

# Pacific Journal of Mathematics

**ON SOME MAPPINGS RELATED TO GRAPHS**

PAUL JOSEPH KELLY

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Let  $N$  denote a set of  $n$  distinct elements  $a_1, a_2, \dots, a_n$  and let  $\mathcal{S}(h) = \{S_1, S_2, \dots, S_m\}$ ,  $m = \binom{n}{h}$  be the collection of all sets formed by selecting  $h$  elements at a time from  $N$ . If  $S_i = \{a_{i_1}, a_{i_2}, \dots, a_{i_h}\}$  is any set in  $\mathcal{S}(h)$  and if  $\Gamma$  is any mapping of  $N$  onto itself, then  $\Gamma$  induces a mapping  $\Psi$  of  $\mathcal{S}(h)$  onto itself defined by  $S_i\Psi = \{a_{i_1}\Gamma, a_{i_2}\Gamma, \dots, a_{i_h}\Gamma\}$ . We seek conditions under which, conversely, a mapping of  $\mathcal{S}(h)$  onto itself must be of this induced type.

If  $\Psi$  is a mapping of  $\mathcal{S}(h)$  onto itself, it will be said to "preserve maximal intersections" if each two of its sets which intersect on  $h - 1$  elements are mapped to two sets which also have  $h - 1$  elements in common. It will be shown that if  $n \neq 2h$  this is sufficient to imply that  $\Psi$  is induced by a mapping of  $N$  onto itself.

We observe first that to each set  $S_i$  in  $\mathcal{S}(h)$  there corresponds a set  $S_i^*$  in  $\mathcal{S}(n - h)$  and which consists of those elements of  $N$  not in  $S_i$ . And to any mapping  $\Psi$  of  $\mathcal{S}(h)$  onto itself there corresponds a mapping  $\Psi^*$  of  $\mathcal{S}(n - h)$  onto itself defined by  $S_i^*\Psi^* = (S_i\Psi)^*$ ,  $i = 1, 2, \dots, m$ . Clearly, if  $\Psi$  preserves maximal intersections so does  $\Psi^*$  and both  $\Psi$  and  $\Psi^*$  are induced mappings or neither is. Thus it suffices always to consider the case  $h \leq n - h$ , that is,  $h \leq n/2$ .

**THEOREM 1.** *If  $n \neq 2h$  and if  $\Psi$  is a mapping of  $\mathcal{S}(h)$  onto itself which preserves maximal intersections, then  $\Psi$  is induced by a mapping of  $N$  onto itself.*

*Proof.* The theorem is trivially correct for  $h = 1$ . For a proof by induction, we suppose the theorem true up to some value  $h - 1$  and consider  $\Psi$  to be a mapping of  $\mathcal{S}(h)$  onto itself, where  $1 < h < n/2$ .

Each set in  $\mathcal{S}(h - 1)$  belongs to exactly  $n - h + 1$  sets in  $\mathcal{S}(h)$  and we wish to show that these sets in  $\mathcal{S}(h)$  must map under  $\Psi$  to  $n - h + 1$  sets which also have a set of  $h - 1$  elements in common. Suppose that this is not the case. Then there exists a set in  $\mathcal{S}(h - 1)$ , which we may take to be  $T = \{a_1, a_2, \dots, a_{h-1}\}$ , such that the sets in  $\mathcal{S}(h)$  which contain  $T$  do not map under  $\Psi$  to a collection of sets with a common intersection of  $h - 1$  elements. Let

$$(1) \quad S_i = \{a_1, a_2, \dots, a_{h-1}, a_{h+i}\}, \quad i = 0, 1, \dots, h, \dots, n - h$$

denote the sets of  $\mathcal{S}(h)$  which contain  $T$ . There is no loss of gener-

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Received March 20, 1963. This work was supported by Contract NSF-G23718.

ality in supposing that it is the intersection of  $S_0\mathcal{P}$  and  $S_1\mathcal{P}$  which is not contained in  $S_2\mathcal{P}$ . Since  $\mathcal{P}$  preserves maximal intersections, we can denote

$$(2) \quad S_0\mathcal{P} = \{b_1, b_2, \dots, b_{h-1}, b_h\}, \quad S_1\mathcal{P} = \{b_1, b_2, \dots, b_{h-1}, b_{h+1}\},$$

where each  $b_i$  is an element from  $N$  and  $i \neq j$  implies  $b_i \neq b_j$ ,  $i, j = 1, 2, \dots, h+1$ . Because  $S_2\mathcal{P}$  does not contain  $\{b_1, b_2, \dots, b_{h-1}\}$ , but must intersect  $S_0\mathcal{P}$  and  $S_1\mathcal{P}$  on  $h-1$  elements,  $S_2\mathcal{P}$  must contain both  $b_h$  and  $b_{h+1}$  and fail to possess just one elements from  $b_1, b_2, \dots, b_{h-1}$ . Since there is nothing to distinguish the possibilities, we may suppose that  $S_2\mathcal{P}$  does not possess  $b_1$ , and hence that

$$(3) \quad S_2\mathcal{P} = \{b_2, \dots, b_{h-1}, b_h, b_{h+1}\}.$$

Because  $n > 2h$ , there are at least  $h+2$  sets  $S_i$  defined by (1) and so at least  $h-1$  sets  $S_i$ , where  $2 < i \leq n-h$ . And the  $\mathcal{P}$  images of all these sets must possess  $b_1, b_h$ , and  $b_{h+1}$ . For suppose  $b_1 \notin S_i\mathcal{P}$ . Since  $S_i\mathcal{P}$  intersects  $S_0\mathcal{P}$  on  $h-1$  elements and not on  $b_1$  then  $\{b_2, b_3, \dots, b_h\} \subset S_i\mathcal{P}$ . And since  $S_i\mathcal{P}$  intersects  $S_1\mathcal{P}$  on  $h-1$  elements and not on  $b_1$ , then  $\{b_2, \dots, b_h, b_{h+1}\} \subset S_i\mathcal{P}$ . But then  $S_i\mathcal{P} = \{b_2, \dots, b_h, b_{h+1}\} = S_2\mathcal{P}$ , which is impossible for  $i \neq 2$ . In the same way,  $b_h \notin S_i\mathcal{P}$  implies  $S_i\mathcal{P} = S_1\mathcal{P}$  and  $b_{h+1} \notin S_i\mathcal{P}$  implies  $S_i\mathcal{P} = S_0\mathcal{P}$ , neither of which is possible for  $2 < i \leq n-h$ .

From the last argument it follows that for  $i > 2$ ,  $S_i\mathcal{P}$  must be of the form

$$(4) \quad S_i\mathcal{P} = \{b_1, b_h, b_{h+1}, x_1, \dots, x_{h-3}\},$$

where  $\{x_1, x_2, \dots, x_{h-3}\}$  is a subset of  $\{b_2, b_3, \dots, b_{h-1}\}$ , which is clearly impossible if  $h=2$ . But in any case, there are at least  $h-1$  different sets  $S_i\mathcal{P}$ , where  $i > 2$ , and each of these is determined by the  $h-3$  order subset of  $\{b_2, \dots, b_{h-1}\}$  which it contains. And since there are only  $h-2$  mutually different such subsets, the sets  $S_i\mathcal{P}$ ,  $i > 2$ , cannot all be distinct, which contradicts the fact that  $\mathcal{P}$  is a one-to-one mapping.

It is now established that for each set  $T$  in  $\mathcal{S}(h-1)$  there exists a set  $T'$  in  $\mathcal{S}(h-1)$  such that all the sets in  $\mathcal{S}(h)$  which contain  $T$  are mapped under  $\mathcal{P}$  to all the sets in  $\mathcal{S}(h)$  which contain  $T'$ . But then the correspondence  $T \rightarrow T'$  is clearly a mapping of  $\mathcal{S}(h-1)$  onto itself, say the mapping  $\Phi$ .

For  $h=2$ ,  $\Phi$  is a mapping of  $N$  onto itself. If  $\{a_i, a_j\}$  is any set in  $\mathcal{S}(2)$ , then  $a_i\Phi$  belongs to the  $\mathcal{P}$  images of all sets which possess  $a_i$ , so  $a_i\Phi$  belongs to  $\{a_i, a_j\}\mathcal{P}$ . By the same argument,  $a_j\Phi$

belongs to  $\{a_i, a_j\}\Psi$ . Since  $a_i\Phi \neq a_j\Phi$ , it follows that  $\{a_i, a_j\}\Psi = \{a_i\Phi, a_j\Phi\}$  and hence that  $\Psi$  is induced by  $\Phi$ .

If  $h > 2$ , consider any two sets in  $\mathcal{S}(h - 1)$ , whose intersection is maximal, say

$$(5) \quad T_1 = \{a_1, a_2, \dots, a_{h-2}, a_{h-1}\}, \quad T_2 = \{a_1, a_2, \dots, a_{h-2}, a_h\}.$$

The set  $S = \{a_1, a_2, \dots, a_h\}$  in  $\mathcal{S}(h)$  maps to a set  $S\Psi = \{b_1, b_2, \dots, b_h\}$ . Since  $T_1$  and  $T_2$  are contained in  $S$ ,  $T_1\Phi$  and  $T_2\Phi$  are  $h - 1$  order subsets of  $S\Psi$ . Since  $T_1 \neq T_2$ , and  $\Phi$  is a one-to-one mapping,  $T_1\Phi \neq T_2\Phi$ , so the order of  $T_1\Phi \cap T_2\Phi$  is  $h - 2$ . Thus  $\Phi$  preserves maximal intersections and so, by the inductive hypothesis,  $\Phi$  is induced by some mapping  $\Gamma$  of  $N$  onto itself.

Now  $S = \{a_1, a_2, \dots, a_h\}$  contains  $T_1$  and  $T_2$  defined in (5) so  $S\Psi$  contains  $T_1\Phi$  and  $T_2\Phi$ . But  $T_1\Phi = \{a_1\Gamma, a_2\Gamma, \dots, a_{h-1}\Gamma\}$ , and  $T_2\Phi = \{a_1\Gamma, \dots, a_{h-1}\Gamma, a_h\Gamma\}$ . Since  $a_i\Gamma \neq a_j\Gamma$  if  $i \neq j$ , it follows that  $S\Psi = \{a_1\Gamma, a_2\Gamma, \dots, a_h\Gamma\}$ , and hence that  $\Psi$  is induced by  $\Gamma$ .

The theorem is not true for  $n = 2h$ , since then the correspondence of  $S_i$  and  $S_i^*$  is a non-induced mapping of  $\mathcal{S}(h)$  onto itself which preserves all orders of intersection.<sup>1</sup>

Consider next an ordinary, finite graph  $G$ , that is, one with  $n$  vertices  $\{p_1, p_2, \dots, p_n\}$  where each two vertices have at most one join and none is joined to itself. Let  $c(p_i, p_j, p_k)$  denote the subgraph of  $G$  induced by  $G$  on the set of vertices which does not include  $p_i, p_j, p_k$ , and let  $m(G)$  be the notation for the join-measure of  $G$ , that is the number of joins in  $G$ .

**THEOREM 2.** *If  $G$  and  $H$  are ordinary  $n$ th order graphs and if there is a mapping of the vertices of  $G$  onto those of  $H$  such that for some integer  $h, 1 < h < n - 1$ , all corresponding subgraphs of order  $h$  have the same join measure, then the mapping is an isomorphism of  $G$  and  $H$ .*

*Proof.* For  $h = 2$  the condition becomes the definition of an isomorphism, so assume that  $2 < h < n - 1$ . Let  $\{p_1, p_2, \dots, p_n\}$  be the vertices of  $G$  and let the vertices  $\{q_1, q_2, \dots, q_n\}$  of  $H$  be labeled so that  $q_i$  is the image of  $p_i$  under the given mapping  $\psi, i = 1, 2, \dots, n$ .

Let  $\{p_{i_1}, p_{i_2}, \dots, p_{i_{h+1}}\}$  be the vertices of any subgraph  $G_i$  of order  $h + 1$  in  $G$ , and let  $c(p_{i_k}; G_i)$  denote the subgraph of  $G_i$  defined on all the vertices of  $G_i$  except  $p_{i_k}$ . Since any join in  $G_i$  belongs to all the  $h$ -order subgraphs of  $G_i$  except two, we have,

$$(1) \quad m(G_i) = \frac{1}{h - 1} \sum_{k=1}^{h+1} m[c(p_{i_k}; G_i)].$$

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<sup>1</sup>This general exception was pointed out to the writer by P. Erdős.

By the same reasoning,

$$(2) \quad m(G_i\Psi) = \frac{1}{h-1} \sum_{k=1}^{h+1} m[c(q_{i_k}; G_i\Psi)] .$$

Since, by assumption,

$$(3) \quad m[c(p_{i_k}; G_i)] = m[c(q_{i_k}; G_i\Psi)] , \quad \text{for all } p_{i_k} \text{ and } q_{i_k} ,$$

it follows that  $m(G_i) = m(G_i\Psi)$ .

Thus if  $\Psi$  preserves the join measure on  $h$ -order subgraphs it does so on  $h+1$  order subgraphs, and, by the same reasoning, preserves the join measure on all subgraphs of order equal to or greater than  $h$ . In particular,  $m(G) = m(H)$ . Then if  $\rho(p_i)$  denotes the degree of  $p_i$ , it follows from

$$(4) \quad \rho(p_i) = m(G) - m[c(p_i)] , \quad i = 1, 2, \dots, n$$

and

$$(5) \quad \rho(q_i) = m(H) - m[c(q_i)] , \quad i = 1, 2, \dots, n$$

that

$$(6) \quad \rho(p_i) = \rho(q_i) , \quad i = 1, 2, \dots, n ,$$

since  $m[c(p_i)] = m[c(q_i)]$ .

Now, corresponding to  $p_i$  and  $p_j$  in  $G$ , let  $\varepsilon_{ij}$  be 1 or 0 according as  $p_i$  and  $p_j$  are or are not joined. Let  $\varepsilon'_{ij}$  be defined in a similar way with respect to  $q_i$  and  $q_j$ . Then, by simple counting,

$$(7) \quad m(G) = m[c(p_i, p_j)] + \rho(p_i) + \rho(p_j) - \varepsilon_{ij} , \quad i \neq j ,$$

and

$$(8) \quad m(H) = m[c(q_i, q_j)] + \rho(q_i) + \rho(q_j) - \varepsilon'_{ij} , \quad i \neq j .$$

Comparing the terms in (7) and (8) it follows that  $\varepsilon_{ij} = \varepsilon'_{ij}$  for all  $i, j$ ,  $i \neq j$ , and hence that  $\Psi$  is an isomorphism of  $G$  and  $H$ .

As a corollary of these theorems it follows that two  $n$ th order graphs are isomorphic if and only if there is a one-to-one correspondence of their subgraphs of some order  $h$ ,  $1 < h < n-1$ , in which corresponding subgraphs have equal join measure and the correspondence preserves maximal intersections.

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# Pacific Journal of Mathematics

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