_n < \delta/2$. Also, there exists $w$ such that $n \geq w$ implies that $p_n$ uses an $s_k$ for some $k \geq c$ (since only a finite number of $p_j$ can be formed from the $s_k$ with $k < c$). Therefore we can write $p_w = \pi + \sum_{j=1}^{n} s_{j_k}$ where $k_n \geq c$. Hence
which is a contradiction to the assumption that $p_w$ is "minimal." Thus, we must have $x \in Ac(S)$ and consequently

$$
\cup \{\pi, \pi + \sigma \} \subset Ac(S).
$$

To show inclusion in the other direction let $x \in Ac(S)$ and suppose that $x \notin \cup_{\pi \in P_{r-1}} \{\pi, \pi + \sigma\}$. Thus, either $x < 0$, $x \geq \sum_{k=1}^{w} s_k$, or there exist $\pi$ and $\pi'$ in $P_{r-1}$ such that $\pi + \sigma \leq x < \pi'$ where no element of $P_{r-1}$ is contained in the interval $[\pi + \sigma, \pi')$. Since the first two possibilities imply that $x \notin Ac(S)$ (contradicting the hypothesis) then we may assume that the third possibility holds. Therefore there exists $\delta > 0$ such that

$$
x \leq \pi' - \delta.
$$

Let $p$ be any element of $P(S)$. Then $p = \sum_{i=1}^{m} s_{i_t} + \sum_{u=1}^{w} s_{j_u}$ for some $m$ and $n$ where

$$
1 \leq i_1 < i_2 < \cdots < i_m \leq r - 1 < j_1 < j_2 < \cdots < j_n.
$$

Thus for $\pi^* = \sum_{t=1}^{m} s_{i_t}$ we have $p \in [\pi^*, \pi^* + \sigma)$. Consequently any element $p$ of $P(S)$ must fall into an interval $[\pi^*, \pi^* + \sigma)$ for some $\pi^* \in P_{r-1}$ and therefore, if $p$ exceeds $x$ then it must exceed $x$ by at least $\delta$ (since $p \notin [\pi + \sigma, \pi')$ and thus by (4) $p > x \in [\pi + \sigma, \pi')$ implies $p \geq \pi' \geq x + \delta$). This contradicts the hypothesis that $x \notin Ac(S)$ and hence we conclude that $Ac(S) \subset \cup_{\pi \in P_{r-1}} \{\pi, \pi + \sigma\}$. Thus, by (3) we have $Ac(S) = \cup_{\pi \in P_{r-1}} \{\pi, \pi + \sigma\}$ and the theorem is proved.

**Theorem 2.** Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers such that:

1. $s_n \downarrow 0$.
2. There exists an integer $r$ such that $n < r$ implies that $s_n$ is not s.r. in $S$ while $n \geq r$ implies that $s_n$ is s.r. in $S$.

Then $Ac(S)$ is the disjoint union of exactly $2^{r-1}$ half-open intervals each of length $\sum_{k=1}^{r} s_k$.

**Proof.** By Theorem 1 we have $Ac(S) = \cup_{\pi \in P_{r-1}} \{\pi, \pi + \sigma\}$ where $\sigma = \sum_{k=1}^{w} s_k$ and $P_{r-1} = P((s_1, \cdots, s_{r-1}))$. Let $\pi = \sum_{k=1}^{r} s_{i_k}$ and $\pi' = \sum_{k=1}^{r} s_{j_k}$ be any two formally distinct sums of the $s_n$ where $1 \leq i_1 < \cdots < i_r \leq r - 1$ and $1 \leq j_1 < \cdots < j_r \leq r - 1$ and we can assume without loss of generality that $\pi \geq \pi'$. Then either there is a least $m \geq 1$ such that $i_m \neq j_m$ or we have $i_k = j_k$ for $k = 1, 2, \cdots, v$ and
In the first case we have
\[ \pi = \sum_{k=1}^{u} s_{i_k} = \sum_{k=1}^{m-1} s_{j_k} + \sum_{k=m}^{u} s_{i_k} > \sum_{k=1}^{m-1} s_{j_k} + \sum_{k=1}^{\infty} s_{i_{m+k}} \text{ (since } s_{i_m} \text{ is not s.r. in } S) \]
\[ > \pi' + \sigma \text{ (since } j_m \geq i_m + 1) . \]

In the second case we have
\[ \pi = \sum_{k=1}^{u} s_{i_k} = \sum_{k=1}^{v} s_{j_k} + \sum_{k=v+1}^{u} s_{i_k} > \sum_{k=1}^{v} s_{j_k} + \sum_{k=1}^{\infty} s_{i_{v+k}} \text{ (since } s_{i_{v+1}} \text{ is not s.r. in } S) \]
\[ \geq \pi' + \sigma \text{ (since } i_{v+1} + 1 \leq i_u + 1 \leq r) . \]

Thus, in either case we see that \( \pi > \pi' + \sigma \). Consequently, any two formally distinct sums in \( P_{r-1} \) are separated by a distance of more than \( \sigma \) and hence, each element \( \pi \) of \( P_{r-1} \) gives rise to a half-open interval \([\pi, \pi + \sigma)\) which is disjoint from any other interval \([\pi', \pi' + \sigma)\) for \( \pi \neq \pi' \in P_{r-1} \). Therefore \( \Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \) is the disjoint union of exactly \( 2^{r-1} \) half-open intervals \([\pi, \pi + \sigma)\), \( \pi \in P_{r-1} \), (since there are exactly \( 2^{r-1} \) formally distinct sums of the form \( \sum_{k=1}^{r-1} \epsilon_k s_k, \epsilon_k = 0 \) or \( 1 \)) where each interval is of length \( \sigma \). This proves the theorem.

We now need three additional lemmas in order to prove the main theorems.

**Lemma 1.** Let \( S = (s_1, s_2, \ldots) \) be a sequence of nonnegative real numbers and suppose that there exists an \( m \) such that \( n \geq m \) implies that \( s_n \leq 2s_{n+1} \). Then \( n \geq m \) implies that \( s_n \) is s.r. in \( S \) (i.e., \( s_n \leq \sum_{k=1}^{\infty} s_{n+k} \)).

**Proof.** If \( \sum_{k=1}^{\infty} s_k = \infty \) then the lemma is immediate. Assume that \( \sum_{k=1}^{\infty} s_k < \infty \). Then
\[
\begin{align*}
n \geq m &\implies s_{n+k} \geq \frac{1}{2} s_{n+k-1}, \\
&\implies \sum_{k=1}^{\infty} s_{n+k} \geq \frac{1}{2} \sum_{k=1}^{\infty} s_{n+k-1} = \frac{1}{2} s_n + \frac{1}{2} \sum_{k=1}^{\infty} s_{n+k} .
\end{align*}
\]
Therefore, \( s_n \leq \sum_{k=1}^{\infty} s_{n+k} \) i.e., \( s_n \) is s.r. in \( S \).

**Lemma 2.** Suppose that \( k \leq (2^{i/n} - 1)^{-1} \) and \( k^{-n} \) is s.r. in \( H^* \) (where \( H^* \) was defined to be the sequence \((1^{-n}, 2^{-n}, \ldots)\)). Then \( (k + 1)^{-n} \) is also s.r. in \( H^* \).
Proof.

\[ k \leq (2^{1/n} - 1)^{-1} \implies \frac{1}{k} \leq 2^{1/n} - 1 \]

(5)

\[ \implies \left( 1 + \frac{1}{k} \right)^n \geq 2 \]

\[ \implies k^{-n} \geq 2(k + 1)^{-n}. \]

Since by hypothesis, \( \sum_{j=k+1}^{\infty} j^{-n} \geq k^{-n} \), then by (5)

\[ \sum_{j=k+2}^{\infty} j^{-n} \geq k^{-n} - (k + 1)^{-n} \geq 2(k + 1)^{-n} - (k + 1)^{-n} = (k + 1)^{-n}. \]

Hence, \((k + 1)^{-n}\) is s.r. in \(H^n\) and the lemma is proved.

**Lemma 3.** Suppose that \( k \geq (2^{1/n} - 1)^{-1} \). Then \( k^{-n} \) is s.r. in \(H_n\).

Proof.

\[ r \geq k \implies r \geq (2^{1/n} - 1)^{-1} \]

\[ \implies \frac{1}{r} \leq 2^{1/n} - 1 \]

\[ \implies \left( 1 + \frac{1}{r} \right)^n \leq 2 \]

\[ \implies r^{-n} \leq 2(r + 1)^{-n}. \]

Therefore, by Lemma 1, \( r^{-n} \) is s.r. in \(H^n\).

**Theorem 3.** Let \( t_n \) denote the largest integer \( k \) such that \( k^{-n} \) is not s.r. in \(H^n\) and let \( P \) denote \( P((1^{-n}, 2^{-n}, \ldots, t_n^{-n})) \). Then

\[ Ac(H^n) = \bigcup_{\pi \in P} [\pi, \pi + \sum_{k=1}^{\infty} (t_n + k)^{-n}] \]

is the disjoint union of exactly \( 2^n \) intervals. Moreover, \( t_n < (2^{1/n} - 1)^{-1} \) and \( t_n \sim n/\ln 2 \) (where \( \ln 2 \) denotes \( \log_e 2 \)).

Proof. With the exception of \( t_n \sim n/\ln 2 \), the theorem follows directly from the preceding results. The following argument, due to L. Shepp, shows that \( t_n \sim n/\ln 2 \).

Consider the function \( f_n(x) \) defined by

(6)

\[ f_n(x) = x^n \left( \sum_{k=1}^{\infty} \frac{1}{(x + k)^n} - \frac{1}{x^n} \right) \]

for \( n = 2, 3, \ldots \) and \( x > 0 \). Since

\[ f_n(x) = \sum_{k=1}^{\infty} \left( 1 + \frac{k}{x} \right)^{-n} - 1 \]

then \( f_n(x) < 0 \) for sufficiently small \( x > 0 \), \( f_n(x) > 0 \) for sufficiently
large $x$, and $f_n(x)$ is continuous and monotone increasing for $x > 0$. Hence the equation $f_n(x) = 0$ has a unique positive root $x_n$ and from the definition of $t_n$ it follows by (6) that $0 < x_n - t_n \leq 1$. Thus, to show that $t_n \sim n/\ln 2$, it suffices to show that $x_n \sim n/\ln 2$. Now it is easily shown (cf., [4], p. 13) that for $a > 0$, $(1 + a/n)^{-n}$ is a decreasing function of $n$. Thus, $f_n(\alpha n)$ is a decreasing function of $n$ and since $f_2(2\alpha) < \infty$ for $\alpha > 0$ then

$$
\lim_{n \to \infty} f_n(\alpha n) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \left(1 + \frac{k}{\alpha n}\right)^{-n} - 1 = \sum_{k=1}^{\infty} \lim_{n \to \infty} \left(1 + \frac{k}{\alpha n}\right)^{-n} - 1 = -1 + \sum_{k=1}^{\infty} e^{-k/\alpha} = (e^{1/\alpha} - 1)^{-1} - 1
$$

since the monotone convergence theorem (cf., [5]) allows us to interchange the sum and limit. Suppose now that for some $\varepsilon > 0$, there exist $n_1 < n_2 < \cdots$ such that $x_{n_t} > n_t(1/\ln 2 + \varepsilon)$. Then

$$
0 = \lim_{t \to \infty} f_{n_t}(x_{n_t}) \geq \lim_{t \to \infty} f_{n_t} \left(n_t \left(\frac{1}{\ln 2} + \varepsilon\right)\right) = (e^{(1/\ln 2 + \varepsilon)^{-1}} - 1)^{-1} - 1 = (2^{1/(1+\varepsilon \ln 2)} - 1)^{-1} - 1 > 0
$$

which is a contradiction. Similarly, if for some $\varepsilon$, $0 < \varepsilon < 1/\ln 2$, there exist $n_1 < n_2 < \cdots$ such that

$$
x_{n_t} < n_t \left(\frac{1}{\ln 2} - \varepsilon\right),
$$

then

$$
0 = \lim_{t \to \infty} f_{n_t}(x_{n_t}) \leq \lim_{t \to \infty} f_{n_t} \left(n_t \left(\frac{1}{\ln 2} - \varepsilon\right)\right) = (e^{(1/\ln 2 - \varepsilon)^{-1}} - 1)^{-1} - 1 = (2^{1/(1-\varepsilon \ln 2)} - 1)^{-1} - 1 < 0
$$

which is again impossible. Hence we have shown that for all $\varepsilon > 0$, there exists an $n_0$ such that $n > n_0$ implies that

$$
n \left(\frac{1}{\ln 2} - \varepsilon\right) \leq x_n \leq n \left(\frac{1}{\ln 2} + \varepsilon\right)
$$

or equivalently

$$
-\varepsilon \leq \frac{x_n}{n} - \frac{1}{\ln 2} \leq \varepsilon.
$$
Therefore, \( \lim_{n \to \infty} x_n/n = 1/\ln 2 \) and the theorem is proved.\(^2\)

The following table gives the values of \( t_n \) for some small values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n )</th>
<th>( [(2^{1/n} - 1)^{-1}] )</th>
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</tr>
<tr>
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<tr>
<td>1000</td>
<td>?</td>
<td>1442</td>
</tr>
</tbody>
</table>

We may now combine Theorem 3 and Theorem A (cf. Eq. (1)) and express the result in ordinary terminology to give:

**Theorem 4.** Let \( n \) be a positive integer, let \( t_n \) be the largest integer \( k \) such that \( k^{-n} > \sum_{j=1}^{\infty} (k+j)^{-n} \) and let \( P \) denote the set \( \{ \sum_{j=1}^{\infty} \varepsilon_j j^{-n} : \varepsilon_j = 0 \text{ or } 1 \} \). Then the rational number \( p/q \) can be written as a finite sum of reciprocals of distinct \( n \)th powers of integers if and only if

\[
\frac{p}{q} \in \bigcup_{n \in P} [\pi, \pi + \sum_{k=1}^{\infty} (t_n + k)^{-n}].
\]

**Corollary 1.** \( p/q \) can expressed as the finite sum of reciprocals of distinct squares if and only if

\[
\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).
\]

**Corollary 2.** \( p/q \) can be expressed as the finite sum of reciprocals of distinct cubes if and only if

\[
\frac{p}{q} \in \left[0, \zeta(3) - \frac{9}{8}\right) \cup \left[\frac{1}{8}, \zeta(3) - 1\right) \cup \left[1, \zeta(3) - \frac{1}{8}\right) \cup \left[\frac{9}{8}, \zeta(3)\right)
\]

where \( \zeta(3) = \sum_{k=1}^{\infty} k^{-3} = 1.2020569\ldots \)

**Remarks.** In theory it should be possible to calculate directly from the relevant theorems (cf., [2], [3]) an explicit bound for the number of terms of \( H^n \) needed to represent \( p/q \) as an element of \( P(H^n) \). However, since the theorems were not designed to minimize such a bound, but rather merely to show its existence, then understandably, this calculated bound would probably be many orders of

---

\(^2\) In fact, it can be shown that \( x_n \) has the expansion \( n/1n2 - 1/2 + c_1n^{-1} + \cdots + c_\kappa n^{-k} + O(n^{-k-1}) \) for any \( k \).
magnitude too large. Erdős and Stein [1] and, independently, van Albada and van Lint [9] have shown that if \( f(n) \) denotes the least number of terms of \( H^1 = (1^{-1}, 2^{-1}, \cdots) \) needed to represent the integer \( n \) as an element of \( P(H^1) \) then \( f(n) \sim e^{n-\gamma} \) where \( \gamma \) is Euler's constant.

It should be pointed out that a more general form of Theorem A may be derived from [2] which can be used to prove results of the following type:

**COROLLARY A.** The rational \( p/q \) with \( (p, q) = 1 \) can be expressed as a finite sum of reciprocals of distinct odd squares if and only if \( q \) is odd and \( p/q \in [0, (\pi^2/8) - 1) \cup [1, \pi^2/8) \).

**COROLLARY B.** The rational \( p/q \) with \( (p, q) = 1 \) can be expressed as a finite sum of reciprocals of distinct squares which are congruent to 4 modulo 5 if and only if \( (q, 5) = 1 \) and

\[
\frac{p}{q} \in \left[ 0, \alpha - \frac{13}{36} \right] \cup \left[ \frac{1}{9}, \alpha - \frac{1}{4} \right] \cup \left[ \frac{1}{4}, \alpha - \frac{1}{9} \right] \cup \left[ \frac{13}{36}, \alpha \right]
\]

where \( \alpha = 2(5 - \sqrt{5})\pi^2/125 = \sum_{k=0}^{\infty} ((5k+2)^{-2} + (5k+3)^{-2}) = 0.43648\cdots \)

It is not difficult to obtain representations of specific rationals as elements of \( P(H^n) \) (for small \( n \)), e.g.,

\[
\frac{1}{2} = 2^{-2} + 3^{-2} + 4^{-2} + 5^{-2} + 6^{-2} + 15^{-2} + 18^{-2} + 36^{-2} + 60^{-2} + 180^{-2},
\]

\[
\frac{1}{3} = 2^{-2} + 4^{-2} + 10^{-2} + 12^{-2} + 20^{-2} + 30^{-2} + 60^{-2},
\]

\[
\frac{5}{37} = 2^{-3} + 5^{-3} + 10^{-3} + 15^{-3} + 16^{-3} + 74^{-3} + 111^{-3} + 185^{-3} + 240^{-3} + 296^{-3} + 444^{-3} + 1480^{-3}, \text{ etc.}
\]

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