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**ON FINITE SUMS OF RECIPROCAL
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RONALD LEWIS GRAHAM

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R. L. GRAHAM

Introduction. It has long been known that every positive rational number can be represented as a finite sum of reciprocals of distinct positive integers (the first proof having been given by Leonardo Pisano [6] in 1202). It is the purpose of this paper to characterize (cf. Theorem 4) those rational numbers which can be written as finite sums of reciprocals of distinct n th powers of integers, where n is an arbitrary (fixed) positive integer and "finite sum" denotes a sum with a finite number of summands. It will follow, for example, that p/q is the finite sum of reciprocals of distinct squares¹ if and only if

$$\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

Our starting point will be the following result:

THEOREM A. *Let n be a positive integer and let H^n denote the sequence $(1^{-n}, 2^{-n}, 3^{-n}, \dots)$. Then the rational number p/q is the finite sum of distinct terms taken from H^n if and only if for all $\varepsilon > 0$, there is a finite sum s of distinct terms taken from H^n such that $0 \leq s - p/q < \varepsilon$.*

Theorem A is an immediate consequence of a result of the author [2, Theorem 4] together with the fact that every sufficiently large integer is the sum of distinct n th powers of positive integers (cf., [8], [7] or [3]).

The main results. We begin with several definitions. Let $S = (s_1, s_2, \dots)$ denote a (possibly finite) sequence of real numbers.

DEFINITION 1. $P(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \varepsilon_k s_k$ where $\varepsilon_k = 0$ or 1 and all but a finite number of the ε_k are 0.

DEFINITION 2. $Ac(S)$ is defined to be the set of all real numbers x such that for all $\varepsilon > 0$, there is an $s \in P(S)$ such that $0 \leq s - x < \varepsilon$. Note that in this terminology Theorem A becomes:

$$(1) \quad P(H^n) = Ac(H^n) \cap Q$$

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¹ This result has also been obtained by P. Erdős (not published).

where Q denotes the set of rational numbers.

DEFINITION 3. A term s_n of S is said to be *smoothly replaceable* in S (abbreviated s.r. in S) if $s_n \leq \sum_{k=1}^{\infty} s_{n+k}$.

THEOREM 1. Let $S = (s_1, s_2, \dots)$ be a sequence of real numbers such that:

1. $s_n \downarrow 0$.
2. There exists an integer r such that $n \geq r$ implies that s_n is smoothly replaceable in S .

Then

$$Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$$

where $P_{r-1} = P((s_1, \dots, s_{r-1}))$ (note that $P_0 = \{0\}$) and $\sigma = \sum_{k=r}^{\infty} s_k$ (where possibly σ is infinite).

Proof. Let $x \in \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$ and assume that $x \notin Ac(S)$. Then $x \in [\pi, \pi + \sigma)$ for some $\pi \in P_{r-1}$. A sum of the form $\pi + \sum_{i=1}^k s_{i_i}$ where $r \leq i_1 < i_2 < \dots < i_k$ will be called "minimal" if

$$(2) \quad \pi + \sum_{t=1}^{k-1} s_{i_t} < x < \pi + \sum_{t=1}^k s_{i_t}$$

(where a sum of the form $\sum_{t=a}^b$ is taken to be 0 for $b < a$). Note that since $x \notin Ac(S) \supset P(S)$ then we never get equality in (2). Let M denote the set of minimal sums. Then M must contain infinitely many elements. For suppose M is a finite set. Let m denote the largest index of any s_j which is used in any element of M and let $p = \pi + \sum_{k=1}^n s_{j_k} + s_m$ be an element of M which uses s_m (where $r \leq j_1 < j_2 < \dots < j_n < m$ and possibly n is zero). Thus we have

$$\pi + \sum_{k=1}^n s_{j_k} < x < \pi + \sum_{k=1}^n s_{j_k} + \sum_{t=1}^{\infty} s_{m+t}$$

since s_m is s.r. in S . Therefore there is a least $d \geq 1$ such that $x < p' = \pi + \sum_{k=1}^n s_{j_k} + \sum_{t=1}^d s_{m+t}$. Hence p' is a "minimal" sum which uses s_{m+d} and $m+d > m$. This is a contradiction to the definition of m and consequently M must be infinite. Now, let $\delta = \inf\{p-x: p \in M\}$. Since $x \notin Ac(S)$ then $\delta > 0$. There exist $p_1, p_2, \dots \in M$ such that $p_n - x < \delta + \delta/2^n$. Since $s_n \downarrow 0$ then there exists c such that $n \geq c$ implies that $s_n < \delta/2$. Also, there exists w such that $n \geq w$ implies that p_n uses an s_k for some $k \geq c$ (since only a finite number of p_j can be formed from the s_k with $k < c$). Therefore we can write $p_w = \pi + \sum_{j=1}^n s_{j_j}$ where $k_n \geq c$. Hence

$$p_w - s_{k_n} - x > p_w - \frac{\delta}{2} - x \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0$$

which is a contradiction to the assumption that p_w is "minimal." Thus, we must have $x \in Ac(S)$ and consequently

$$(3) \quad \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma) \subset Ac(S).$$

To show inclusion in the other direction let $x \in Ac(S)$ and suppose that $x \notin \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$. Thus, either $x < 0$, $x \geq \sum_{k=1}^{\infty} s_k$, or there exist π and π' in P_{r-1} such that $\pi + \sigma \leq x < \pi'$ where no element of P_{r-1} is contained in the interval $[\pi + \sigma, \pi')$. Since the first two possibilities imply that $x \notin Ac(S)$ (contradicting the hypothesis) then we may assume that the third possibility holds. Therefore there exists $\delta > 0$ such that

$$(4) \quad x \leq \pi' - \delta.$$

Let p be any element of $P(S)$. Then $p = \sum_{i=1}^m s_{i_t} + \sum_{u=1}^n s_{j_u}$ for some m and n where

$$1 \leq i_1 < i_2 < \dots < i_m \leq r-1 < j_1 < j_2 < \dots < j_n.$$

Thus for $\pi^* = \sum_{i=1}^m s_{i_t}$ we have $p \in [\pi^*, \pi^* + \sigma)$. Consequently any element p of $P(S)$ must fall into an interval $[\pi^*, \pi^* + \sigma)$ for some $\pi^* \in P_{r-1}$ and therefore, if p exceeds x then it must exceed x by at least δ (since $p \notin [\pi + \sigma, \pi')$ and thus by (4) $p > x \in [\pi + \sigma, \pi')$ implies $p \geq \pi' \geq x + \delta$). This contradicts the hypothesis that $x \in Ac(S)$ and hence we conclude that $Ac(S) \subset \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$. Thus, by (3) we have $Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$ and the theorem is proved.

THEOREM 2. *Let $S = (s_1, s_2, \dots)$ be a sequence of real numbers such that:*

1. $s_n \downarrow 0$.
2. *There exists an integer r such that $n < r$ implies that s_n is not s.r. in S while $n \geq r$ implies that s_n is s.r. in S .*

Then $Ac(S)$ is the disjoint union of exactly 2^{r-1} half-open intervals each of length $\sum_{k=r}^{\infty} s_k$.

Proof. By Theorem 1 we have $Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$ where $\sigma = \sum_{k=r}^{\infty} s_k$ and $P_{r-1} = P((s_1, \dots, s_{r-1}))$. Let $\pi = \sum_{k=1}^u s_{i_k}$ and $\pi' = \sum_{k=1}^v s_{j_k}$ be any two formally distinct sums of the s_n where $1 \leq i_1 < \dots < i_u \leq r-1$ and $1 \leq j_1 < \dots < j_v \leq r-1$ and we can assume without loss of generality that $\pi \geq \pi'$. Then either there is a *least* $m \geq 1$ such that $i_m \neq j_m$ or we have $i_k = j_k$ for $k = 1, 2, \dots, v$ and

$u > v$. In the first case we have

$$\begin{aligned} \pi &= \sum_{k=1}^u s_{i_k} = \sum_{k=1}^{m-1} s_{j_k} + \sum_{k=m}^u s_{i_k} \\ &> \sum_{k=1}^{m-1} s_{j_k} + \sum_{k=1}^{\infty} s_{i_m+k} \quad (\text{since } s_{i_m} \text{ is not s.r. in } S) \\ &\geq \pi' + \sigma \quad (\text{since } j_m \geq i_m + 1). \end{aligned}$$

In the second case we have

$$\begin{aligned} \pi &= \sum_{k=1}^u s_{i_k} = \sum_{k=1}^v s_{j_k} + \sum_{k=v+1}^u s_{i_k} \\ &> \sum_{k=1}^v s_{j_k} + \sum_{k=1}^{\infty} s_{i_{v+1}+k} \quad (\text{since } s_{i_{v+1}} \text{ is not s.r. in } S) \\ &\geq \pi' + \sigma \quad (\text{since } i_{v+1} + 1 \leq i_u + 1 \leq r). \end{aligned}$$

Thus, in either case we see that $\pi > \pi' + \sigma$. Consequently, any two formally distinct sums in P_{r-1} are separated by a distance of more than σ and hence, each element π of P_{r-1} gives rise to a half-open interval $[\pi, \pi + \sigma)$ which is disjoint from any other interval $[\pi', \pi' + \sigma)$ for $\pi \neq \pi' \in P_{r-1}$. Therefore $Ac(S) = \bigcup_{\pi \in P_{r-1}} [\pi, \pi + \sigma)$ is the disjoint union of exactly 2^{r-1} half-open intervals $[\pi, \pi + \sigma)$, $\pi \in P_{r-1}$, (since there are exactly 2^{r-1} formally distinct sums of the form $\sum_{k=1}^{r-1} \varepsilon_k s_k$, $\varepsilon_k = 0$ or 1) where each interval is of length σ . This proves the theorem.

We now need three additional lemmas in order to prove the main theorems.

LEMMA 1. *Let $S = (s_1, s_2, \dots)$ be a sequence of nonnegative real numbers and suppose that there exists an m such that $n \geq m$ implies that $s_n \leq 2s_{n+1}$. Then $n \geq m$ implies that s_n is s.r. in S (i.e., $s_n \leq \sum_{k=1}^{\infty} s_{n+k}$).*

Proof. If $\sum_{k=1}^{\infty} s_k = \infty$ then the lemma is immediate. Assume that $\sum_{k=1}^{\infty} s_k < \infty$. Then

$$\begin{aligned} n \geq m &\implies s_{n+k} \geq \frac{1}{2} s_{n+k-1}, & k = 1, 2, 3, \dots \\ &\implies \sum_{k=1}^{\infty} s_{n+k} \geq \frac{1}{2} \sum_{k=1}^{\infty} s_{n+k-1} = \frac{1}{2} s_n + \frac{1}{2} \sum_{k=1}^{\infty} s_{n+k}. \end{aligned}$$

Therefore, $s_n \leq \sum_{k=1}^{\infty} s_{n+k}$, i.e., s_n is s.r. in S .

LEMMA 2. *Suppose that $k \leq (2^{1/n} - 1)^{-1}$ and k^{-n} is s.r. in H^n (where H^n was defined to be the sequence $(1^{-n}, 2^{-n}, \dots)$). Then $(k+1)^{-n}$ is also s.r. in H^n .*

Proof.

$$\begin{aligned}
 (5) \quad k &\leq (2^{1/n} - 1)^{-1} \implies \frac{1}{k} \leq 2^{1/n} - 1 \\
 &\implies \left(1 + \frac{1}{k}\right)^n \geq 2 \\
 &\implies k^{-n} \geq 2(k+1)^{-n}.
 \end{aligned}$$

Since by hypothesis, $\sum_{j=k+1}^{\infty} j^{-n} \geq k^{-n}$, then by (5)

$$\sum_{j=k+2}^{\infty} j^{-n} \geq k^{-n} - (k+1)^{-n} \geq 2(k+1)^{-n} - (k+1)^{-n} = (k+1)^{-n}.$$

Hence, $(k+1)^{-n}$ is s.r. in H^n and the lemma is proved.

LEMMA 3. Suppose that $k \geq (2^{1/n} - 1)^{-1}$. Then k^{-n} is s.r. in H_n .

Proof.

$$\begin{aligned}
 r \geq k &\implies r \geq (2^{1/n} - 1)^{-1} \\
 &\implies \frac{1}{r} \leq 2^{1/n} - 1 \\
 &\implies \left(1 + \frac{1}{r}\right)^n \leq 2 \\
 &\implies r^{-n} \leq 2(r+1)^{-n}.
 \end{aligned}$$

Therefore, by Lemma 1, r^{-n} is s.r. in H^n .

THEOREM 3. Let t_n denote the largest integer k such that k^{-n} is not s.r. in H^n and let P denote $P((1^{-n}, 2^{-n}, \dots, t_n^{-n}))$. Then

$$Ac(H^n) = \bigcup_{\pi \in P} [\pi, \pi + \sum_{k=1}^{\infty} (t_n + k)^{-n}]$$

is the disjoint union of exactly 2^{t_n} intervals. Moreover, $t_n < (2^{1/n} - 1)^{-1}$ and $t_n \sim n/\ln 2$ (where $\ln 2$ denotes $\log_e 2$).

Proof. With the exception of $t_n \sim n/\ln 2$, the theorem follows directly from the preceding results. The following argument, due to L. Shepp, shows that $t_n \sim n/\ln 2$.

Consider the function $f_n(x)$ defined by

$$(6) \quad f_n(x) = x^n \left(\sum_{k=1}^{\infty} \frac{1}{(x+k)^n} - \frac{1}{x^n} \right)$$

for $n = 2, 3, \dots$ and $x > 0$. Since

$$f_n(x) = \sum_{k=1}^{\infty} \left(1 + \frac{k}{x}\right)^{-n} - 1$$

then $f_n(x) < 0$ for sufficiently small $x > 0$, $f_n(x) > 0$ for sufficiently

large x , and $f_n(x)$ is continuous and monotone increasing for $x > 0$. Hence the equation $f_n(x) = 0$ has a unique positive root x_n and from the definition of t_n it follows by (6) that $0 < x_n - t_n \leq 1$. Thus, to show that $t_n \sim n/\ln 2$, it suffices to show that $x_n \sim n/\ln 2$. Now it is easily shown (cf., [4], p. 13) that for $a > 0$, $(1 + a/n)^{-n}$ is a decreasing function of n . Thus, $f_n(\alpha n)$ is a decreasing function of n and since $f_2(2\alpha) < \infty$ for $\alpha > 0$ then

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(\alpha n) &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(1 + \frac{k}{\alpha n}\right)^{-n} - 1 \\ &= \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} \left(1 + \frac{k}{\alpha n}\right)^{-n} - 1 \\ &= -1 + \sum_{k=1}^{\infty} e^{-k/\alpha} = (e^{1/\alpha} - 1)^{-1} - 1 \end{aligned}$$

since the monotone convergence theorem (cf., [5]) allows us to interchange the sum and limit. Suppose now that for some $\varepsilon > 0$, there exist $n_1 < n_2 < \dots$ such that $x_{n_i} > n_i(1/\ln 2 + \varepsilon)$. Then

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} f_{n_i}(x_{n_i}) \geq \lim_{i \rightarrow \infty} f_{n_i}\left(n_i\left(\frac{1}{\ln 2} + \varepsilon\right)\right) \\ &= (e^{(1/\ln 2 + \varepsilon)} - 1)^{-1} - 1 \\ &= (2^{1/(1 + \varepsilon \ln 2)} - 1)^{-1} - 1 > 0 \end{aligned}$$

which is a contradiction. Similarly, if for some ε , $0 < \varepsilon < 1/\ln 2$, there exist $n_1 < n_2 < \dots$ such that

$$x_{n_i} < n_i\left(\frac{1}{\ln 2} - \varepsilon\right),$$

then

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} f_{n_i}(x_{n_i}) \leq \lim_{i \rightarrow \infty} f_{n_i}\left(n_i\left(\frac{1}{\ln 2} - \varepsilon\right)\right) \\ &= (e^{(1/\ln 2 - \varepsilon)} - 1)^{-1} - 1 \\ &= (2^{1/(1 - \varepsilon \ln 2)} - 1)^{-1} - 1 < 0 \end{aligned}$$

which is again impossible. Hence we have shown that for all $\varepsilon > 0$, there exists an n_0 such that $n > n_0$ implies that

$$n\left(\frac{1}{\ln 2} - \varepsilon\right) \leq x_n \leq n\left(\frac{1}{\ln 2} + \varepsilon\right)$$

or equivalently

$$-\varepsilon \leq \frac{x_n}{n} - \frac{1}{\ln 2} \leq \varepsilon.$$

Therefore, $\lim_{n \rightarrow \infty} x_n/n = 1/\ln 2$ and the theorem is proved.²

The following table gives the values of t_n for some small values of n .

n	t_n	$[(2^{1/n} - 1)^{-1}]$
1	0	1
2	1	2
3	2	3
4	4	5
5	5	6
10	12	13
100	?	143
1000	?	1442

We may now combine Theorem 3 and Theorem A (cf. Eq. (1)) and express the result in ordinary terminology to give:

THEOREM 4. *Let n be a positive integer, let t_n be the largest integer k such that $k^{-n} > \sum_{j=1}^{\infty} (k+j)^{-n}$ and let P denote the set $\{\sum_{j=1}^t \epsilon_j j^{-n} : \epsilon_j = 0 \text{ or } 1\}$. Then the rational number p/q can be written as a finite sum of reciprocals of distinct n th powers of integers if and only if*

$$\frac{p}{q} \in \bigcup_{\pi \in P} [\pi, \pi + \sum_{k=1}^{\infty} (t_n + k)^{-n}].$$

COROLLARY 1. *p/q can be expressed as the finite sum of reciprocals of distinct squares if and only if*

$$\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

COROLLARY 2. *p/q can be expressed as the finite sum of reciprocals of distinct cubes if and only if*

$$\frac{p}{q} \in \left[0, \zeta(3) - \frac{9}{8}\right) \cup \left[\frac{1}{8}, \zeta(3) - 1\right) \cup \left[1, \zeta(3) - \frac{1}{8}\right) \cup \left[\frac{9}{8}, \zeta(3)\right)$$

where $\zeta(3) = \sum_{k=1}^{\infty} k^{-3} = 1.2020569 \dots$

REMARKS. In theory it should be possible to calculate directly from the relevant theorems (cf., [2], [3]) an explicit bound for the number of terms of H^n needed to represent p/q as an element of $P(H^n)$. However, since the theorems were not designed to minimize such a bound, but rather merely to show its existence, then understandably, this calculated bound would probably be many orders of

² In fact, it can be shown that x_n has the expansion $n/\ln 2 - 1/2 + c_1 n^{-1} + \dots + c_k n^{-k} + O(n^{-k-1})$ for any k .

magnitude too large. Erdős and Stein [1] and, independently, van Albada and van Lint [9] have shown that if $f(n)$ denotes the least number of terms of $H^1 = (1^{-1}, 2^{-1}, \dots)$ needed to represent the integer n as an element of $P(H^1)$ then $f(n) \sim e^{n-\gamma}$ where γ is Euler's constant.

It should be pointed out that a more general form of Theorem A may be derived from [2] which can be used to prove results of the following type:

COROLLARY A. *The rational p/q with $(p, q) = 1$ can be expressed as a finite sum of reciprocals of distinct odd squares if and only if q is odd and $p/q \in [0, (\pi^2/8) - 1) \cup [1, \pi^2/8)$.*

COROLLARY B. *The rational p/q with $(p, q) = 1$ can be expressed as a finite sum of reciprocals of distinct squares which are congruent to 4 modulo 5 if and only if $(q, 5) = 1$ and*

$$\frac{p}{q} \in \left[0, \alpha - \frac{13}{36}\right) \cup \left[\frac{1}{9}, \alpha - \frac{1}{4}\right) \cup \left[\frac{1}{4}, \alpha - \frac{1}{9}\right) \cup \left[\frac{13}{36}, \alpha\right)$$

where $\alpha = 2(5 - \sqrt{5})\pi^2/125 = \sum_{k=0}^{\infty} ((5k+2)^{-2} + (5k+3)^{-2}) = 0.43648\dots$

It is not difficult to obtain representations of specific rationals as elements of $P(H^n)$ (for small n), e.g.,

$$\begin{aligned} \frac{1}{2} &= 2^{-2} + 3^{-2} + 4^{-2} + 5^{-2} + 6^{-2} + 15^{-2} + 18^{-2} + 36^{-2} + 60^{-2} + 180^{-2}, \\ \frac{1}{3} &= 2^{-2} + 4^{-2} + 10^{-2} + 12^{-2} + 20^{-2} + 30^{-2} + 60^{-2}, \\ \frac{5}{37} &= 2^{-3} + 5^{-3} + 10^{-3} + 15^{-3} + 16^{-3} + 74^{-3} + 111^{-3} + 185^{-3} + 240^{-3} \\ &\quad + 296^{-3} + 444^{-3} + 1480^{-3}, \text{ etc.}! \end{aligned}$$

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Richard Arens, <i>Normal form for a Pfaffian</i>	1
Charles Vernon Coffman, <i>Non-linear differential equations on cones in Banach spaces</i>	9
Ralph DeMarr, <i>Order convergence in linear topological spaces</i>	17
Peter Larkin Duren, <i>On the spectrum of a Toeplitz operator</i>	21
Robert E. Edwards, <i>Endomorphisms of function-spaces which leave stable all translation-invariant manifolds</i>	31
Erik Maurice Ellentuck, <i>Infinite products of isols</i>	49
William James Firey, <i>Some applications of means of convex bodies</i>	53
Haim Gaifman, <i>Concerning measures on Boolean algebras</i>	61
Richard Carl Gilbert, <i>Extremal spectral functions of a symmetric operator</i>	75
Ronald Lewis Graham, <i>On finite sums of reciprocals of distinct nth powers</i>	85
Hwa Suk Hahn, <i>On the relative growth of differences of partition functions</i>	93
Isidore Isaac Hirschman, Jr., <i>Extreme eigen values of Toeplitz forms associated with Jacobi polynomials</i>	107
Chen-jung Hsu, <i>Remarks on certain almost product spaces</i>	163
George Seth Innis, Jr., <i>Some reproducing kernels for the unit disk</i>	177
Ronald Jacobowitz, <i>Multiplicativity of the local Hilbert symbol</i>	187
Paul Joseph Kelly, <i>On some mappings related to graphs</i>	191
William A. Kirk, <i>On curvature of a metric space at a point</i>	195
G. J. Kurowski, <i>On the convergence of semi-discrete analytic functions</i>	199
Richard George Laatsch, <i>Extensions of subadditive functions</i>	209
V. Marić, <i>On some properties of solutions of $\Delta\psi + A(r^2)X\nabla\psi + C(r^2)\psi = 0$</i> ...	217
William H. Mills, <i>Polynomials with minimal value sets</i>	225
George James Minty, Jr., <i>On the monotonicity of the gradient of a convex function</i>	243
George James Minty, Jr., <i>On the solvability of nonlinear functional equations of 'monotonic' type</i>	249
J. B. Muskat, <i>On the solvability of $x^e \equiv e \pmod{p}$</i>	257
Zeev Nehari, <i>On an inequality of P. R. Bessack</i>	261
Raymond Moos Redheffer and Ernst Gabor Straus, <i>Degenerate elliptic equations</i>	265
Abraham Robinson, <i>On generalized limits and linear functionals</i>	269
Bernard W. Roos, <i>On a class of singular second order differential equations with a non linear parameter</i>	285
Tôru Saitô, <i>Ordered completely regular semigroups</i>	295
Edward Silverman, <i>A problem of least area</i>	309
Robert C. Sine, <i>Spectral decomposition of a class of operators</i>	333
Jonathan Dean Swift, <i>Chains and graphs of Ostrom planes</i>	353
John Griggs Thompson, <i>2-signalizers of finite groups</i>	363
Harold Widom, <i>On the spectrum of a Toeplitz operator</i>	365