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ON SOME MAPPINGS RELATED TO GRAPHS

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Let N denote a set of n distinct elements a_1, a_2, \dots, a_n and let $\mathcal{S}(h) = \{S_1, S_2, \dots, S_m\}$, $m = \binom{n}{h}$ be the collection of all sets formed by selecting h elements at a time from N . If $S_i = \{a_{i_1}, a_{i_2}, \dots, a_{i_h}\}$ is any set in $\mathcal{S}(h)$ and if Γ is any mapping of N onto itself, then Γ induces a mapping Ψ of $\mathcal{S}(h)$ onto itself defined by $S_i\Psi = \{a_{i_1}\Gamma, a_{i_2}\Gamma, \dots, a_{i_h}\Gamma\}$. We seek conditions under which, conversely, a mapping of $\mathcal{S}(h)$ onto itself must be of this induced type.

If Ψ is a mapping of $\mathcal{S}(h)$ onto itself, it will be said to “preserve maximal intersections” if each two of its sets which intersect on $h - 1$ elements are mapped to two sets which also have $h - 1$ elements in common. It will be shown that if $n \neq 2h$ this is sufficient to imply that Ψ is induced by a mapping of N onto itself.

We observe first that to each set S_i in $\mathcal{S}(h)$ there corresponds a set S_i^* in $\mathcal{S}(n - h)$ and which consists of those elements of N not in S_i . And to any mapping Ψ of $\mathcal{S}(h)$ onto itself there corresponds a mapping Ψ^* of $\mathcal{S}(n - h)$ onto itself defined by $S_i^*\Psi^* = (S_i\Psi)^*$, $i = 1, 2, \dots, m$. Clearly, if Ψ preserves maximal intersections so does Ψ^* and both Ψ and Ψ^* are induced mappings or neither is. Thus it suffices always to consider the case $h \leq n - h$, that is, $h \leq n/2$.

THEOREM 1. *If $n \neq 2h$ and if Ψ is a mapping of $\mathcal{S}(h)$ onto itself which preserves maximal intersections, then Ψ is induced by a mapping of N onto itself.*

Proof. The theorem is trivially correct for $h = 1$. For a proof by induction, we suppose the theorem true up to some value $h - 1$ and consider Ψ to be a mapping of $\mathcal{S}(h)$ onto itself, where $1 < h < n/2$.

Each set in $\mathcal{S}(h - 1)$ belongs to exactly $n - h + 1$ sets in $\mathcal{S}(h)$ and we wish to show that these sets in $\mathcal{S}(h)$ must map under Ψ to $n - h + 1$ sets which also have a set of $h - 1$ elements in common. Suppose that this is not the case. Then there exists a set in $\mathcal{S}(h - 1)$, which we may take to be $T = \{a_1, a_2, \dots, a_{h-1}\}$, such that the sets in $\mathcal{S}(h)$ which contain T do not map under Ψ to a collection of sets with a common intersection of $h - 1$ elements. Let

$$(1) \quad S_i = \{a_1, a_2, \dots, a_{h-1}, a_{h+i}\}, \quad i = 0, 1, \dots, h, \dots, n - h$$

denote the sets of $\mathcal{S}(h)$ which contain T . There is no loss of gener-

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ality in supposing that it is the intersection of $S_0\mathcal{P}$ and $S_1\mathcal{P}$ which is not contained in $S_2\mathcal{P}$. Since \mathcal{P} preserves maximal intersections, we can denote

$$(2) \quad S_0\mathcal{P} = \{b_1, b_2, \dots, b_{h-1}, b_h\}, \quad S_1\mathcal{P} = \{b_1, b_2, \dots, b_{h-1}, b_{h+1}\},$$

where each b_i is an element from N and $i \neq j$ implies $b_i \neq b_j$, $i, j = 1, 2, \dots, h+1$. Because $S_2\mathcal{P}$ does not contain $\{b_1, b_2, \dots, b_{h-1}\}$, but must intersect $S_0\mathcal{P}$ and $S_1\mathcal{P}$ on $h-1$ elements, $S_2\mathcal{P}$ must contain both b_h and b_{h+1} and fail to possess just one elements from b_1, b_2, \dots, b_{h-1} . Since there is nothing to distinguish the possibilities, we may suppose that $S_2\mathcal{P}$ does not possess b_1 , and hence that

$$(3) \quad S_2\mathcal{P} = \{b_2, \dots, b_{h-1}, b_h, b_{h+1}\}.$$

Because $n > 2h$, there are at least $h+2$ sets S_i defined by (1) and so at least $h-1$ sets S_i , where $2 < i \leq n-h$. And the \mathcal{P} images of all these sets must possess b_1, b_h , and b_{h+1} . For suppose $b_1 \notin S_i\mathcal{P}$. Since $S_i\mathcal{P}$ intersects $S_0\mathcal{P}$ on $h-1$ elements and not on b_1 then $\{b_2, b_3, \dots, b_h\} \subset S_i\mathcal{P}$. And since $S_i\mathcal{P}$ intersects $S_1\mathcal{P}$ on $h-1$ elements and not on b_1 , then $\{b_2, \dots, b_h, b_{h+1}\} \subset S_i\mathcal{P}$. But then $S_i\mathcal{P} = \{b_2, \dots, b_h, b_{h+1}\} = S_2\mathcal{P}$, which is impossible for $i \neq 2$. In the same way, $b_h \notin S_i\mathcal{P}$ implies $S_i\mathcal{P} = S_1\mathcal{P}$ and $b_{h+1} \notin S_i\mathcal{P}$ implies $S_i\mathcal{P} = S_0\mathcal{P}$, neither of which is possible for $2 < i \leq n-h$.

From the last argument it follows that for $i > 2$, $S_i\mathcal{P}$ must be of the form

$$(4) \quad S_i\mathcal{P} = \{b_1, b_h, b_{h+1}, x_1, \dots, x_{h-3}\},$$

where $\{x_1, x_2, \dots, x_{h-3}\}$ is a subset of $\{b_2, b_3, \dots, b_{h-1}\}$, which is clearly impossible if $h = 2$. But in any case, there are at least $h-1$ different sets $S_i\mathcal{P}$, where $i > 2$, and each of these is determined by the $h-3$ order subset of $\{b_2, \dots, b_{h-1}\}$ which it contains. And since there are only $h-2$ mutually different such subsets, the sets $S_i\mathcal{P}$, $i > 2$, cannot all be distinct, which contradicts the fact that \mathcal{P} is a one-to-one mapping.

It is now established that for each set T in $\mathcal{S}(h-1)$ there exists a set T' in $\mathcal{S}(h-1)$ such that all the sets in $\mathcal{S}(h)$ which contain T are mapped under \mathcal{P} to all the sets in $\mathcal{S}(h)$ which contain T' . But then the correspondence $T \rightarrow T'$ is clearly a mapping of $\mathcal{S}(h-1)$ onto itself, say the mapping Φ .

For $h = 2$, Φ is a mapping of N onto itself. If $\{a_i, a_j\}$ is any set in $\mathcal{S}(2)$, then $a_i\Phi$ belongs to the \mathcal{P} images of all sets which possess a_i , so $a_i\Phi$ belongs to $\{a_i, a_j\}\mathcal{P}$. By the same argument, $a_j\Phi$

belongs to $\{a_i, a_j\}\mathcal{P}$. Since $a_i\mathcal{P} \neq a_j\mathcal{P}$, it follows that $\{a_i, a_j\}\mathcal{P} = \{a_i\mathcal{P}, a_j\mathcal{P}\}$ and hence that \mathcal{P} is induced by \mathcal{P} .

If $h > 2$, consider any two sets in $\mathcal{S}(h - 1)$, whose intersection is maximal, say

$$(5) \quad T_1 = \{a_1, a_2, \dots, a_{h-2}, a_{h-1}\}, \quad T_2 = \{a_1, a_2, \dots, a_{h-2}, a_h\}.$$

The set $S = \{a_1, a_2, \dots, a_h\}$ in $\mathcal{S}(h)$ maps to a set $S\mathcal{P} = \{b_1, b_2, \dots, b_h\}$. Since T_1 and T_2 are contained in S , $T_1\mathcal{P}$ and $T_2\mathcal{P}$ are $h - 1$ order subsets of $S\mathcal{P}$. Since $T_1 \neq T_2$, and \mathcal{P} is a one-to-one mapping, $T_1\mathcal{P} \neq T_2\mathcal{P}$, so the order of $T_1\mathcal{P} \cap T_2\mathcal{P}$ is $h - 2$. Thus \mathcal{P} preserves maximal intersections and so, by the inductive hypothesis, \mathcal{P} is induced by some mapping Γ of N onto itself.

Now $S = \{a_1, a_2, \dots, a_h\}$ contains T_1 and T_2 defined in (5) so $S\mathcal{P}$ contains $T_1\mathcal{P}$ and $T_2\mathcal{P}$. But $T_1\mathcal{P} = \{a_1\Gamma, a_2\Gamma, \dots, a_{h-1}\Gamma\}$, and $T_2\mathcal{P} = \{a_1\Gamma, \dots, a_{h-1}\Gamma, a_h\Gamma\}$. Since $a_i\Gamma \neq a_j\Gamma$ if $i \neq j$, it follows that $S\mathcal{P} = \{a_1\Gamma, a_2\Gamma, \dots, a_h\Gamma\}$, and hence that \mathcal{P} is induced by Γ .

The theorem is not true for $n = 2h$, since then the correspondence of S_i and S_i^* is a non-induced mapping of $\mathcal{S}(h)$ onto itself which preserves all orders of intersection.¹

Consider next an ordinary, finite graph G , that is, one with n vertices $\{p_1, p_2, \dots, p_n\}$ where each two vertices have at most one join and none is joined to itself. Let $c(p_i, p_j, p_k)$ denote the subgraph of G induced by G on the set of vertices which does not include p_i, p_j, p_k , and let $m(G)$ be the notation for the join-measure of G , that is the number of joins in G .

THEOREM 2. *If G and H are ordinary n th order graphs and if there is a mapping of the vertices of G onto those of H such that for some integer h , $1 < h < n - 1$, all corresponding subgraphs of order h have the same join measure, then the mapping is an isomorphism of G and H .*

Proof. For $h = 2$ the condition becomes the definition of an isomorphism, so assume that $2 < h < n - 1$. Let $\{p_1, p_2, \dots, p_n\}$ be the vertices of G and let the vertices $\{q_1, q_2, \dots, q_n\}$ of H be labeled so that q_i is the image of p_i under the given mapping ψ , $i = 1, 2, \dots, n$.

Let $\{p_{i_1}, p_{i_2}, \dots, p_{i_{h+1}}\}$ be the vertices of any subgraph G_i of order $h + 1$ in G , and let $c(p_{i_k}; G_i)$ denote the subgraph of G_i defined on all the vertices of G_i except p_{i_k} . Since any join in G_i belongs to all the h -order subgraphs of G_i except two, we have,

$$(1) \quad m(G_i) = \frac{1}{h - 1} \sum_{k=1}^{k=h+1} m[c(p_{i_k}; G_i)].$$

¹This general exception was pointed out to the writer by P. Erdős.

By the same reasoning,

$$(2) \quad m(G_i\mathcal{P}) = \frac{1}{h-1} \sum_{k=1}^{k=h+1} m[c(q_{i_k}; G_i\mathcal{P})].$$

Since, by assumption,

$$(3) \quad m[c(p_{i_k}; G_i)] = m[c(q_{i_k}; G_i\mathcal{P})], \quad \text{for all } p_{i_k} \text{ and } q_{i_k},$$

it follows that $m(G_i) = m(G_i\mathcal{P})$.

Thus if \mathcal{P} preserves the join measure on h -order subgraphs it does so on $h+1$ order subgraphs, and, by the same reasoning, preserves the join measure on all subgraphs of order equal to or greater than h . In particular, $m(G) = m(H)$. Then if $\rho(p_i)$ denotes the degree of p_i , it follows from

$$(4) \quad \rho(p_i) = m(G) - m[c(p_i)], \quad i = 1, 2, \dots, n$$

and

$$(5) \quad \rho(q_i) = m(H) - m[c(q_i)], \quad i = 1, 2, \dots, n$$

that

$$(6) \quad \rho(p_i) = \rho(q_i), \quad i = 1, 2, \dots, n,$$

since $m[c(p_i)] = m[c(q_i)]$.

Now, corresponding to p_i and p_j in G , let ε_{ij} be 1 or 0 according as p_i and p_j are or are not joined. Let ε'_{ij} be defined in a similar way with respect to q_i and q_j . Then, by simple counting,

$$(7) \quad m(G) = m[c(p_i, p_j)] + \rho(p_i) + \rho(p_j) - \varepsilon_{ij}, \quad i \neq j,$$

and

$$(8) \quad m(H) = m[c(q_i, q_j)] + \rho(q_i) + \rho(q_j) - \varepsilon'_{ij}, \quad i \neq j.$$

Comparing the terms in (7) and (8) it follows that $\varepsilon_{ij} = \varepsilon'_{ij}$ for all i, j , $i \neq j$, and hence that \mathcal{P} is an isomorphism of G and H .

As a corollary of these theorems it follows that two n th order graphs are isomorphic if and only if there is a one-to-one correspondence of their subgraphs of some order h , $1 < h < n-1$, in which corresponding subgraphs have equal join measure and the correspondence preserves maximal intersections.

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