ON THE CONVERGENCE OF SEMI-DISCRETE ANALYTIC FUNCTIONS

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1. Introduction. In a previous paper [3], the author has presented the basic concepts and definitions for semi-discrete analytic functions. These functions are defined on two types of semi-lattices (sets of lines in the \(xy\)-plane, parallel to the \(x\)-axis)—one of which leads to a symmetric theory. We will concern ourselves here only with the symmetric case. These functions satisfy the following defining equation [3] on a region of the semi-lattice

\[
\frac{\partial f(z)}{\partial z} = \frac{f(z + i\ell/2) - f(z - i\ell/2)}{i\ell},
\]

where \(\ell > 0\) is the spacing of the semi-lattice. For convenience, we will repeat the definition of the symmetric semi-lattice and its associated odd and even semi-lattices. A grid-line, \(a_m\), is the set of points in the \(xy\)-plane such that \(y = mh\) where \(\ell > 0\). The union \(G(2k, \ell)\) of the \(a_m\) for \(m = k (k = 0, \pm 1, \pm 2, \cdots)\) is called the even semi-lattice; the union \(G(2k + 1, \ell)\) of the \(a_m\) for \(m = (2k + 1)/2\) is called the odd semi-lattice. The semi-discrete \(z\)-plane is the union of \(G(2k, \ell)\) and \(G(2k + 1, \ell)\). It will be denoted by \(L(\ell)\). Additional concepts such as domains, paths, path-integrals, etc., are defined in [3]. The following notational conventions will be employed:

\[
f_k = f(x + i\ell k) = f_k(x),
\]

and the abbreviation \(SD\) will be used to stand for semi-discrete.

2. Sub and super harmonic semi-discrete functions. In the continuous case, it is well-known that if a function \(u(x, y)\) is defined over a region \(R\) of the plane and if, further, \(\Delta u \geq 0\) for all \((x, y) \in R\), where \(\Delta\) denotes the two dimensional Laplacian; then \(u(x, y)\) cannot have a maximum on the interior of \(R\). Such a function is said to be sub-harmonic in \(R\) [2]. Similarly, if the function \(u(x, y)\) defined on \(R\) satisfies the equation \(\Delta u \leq 0\) for all \((x, y) \in R\); then \(u(x, y)\) cannot have a minimum on the interior of \(R\). Such a function is said to be super-harmonic in \(R\) [2]. An analogous result holds for semi-discrete functions which are defined on domains of either the even or odd semi-lattice. To be specific, we will consider functions \(u(x, y)\) defined on

domains of $G(2k, h)$ and introduce the notation

\begin{align}
(a) & \quad hEu(x, y) = u(x, y + h) - u(x, y), \\
(b) & \quad h\tilde{E}u(x, y) = u(x, y) - u(x, y - h).
\end{align}

The semi-discrete Laplacian operators for $G(2k)$ is then

\begin{equation}
\mathcal{V}u(x, y) = \frac{\partial^2 u(x, y)}{\partial x^2} + E\tilde{E}u(x, y).
\end{equation}

**Theorem 2.1.** Let $u(x, y)$ be a SD-function defined on a semi-discrete domain $R$ of $G(2k, h)$. If $\mathcal{V}u \geq 0$ for all $(x, y) \in R$, then on $R$

\begin{equation}
u(x, y) \leq M,
\end{equation}

where $M$ is the supremum of $u(x, y)$ on $C$, the boundary of $R$.

**Proof.** The proof of this statement is obtained by a suitable modification of the proof for the "weak maximum theorem" established by Helmbold [1] for semi-discrete harmonic functions. Let $C$ denote the boundary of the SD-domain $R$ of $G(2k, h)$, let $u(x, y)$ be a SD-function on $R$ such that $\mathcal{V}u \geq 0$ for all $(x, y) \in R$, and let $M'$ denote the supremum of $u(x, y)$ on $R$. Assume that $u$ takes the value $M'$ at a point $(t, nh)$ of the interior $R^o = R - C$ of $R$. If the adjacent points $(t, (n \pm 1)h)$ are points of $R^o$, $\partial^2 u/\partial x^2 = u''$ will be continuous at $(t, nh)$ and further $u''(t) \leq 0$. By assumption $\mathcal{V}u(t) \geq 0$ which, together with the previous remarks, implies that

\begin{enumerate}[(a)]
    \item $u_n(t) = u_{n+1}(t) = u_{n-1}(t) = M'$.
\end{enumerate}

This argument may be repeated for the sequence of points $(t, (n \pm 1)h)$, $(t, (n \pm 2)h), \cdots$ until a point $(t, ph)$ is reached such that one of its adjacent points is a point of $C$. If $u''_p$ is continuous, the proof is complete. Otherwise, since $u''_p$ is then at least piecewise continuous, integration of $\mathcal{V}u_p \geq 0$ shows that for some range of values of $\varepsilon > 0$

\begin{enumerate}[(b)]
    \item $u'_p(t + \varepsilon) - u'_p(t) \geq \varepsilon h^{-2} [2u_p(\theta) - u_{p+1}(\theta) - u_{p-1}(\theta)],$
\end{enumerate}

where $t \leq \theta \leq t + \varepsilon$. Since $u_p = M'$ is a maximum, the left side of (b) is negative. Hence, the bracketed term is negative. Taking the limit of this term as $\varepsilon \to 0$, $\varepsilon > 0$ shows that

\begin{enumerate}[(c)]
    \item $2M' \leq u_{p+1}(t^+) + u_{p-1}(t^+).$
\end{enumerate}

Similarly, we obtain

\begin{enumerate}[(d)]
    \item $2M' \leq u_{p+1}(t^-) + u_{p-1}(t^-).$
\end{enumerate}

Addition of (c) and (d) shows that $M' \leq M$ where $M$ is the maximum
value of \( u(x, y) \) on \( C \).

In an identical manner, we establish the following result for super SD-harmonic functions.

**Theorem 2.2.** Let \( u(x, y) \) be a SD-function defined on a semi-discrete domain \( R \) of \( G(2k, h) \). If \( \forall u \leq 0 \) for all \( (x, y) \in R \), then on \( R \)

\[
(2.4) \quad u(x, y) \geq m,
\]

where \( m \) is the infimum of \( u(x, y) \) on \( C \), the boundary of \( R \).

3. Limit theorem for semi-discrete analytic functions. A SD-function \( f(z) \) of the complex variable \( z = x + inh \) which is continuous and single-valued on a SD-domain \( R \) of \( L(h) \) is said to be SD-analytic if it satisfies (1.1) for all points \( z \in R \) [3]. In addition, if we write \( f = u + iv \), then \( \forall u = \forall v = 0 \) on \( R \); that is, \( u \) and \( v \) are SD-harmonic.

Let us suppose that \( L(h) \) is superimposed upon the continuous \( z \)-plane, denoted by \( L_\epsilon \), with their \( x \) and \( y \) axes coinciding. Let \( R_\epsilon \) be a simply-connected finite domain of \( L_\epsilon \) whose boundary is a Jordan curve. A covering set of rectangles, \( Q_\kappa \), is defined as follows,

\[
Q_\kappa = \{ (x, y) : \alpha_\kappa \leq x \leq \beta_\kappa; (kh - h) \leq 2y \leq (kh + h) \},
\]

where \( \alpha_\kappa \) is the least value of \( x \) in \( R \) taken on the strip \( kh - h \leq 2y \leq kh + h \), and \( \beta_\kappa \) is the greatest value of \( x \) in \( R \) on this strip.

By construction, each point of \( R_\epsilon \) is also a point of \( Q = \bigcup Q_\kappa \). The intersection of \( Q \) with \( L(h) \) forms a SD-domain, \( R(h) \), which approximates \( R_\epsilon \). We consider the sequence of SD-domains \( \{ R(h_j) ; h_1 > h_2 > \cdots \} \) obtained by the above procedure upon successive refinements of the semi-lattice retaining at each step the lines of the previous semi-lattice. In the limit, \( R(h_3) \rightarrow R_\epsilon \). It is shown in [3] that a SD-analytic function is completely determined in \( R(h) \) by its values on \( C(h) \), the total-boundary of \( R(h) \). Therefore, let us assume that an interpolation scheme is established to provide such boundary values for a SD-analytic function \( f^{(h_j)}(z) \) on \( R(h) \) from the boundary values of an analytic function \( \zeta(z) \) on \( R_\epsilon \) such that these approximate boundary values tend uniformly to the true boundary values. We consider the sequence of SD-analytic functions \( \{ f^{(h_j)}(z) \} \) so determined on \( \{ R(h_j) \} \) and will prove that as \( h_j \rightarrow 0 \), \( f^{(h_j)}(z) \rightarrow \zeta(z) \).

**Theorem 3.1.** Let \( R \) be a domain whose boundary \( C \) is a Jordan curve and let \( R' \) be a subdomain of \( R \) which is bounded by a Jordan curve \( C' \subset R \). Consider the set of all possible semi-lattices \( G(2k, h) \) parallel to the real axis of the \( z \)-plane. Consider also the set of all SD-functions \( u^{(h)}(x, y) \) which are uniformly bounded, \( |u| \leq M \) in \( R \),
and which satisfy in \( R \) the equation \( \nabla u = 0 \). Then, for \( h \) sufficiently small, there exists a constant \( M' \) such that

\[
|\frac{\partial u^{(k)}}{\partial x}| \leq M' \quad \text{and} \quad |\nabla u^{(k)}| \leq M'
\]

for all \((x, y) \in R\).

**Proof.** The proof of this statement follows the proof given by Fellows [4] for the discrete case. The sub-domain \( R' \) can be covered by a finite number of rectangles contained in \( R \) and each of these rectangles can be inclosed in a larger rectangle also contained in \( R \). Following the argument of Feller [4], it will be sufficient to consider, for an arbitrary \( \delta > 0 \), the two concentric rectangles

\[
R = \{(x, y) : |x| < a - \delta, |y| < b\} \\
R' = \{(x, y) : |x| < a - \delta, |y| < b - \delta/3\}
\]

where \( b \) is a multiple of the gap \( h \), and \( h < \delta/3 \).

To prove the assertion, we shall show that the function

\[
\psi(x, y) = \left(\frac{\partial u}{\partial x}\right)^2 \Phi(x, y) + C\{u^i(x, y) + u^i(x, y + h) + u^i(x, y - h)\}
\]

where \( \Phi(x, y) = (x^2 - a^2)(y^2 - b^2) \) and \( C \) is a large positive constant, to be determined later, satisfies the inequality \( F(\psi) \geq 0 \).

Assume for the moment that this has been established. Then, by Theorem 2.1, it follows that \( \psi \) attains its maximum value on the boundary. However, by definition, \( \Phi = 0 \) on the boundary and thus in the entire rectangle

\[
0 \leq \psi(P) \leq 3CM^3
\]

where \( M \) is the uniform bound on \( u \). Since the second term of \( \psi \) is nonnegative, we may conclude that for all \( P \in R' \)

\[
\left(\frac{\partial u}{\partial x}\right)^2 \Phi \leq 3CM^3 |\Phi| \leq 3CM^3/\delta/3)
\]

[since for small \( \delta \), \( \Phi \geq (\delta/2)^4(\delta/3)^4 \geq (\delta/3)^4 \)].

Since \( (\delta/3)^4 > 0 \), taking the last expression for \( M' \) establishes the theorem, subject to showing that \( F(\psi) \geq 0 \). Only the outline of this calculation will be presented. The complete sequence of steps follows the argument given by Feller [4] using the differential rather than the difference operator on \( x \).

Calculation of \( F\psi \) using the fact that \( u \) is SD-harmonic [as is \( u' \)] gives
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\[ F(\psi) = (u')^2 \psi(\Phi) + \Phi[2(u'')^2 + (Eu')^2 + (\tilde{E}u')^2] \]
\[ + \Phi'(4u'u'') + E\Phi[u'_t(Ew' + u'Ew') \]
\[ + \tilde{E}\Phi[u'_t(E\tilde{u}' + u'\tilde{E}u')] + C[2(u')^2 + (Eu')^2 + (\tilde{E}u')^2] \]
\[ + C(2(u'_t)^2 + (Ew'_t)^2 + (\tilde{E}u'_t)^2 + 2(u'_t)^2 + (Ew)'' + (\tilde{E}u)'' + (E\tilde{u})''] \]

where \( u_{\pm 1} = u(x, y \pm h) \). Since \( |\partial \Phi/\partial x| = 4|x(y^3 - b^3)|v_\Phi \), a constant \( \lambda \) exists such that for all points of \( R |\Phi'| < \lambda \sqrt{\Phi} \). Similar bounds exist for \( E\Phi \) and \( \tilde{E}\Phi \). Further, in \( R, F(\Phi) \) is bounded. Accordingly we assume that \( \lambda \) is so chosen that on \( R \)
\[ |F(\Phi)| < \lambda, \quad |\Phi'| < \lambda \sqrt{\Phi}, \quad |E\Phi| < \lambda \sqrt{\Phi}, \quad |\tilde{E}\Phi| < \lambda \sqrt{\Phi}. \]

For an arbitrary \( \varepsilon > 0 \), we see that
\[ |u'u''\Phi'| \leq \left( \frac{u'}{\varepsilon} \right)^2 + \varepsilon^3 \lambda^2 (u''\Phi')^2. \]

With such bounds established for the various terms which appear in (a), the following inequality is obtained.

\[ F(\psi) \geq [(Eu')^2 + (\tilde{E}u')^2 + 2(u''\Phi')] \Phi(1 - 2\varepsilon^2 \lambda^2) \]
\[ + 2(u')^2[C - 3/\varepsilon^2] + C[(Eu')^2 + (\tilde{E}u')^2 + (Ew')^2] \]
\[ + C[(\tilde{E}u')^2 + (Ew')^2 + (\tilde{E}w')^2] + (u'_t)^2[2C - 1/\varepsilon^2] \]
\[ + (\tilde{u}_t)^2[2C - 1/\varepsilon^2]. \]

Selecting \( \varepsilon \) so that \( \varepsilon^2 \lambda^2 = 1/2 \), the first term on the right in (b) vanishes. Finally, choosing \( C \geq 3/\varepsilon^2 \), the remaining terms on the right in (b) will be positive. That is, \( F(\psi) \geq 0 \).

**Theorem 3.2.** Let \( \{u^{(h)}(x, y)\} \) be the set of uniformly bounded SD-functions considered in Theorem 3.1. This set is a family of equi-continuous functions on \( R \).

**Proof.** In Theorem 3.1 we established the existence of a uniform bound for the set \( \{\partial u^{(h)}/\partial x\} \) and also \( \{Ew^{(h)}\} \). Let \( M \) denote this bound. (1) Given \( \varepsilon > 0 \), let \( P, Q \) be two points on a line of the semi-lattice such that \( PQ < \varepsilon/M \); that is, \( |x_P - x_Q| < \varepsilon/M \), where \( x_P \) denotes the \( x \)-coordinate of \( P \) and \( x_Q \) denotes the \( x \)-coordinate of \( Q \). Then
\[ |u^{(h)}(P) - u^{(h)}(Q)| = \left| \int_{x_Q}^{x_P} \frac{\partial u^{(h)}}{\partial t} \, dt \right| \leq [M^2(x_P - x_Q)^2]^{1/2} \leq \varepsilon. \]

(2) Given \( \varepsilon > 0 \), let \( P, Q \) be two points of \( R \) which lie on a vertical line in \( R \) such that \( |y_P - y_Q| < \varepsilon/Mh \).
\[ |u^{(h)}(P) - u^{(h)}(Q)| = \varepsilon \left| \sum_{y_p, y_q} Ew^{(h)} \right|. \]
Thus,

\[ |u^{(k)}(P) - u^{(h)}(Q)| \leq |y_P - y_Q| Mh \leq \varepsilon. \]

(3) Given \( \varepsilon > 0 \), let \( P, Q \) be two arbitrary points of \( R \) such that \( PQ < \delta(\varepsilon) \). Let \( P' \) lie on the same vertical line as \( P \) and have the same \( y \)-coordinate as \( Q \); i.e., \( P' = (x_P, y_Q) \). Then

\[ |u^{(k)}(P) - u^{(h)}(Q)| \leq |u^{(h)}(P) - u^{(h)}(P')| + |u^{(h)}(P') - u^{(h)}(Q)|. \]

Application of the two previous cases completes the proof.

By Theorem 3.2, if \( \{f^{(h)} = u^{(h)} + iv^{(h)}\} \) is a set of uniformly bounded SDA functions, this set is a family of equicontinuous functions which, by Kellogg [2], contains a subsequence converging uniformly in \( R' \) to a continuous limit. Since \( R' \) was an arbitrary closed sub-domain of \( R \), we can choose a sequence of such regions \( R' \subset R'' \subset \cdots \subset R \) whose sum is \( R \) and find successive subsequences of \( f^{(h_1)}, f^{(h_2)}, \ldots \) which converge in each of these regions to a continuous function. The resultant diagonal subsequence will converge uniformly to a continuous function in all of \( R \). Since successive differences and derivatives of SD-harmonic functions are again SD-harmonic, the arguments in Theorems 3.1 and 3.2 can be repeated to show that there is a subsequence of the final subsequence whose first derivative and first difference ratio also converge in \( R \). Thus, we can find a final subsequence which will have an arbitrary number of successive derivatives or differences which converge in \( R \). Denote this final convergent subsequence by \( \{f_\ast^{(h)}\} \) and let \( \zeta(z) \) be the continuous function in \( R \) to which it converges.

Let \( C \) be a rectifiable curve in \( L_\ast \). By the construction of \( Q \), each point of \( C \) is a point of \( Q \). Consider a rectangle \( Q_\ast \) of \( Q \) which contains a segment \( C_\ast \) of \( C \). To be explicit, we will assume that \( C_\ast \) intersects \( Q_\ast \cap L(h) \) at the three points \( p_1 = (x_1, h(k - 1)/2), p_2 = (x_2, h(k + 1)/2), \) and \( p_3 = (x_3, h(k - l)/2) \), and that the positive direction is from \( p_1 \) to \( p_3 \). The remaining possibilities can be treated by suitable modifications of the following discussion. On \( Q_\ast \cap L(h) \), three SD-paths may be defined. The upper SD-path consists of the points \( p_1, (x_1, h(k/2)), \) and the line segment from \( x_1 \) to \( x_2 \) with \( y = h(k + 1)/2 \). The lower SD-path is the line segment from \( x_2 \) to \( x_3 \) with \( y = h(k - 1)/2 \), the points \( (x_3, h(k)/2) \), and \( p_3 \). The mixed SD-path consists of the line segment from \( x_1 \) to \( x_2 \) with \( y = h(k - 1)/2 \), the point \( p_2 \), and the line segment from \( x_2 \) to \( x_3 \) with \( y = h(k + 1)/2 \). At least one of these SD-paths must lie within \( R(h) \) and will be chosen to be the SD approximation of the segment \( C_\ast \). The SD-Cauchy theorem [3] shows that it is immaterial which SD-path is chosen if more than one of these approximating SD-paths lies within \( R(h) \). The SD-path on \( R(h) \) which approximates \( C \) is the union of the SD-paths chosen to approximate its segments, \( C_\ast \).
THEOREM 3.3. Let \( \zeta(z) \) be a continuous function on a domain \( R \) and let \( C \) be a rectifiable [or Jordan] curve which is contained in \( R \). If \( C_h \) is a SD-path contained in \( R_h \) which approximates \( C \), then

\[
\lim_{h \to 0} \int_{C_h} \zeta(z) \, dz = \int_C \zeta(z) \, dz.
\]

Proof. By the definition for SD-path integration [3],

\[
\int_{C_h} \zeta \, dz = \sum_{p=M}^{N-1} \int_{x_p}^{x_{p+1}} \zeta(t) \, dt + i h \sum_{p=M}^{N-2} \zeta_{p+1/2}(x_{p+1}) ,
\]

where \( C_h \) is a SD-path joining \( z_M = x_M + iM \) and \( z_N = x_N + iN \). We note that as \( h \to 0 \), so must \( |x_p - x_{p+1}| \to 0 \). Since \( \zeta \) is continuous, there exists a value \( \lambda_p \) where \( x_p \leq \lambda_p \leq x_{p+1} \) such that

\[
\int_{C_h} \zeta \, dz = \sum_{p=M}^{N-1} [x_{p+1} - x_p] \zeta(\lambda_p) + i h \sum_{p=M}^{N-2} \zeta_{p+1/2}(x_{p+1}) .
\]

As \( h \to 0 \) the right side of the above converges to the value of the path-integral of the continuous function \( \zeta \) along the path \( C \).

THEOREM 3.4. Let \( R(h) \) denote a sequence of semi-lattices on a domain \( R \) such that \( h \to 0 \), and let \( f^{(h)} \) be semi-discrete analytic on \( R(h) \). If the collection of these \( f^{(h)} \) is uniformly bounded in \( R \), then it contains a subsequence that converges everywhere in \( R \) to a function \( \zeta(z) \) that is analytic in \( R \).

Proof. This subsequence is the final subsequence obtained in the previous discussion. Let \( C \) denote an arbitrary closed rectifiable path in \( R \) and let \( C_h \) be a closed SD-path on \( R(h) \) which approximates \( C \). Then
(a) \[ \lim_{h \to 0} \oint_{C_h} f_\ast^{(h)} \, dz = \oint_{C} \zeta(z) \, dz , \]

where \( \{f_\ast^{(h)}\} \) is the subsequence which converges to \( \zeta \). To establish (a) we consider

(b) \[ \left| \oint_{C_h} f_\ast^{(h)} \, dz - \oint_{C} \zeta(z) \, dz \right| \leq \left| \oint_{C_h} (f_\ast^{(h)} - \zeta) \, dz \right| + \left| \oint_{C_h} \zeta \, dz - \oint_{C} \zeta \, dz \right| . \]

Since \( f_\ast^{(h)} \to \zeta \), given \( \varepsilon > 0 \) there exists \( \delta_1(\varepsilon) > 0 \) such that the first term on the right side of (b) is smaller than \( \varepsilon/2 \) provided \( h \leq \delta_1 \).

Similarly by Theorem 3.3, there exists \( \delta_2(\varepsilon) > 0 \) such that the second term on the right side of (b) is smaller than \( \varepsilon/2 \) provided \( h \leq \delta_2 \).

Thus, on letting \( \delta = \max(\delta_1, \delta_2) \)

(c) \[ \left| \oint_{C_h} f_\ast^{(h)} \, dz - \oint_{C} \zeta \, dz \right| < \varepsilon , \]

provided \( h \leq \delta \). This establishes (a). However, since \( f_\ast^{(h)} \) is SDA for each \( h \), the left side of (a) is always zero. Thus

(d) \[ \oint_{C} \zeta(z) \, dz = 0 . \]

Since \( C \) is an arbitrary closed rectifiable curve of \( R \) and \( \zeta \) is continuous, by Morera’s theorem \( \zeta(z) \) is analytic in \( R \).

To complete the discussion we must show that the limit function \( \zeta(z) = U(z) + iV(z) \) of the chosen subsequence \( \{f_\ast^{(h)}\} \) satisfies the given boundary condition \( \zeta = \psi(s) \) on \( C \), the boundary of \( R \). It is sufficient for this purpose to consider the real-valued function \( U = \text{Re}\{\zeta\} \) and show that \( U = \text{Re}\{\psi(s)\} \) on \( C \). Let \( Q \) be a fixed point of \( C \). By hypothesis we can construct a circle lying outside \( C \) and intersecting \( C \) only at the point \( Q \), see Feller [4]. We denote the center of this circle by \( A \), its radius by \( \rho \), and let \( P \) denote an arbitrary point of \( R \) whose distance from \( A \) is \( r \).

For an arbitrary \( \varepsilon > 0 \), we define the functions [4]

(3.2) \[ U_i(P) = F(Q) + \varepsilon + K\left( \frac{1}{\rho} - \frac{1}{r} \right) , \]

and

(3.3) \[ \varphi U_i(P) = -K[r^{-3} + 0(h)] < 0 , \]

where \( F = \text{Re}\{\psi\} \) and \( K \) is a positive constant to be determined later. On any semi-lattice
in $\mathcal{R}$ provided that $h$ is sufficiently small. Now if $u(P)$ is a solution of the differential-difference equation $\nabla u = 0$ for the semi-lattice, by (3.3) the function $U_i(P) - u(P)$ is SD super-harmonic for $P \in \mathcal{R}$. Accordingly, by Theorem 2.2, it assumes its minimum on $C$. Similarly, the function $U_i(P) - u(P)$ is SD sub-harmonic and by Theorem 2.1 assumes its maximum on $C$.

The argument given by Feller [4] now applies directly. We consequently establish that

$$\lim_{P \to Q} U(P) \leq F(Q),$$

and

$$\lim_{P \to Q} U(P) \geq F(Q),$$

which completes the proof.

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