POLYNOMIALS WITH MINIMAL VALUE SETS

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Let \( \mathbb{F} \) be a finite field of characteristic \( p \) that contains exactly \( q \) elements. Let \( F(x) \) be a polynomial over \( \mathbb{F} \) of degree \( f \), \( f > 0 \), and let \( r + 1 \) denote the number of distinct values \( F(\tau) \) as \( \tau \) ranges over \( \mathbb{F} \). Carlitz, Lewis, Mills, and Straus [1] pointed out that \( r \geq [(q - 1)/f] \), and raised the question of determining all polynomials for which \( r = [(q - 1)/f] \). The cases \( r = 0 \) and \( r = 1 \) are special cases that do not fit into the general pattern. These are treated in [1], and do not concern us here. Thus we arrive at the statement of our main problem: For what polynomials \( F(x) \) do we have

\[
(1) \quad r = [(q - 1)/f] \geq 2?
\]

Carlitz, Lewis, Mills, and Straus [1] determined all polynomials with \( f < 2p + 2 \) for which (I) holds. In the present paper this result is extended—all polynomials with \( f \leq \sqrt{q} \) for which (I) holds are determined. These are polynomials of the form

\[
F(x) = \alpha L^v + \gamma,
\]

where \( L \) is a polynomial that factors into distinct linear factors over \( \mathbb{F} \) and that has the form

\[
L = \beta + \sum_i \varphi_i x^{p^{ki}},
\]

and where \( v \) and \( k \) are integers such that \( v \mid (p^k - 1) \) and \( q \) is a power of \( p^k \). Regardless of the size of \( f \) our present methods give a great deal of information about \( F(x) \). Furthermore many of the proofs of [1] can be shortened and simplified by using the results of \( \S \) 1 of the present paper.

The results of [1] provide a complete answer for the case \( q = p \). In the present paper the problem is completely solved for the case \( q = p^2 \).

1. Preliminaries. Let \( \mathbb{F} \) be a finite field with \( q \) elements and characteristic \( p \). We use Greek letters for elements of \( \mathbb{F} \), and small Latin letters, other than \( x \), for nonnegative integers. We use capital letters for polynomials in one variable over \( \mathbb{F} \). The polynomials denoted by \( A, B, C, D, E \) and the integers denoted by \( a, b, c, d, e \)

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vary from proof to proof. The polynomials and integers denoted by other letters, except \( i \) and \( j \), remain the same throughout the paper.

Let \( F = F(x) \) be a polynomial over \( \mathbb{F} \) of degree \( f, f > 0 \). Let \( \gamma_0, \gamma_1, \ldots, \gamma_r \) denote the distinct values assumed by \( F(\tau) \) as \( \tau \) ranges over \( \mathbb{F} \). It follows easily from the fact that a polynomial of degree \( f \) has at most \( f \) roots, that \( r + 1 \geq q/f \). This is equivalent to \( r \geq [ (q - 1)/f ] \). We intend to study the question raised in [1] of characterizing those polynomials for which \( r = [ (q - 1)/f ] \). The cases \( r = 0 \) and \( r = 1 \) were fully treated in [1]. Hence we make the assumption that

\[
(1) \quad r = [ (q - 1)/f ] \geq 2.
\]

Subtracting the constant \( \gamma_0 \) from \( F \) does not change the value of \( r \). Thus it is sufficient to consider the case \( \gamma_0 = 0 \). In the first two sections of this paper, we assume that

\[
\gamma_0 = 0.
\]

Then \( \gamma_i \neq 0 \) for \( i > 0 \). We now set

\[
F_i = F - \gamma_i, \quad 0 \leq i \leq r.
\]

The polynomials \( F_i \) are relatively prime in pairs, and each of them has at least one root in \( \mathbb{F} \). Let \( \pi_{i1}, \pi_{i2}, \ldots, \pi_{ir} \) be the distinct roots of \( F_i \) that lie in \( \mathbb{F} \) and set

\[
L_i = \prod_{j=1}^{i} (x - \pi_{ij}), \quad 0 \leq i \leq r.
\]

Then \( l_i = \deg L_i \geq 1, \ 0 \leq i \leq r, \) and

\[
(2) \quad x^q - x = \prod_{i=0}^{r} L_i.
\]

Now set \( F_i = L_i U_i, \ 0 \leq i \leq r, \) and

\[
(3) \quad G = \prod_{i=0}^{r} U_i.
\]

Then the \( L_i \), the \( U_i \) and \( G \) are polynomials over \( \mathbb{F} \), and

\[
(4) \quad (x^q - x)G = \prod_{i=0}^{r} F_i.
\]

Now (4) and (1) give us an upper bound on the degree of \( G \), namely

\[
\deg G = (r + 1)f - q \leq q - 1 + f - q = f - 1.
\]

\[1\] The relations (2), (3), (4), (5), (6), and (7) can all be found in [1] under the assumption that the leading coefficient of \( F \) is 1.
Thus we have

(5) \[ \deg G < f. \]

Set \( u_i = \deg U_i, 0 \leq i \leq r. \) We already have \( F = F_0 \) by the assumption \( \gamma_0 = 0. \) We set \( L = L_0, \ U = U_0, \ l = l_0, \) and \( u = u_0. \)

We now differentiate both sides of (2) and obtain \(-1 \equiv L'L^\ast ( \mod L), \) where \( L^\ast = L_1 L_2 \cdots L_r. \) Hence \( G \equiv -L'L^\ast G \pmod{LG}. \) Since \( F = LU \) and \( U \mid G, \) it follows that \( F \mid LG \) and thus

\[ G \equiv -L'L^\ast G \pmod{F}. \]

Now

\[ L^\ast G = U \prod_{i=1}^{r} (L_i U_i) = U \prod_{i=1}^{r} (F - \gamma_i) \equiv -\zeta U \pmod{F}, \]

where

\[ \zeta = -\prod_{i=1}^{r} (-\gamma_i) \neq 0. \]

Hence \( G \equiv \zeta L'U \pmod{F}. \) Since \( \deg (\zeta L'U) < \deg (LU) = f \) and \( \deg G < f, \) we must have

(6) \[ G = \zeta L'U. \]

By symmetry it follows that

(7) \[ G = \zeta_i L'_i U_i, \quad 0 \leq i \leq r, \]

for suitable nonzero elements \( \zeta_i \) of \( \mathcal{K}. \)

We next derive another expression for \( G. \)

**Lemma 1.** There exists a nonzero element \( \theta \) in \( \mathcal{K} \) such that \( G = \theta F''. \)

**Proof.** Since \( F'' = F'_i = L_i U_i + L_i U'_i, \) it follows from (7) that

\[ L_i U'_i = F'' - G/\zeta_i, \quad 0 \leq i \leq r. \]

Therefore \( L_0 U'_0 = LU', \ L_i U'_i, \) and \( L_2 U'_2 \) are linearly dependent. Thus there exist \( \lambda, \lambda_1, \) and \( \lambda_2 \) in \( \mathcal{K}, \) not all zero, such that

\[ \lambda LU' + \lambda_1 L_1 U'_1 + \lambda_2 L_2 U'_2 = 0. \]

Multiplying this relation by \( UU_1 U_2 \) and noting that \( LU = F', \ L_1 U_1 = F - \gamma_1, \ L_2 U_2 = F - \gamma_2, \) we obtain

(8) \[ (\lambda U'U_1 U_2 + \lambda_1 UU'_1 U_2 + \lambda_2 UU'_1 U_2)F = \lambda_1 UU'_1 U_2 + \lambda_2 \gamma_2 UU_1 U_2'. \]

Now the degree of the right side of (8) is less than \( u + u_1 + u_2 \) and
\[ u + u_1 + u_2 \leq \deg G < f = \deg F. \]

This is possible only if we have

\[ (9) \quad \lambda U'U_1U_2 + \lambda_1UU_1U_2 + \lambda_2UU_1U_2' = 0. \]

The constants \( \lambda, \lambda_1, \) and \( \lambda_2 \) are not all zero. Without loss of generality we suppose \( \lambda_2 \neq 0. \) Then (9) gives us \( U_2 \mid UU_1U_2'. \) Since \( U_2 \mid F, \) \( U_2 \) must be relatively prime to both \( F \) and \( F_1. \) Hence \( U_2 \) is relatively prime to \( UU_1, \) and \( U_2 \mid U_2'. \) This implies that \( U_2' = 0. \) Hence

\[ F' = F_2' = L_2U_2 + L_2U_2' = L_2U_2 = G/\zeta_2. \]

Thus \( G = \zeta_2F', \) which completes this proof.

Lemma 1 is false for \( r \leq 1— \) counter examples can be readily constructed.

**Lemma 2.** For each \( j, \) \( 0 \leq j \leq r, \) \( U_j \) is of the form

\[ U_j = L_j^wH_j^p, \]

where \( w_j \) is a nonnegative integer, \( H_j \) is a polynomial over \( \mathbb{R}, \) and \( L_j \nmid H_j. \)

**Proof.** By symmetry it is sufficient to prove the lemma for the case \( j = 0. \) Combining (6) with Lemma 1 we obtain

\[ \zeta L'U = G = \theta F' = \theta L'U + \theta LU'. \]

Thus

\[ (10) \quad \theta LU' = (\zeta - \theta)L'U. \]

We set \( U = L^wA, \) where \( L \nmid A \) and \( w \geq 0. \) Then substitution in (10) gives us

\[ \theta wL^wL'A + \theta L^w+1A' = (\zeta - \theta)L'L^wA. \]

This reduces to

\[ \theta LA' = (\zeta - \theta - w\theta)L'A. \]

Thus \( L \mid (\zeta - \theta - w\theta)L'A. \) Since \( L \) is the product of distinct linear factors, it follows that \( L \) and \( L' \) are relatively prime. Since \( L \nmid A, \) this implies that \( \zeta - \theta - w\theta = 0. \) Therefore \( \theta LA' = 0. \) It follows that \( A' = 0. \) Hence \( A = H^p \) for some polynomial \( H. \) Then we have \( L \nmid H \) and \( U = L^wH^p, \) which completes this proof.

We now suppose, without loss of generality, that

\[ (11) \quad l \leq l_j, \quad 0 \leq j \leq r. \]
**Lemma 3.** Under the assumption (11), the constants \( w_j \) of Lemma 2 satisfy
\[
w_1 = w_2 = \cdots = w_r = 0.
\]

*Proof.* Combining (3) and (6) we obtain
\[
\zeta L'U = G = UU_1U_2 \cdots U_r.
\]
Now suppose \( 1 \leq j \leq r \). Then \( U_j \parallel L' \), and hence
\[
u_j \leq \deg L' < l \leq l_j.
\]
Therefore \( L_j \nmid U_j \), so that we have \( w_j = 0 \). This completes the proof.

Set \( H = H_0 \) and \( v = w_0 + 1 \). Then from Lemmas 2 and 3 we obtain
\[
(12) \quad F = LU = L^vH^p,
\]
and
\[
(13) \quad F_i = L_iU_i = L_iH_i^p, \quad 1 \leq i \leq r,
\]
where \( L \nmid H, \quad L_i \nmid H_i \). Moreover
\[
\zeta L' = G|U = UU_1U_2 \cdots U_r = (H_1H_2 \cdots H_r)^p.
\]
Thus \( L' = S^p \), where \( S = \zeta^{-1/p}H_1H_2 \cdots H_r \). Therefore \( L \) is of the form
\[
(14) \quad L = xS^p + T^p,
\]
where \( T \), as well as \( S \), is a polynomial over \( \mathcal{K} \).

2. **The polynomial** \( R(x) \). Set
\[
R(x) = \prod_{i=1}^r (x - \gamma_i) = \sum_{j=0}^r \rho_j x^j,
\]
where \( \rho_j \in \mathcal{K}, \quad 0 \leq j \leq r, \quad \rho_r = 1 \). From (4) and (6) we obtain
\[
LUR(F) = FR(F) = \prod_{i=0}^r F_i = (x^q - x)G = \zeta(x^q - x)L'U.
\]
These identities and (12) give us
\[
\sum_{j=0}^r \rho_j L^{1+sj}H^{pj} = LR(F) = \zeta(x^q - x)L'.
\]
Differentiating both sides of (15) and noting that \( L'' = 0 \) by (14), we get the congruence
\[ \rho_0 L' \equiv -\zeta L' \pmod{L}. \]

Since \( L' \neq 0 \), we obtain

(16) \[ \rho_0 = -\zeta. \]

By Lemma 1 we have \( F' = G/\theta \neq 0 \). Hence \( p \nmid v \).

Let \( k \) be the smallest positive integer such that \( v \mid (p^k - 1) \). The main objective of this section is to show that \( 1 + vj \) is a power of \( p^k \) for every nonzero coefficient \( \rho_i \) of \( R(x) \).

In the proof of the following lemma the notation \( A \upharpoonright B \) means that \( A \mid B \) and \( (A, B/A) = 1 \).

**Lemma 4.** Let \( d \) be a nonnegative integer such that \( L' \) is a \( p^d \)-th power and \( 1 + vr > p^{d-1} \). If \( j \) is an integer such that \( \rho_j \neq 0 \), then either (i) \( 1 + vj \) is a power of \( p^k \), or (ii) \( p^a \mid (1 + vj) \). Moreover \( H \) is a \( p^{d-1} \)-st power.

**Proof by induction on \( d \).** The desired result is trivial for \( d = 0 \). We suppose that it is true for an integer \( d \) and show that this implies that it is true for \( d + 1 \). Thus we assume that \( L' \) is a \( p^d \)-st power and \( 1 + vr > p^d \). Then the induction hypothesis applies so that \( R(x) \) is of the form

(17) \[ R(x) = \sum_{i=0}^{c} \omega_i x^{(p^k i - 1)/y} + \sum' \rho_j x^j, \]

where \( \omega_i \in \mathcal{N}, \ 0 \leq i \leq c, \ c = [d/k] \), and the second summation \( \sum' \) is over all \( j \) such that

\[ p^d \mid (1 + vj), \ p^d < 1 + vj, \ j \leq r. \]

Moreover \( H \) is a \( p^{d-1} \)-st power. Thus

\[ H = A^{p^{d-1}} \quad \text{and} \quad F = L^a A^{p^d} \]

for some polynomial \( A \) over \( \mathcal{N} \). Substitution in (15) gives us

(18) \[ \sum' \rho_j L^{1+vj} A^{p^d} = \zeta x^a L' + B, \]

where

\[ B = -\zeta x L' - \sum_{i=0}^{c} \omega_i L^{p^k i} A^{p^d (p^k i - 1)/y}. \]

The left side of (18) is a \( p^d \)-th power. Since

\[ q \geq 1 + fr \geq 1 + vr > p^d \]

and \( q \) is a power of \( p \), it follows that \( p^{d+1} \mid q \). Hence \( \zeta x^a L' \) is a \( p^{d+1} \)-st power. Therefore \( B \) is a \( p^{d+1} \)-th power. Thus we can set
\[ \zeta x^q L' = C^{p^d+1} \text{ and } B = D^{p^d}. \]

Since \( 1 + vr > p^d \) and \( \rho_r \neq 0 \), it follows that the left side of (18) does not vanish identically. Let the term corresponding to \( j = a \) be the nonzero term of lowest degree in the left side of (18). Thus \( a \) is the least integer such that \( \rho_a \neq 0 \) and \( 1 + va > p^d \). Then \( p^d | (1 + va) \), and hence \( 1 + va \geq 2p^d \). Because of the way \( a \) was chosen we have

\[ L^{1+va} A^{a^d} \| (\zeta x^q L' + B). \]

Extracting the \( p^d \)th roots of both sides of (19) we get

\[ L^{(1+va)p^{-d}} A^a \| (C^p + D). \]

Since \( 1 + va \geq 2p^d \) this gives us \( L^a A^a | (C^p + D) \). By differentiation we obtain

\[ LA^{a-1} | D'. \]

Now

\[ \deg D' < p^{-d} \deg B \leq p^{-a} \deg \{L^{p^k a} A^{p^d(p^{k-1})/v}\} \leq \deg \{LA^{(p^{k-1})/v}\}. \]

Since

\[ a > (p^d - 1)/v \geq (p^{k-1} - 1)/v, \]

we have \( (p^{k-1} - 1)/v \leq a - 1 \), and

\[ \deg D' < \deg (LA^{a-1}). \]

Combining this with (20) we get \( D' = 0 \). Thus \( D \) must be a \( p^d \)th power, and \( B \) a \( p^{d+1} \)st power. Thus the right side of (19) is a \( p^{d+1} \)st power. Hence the left side of (19) is also a \( p^{d+1} \)st power. Now \( L \mid H \). Since \( L \) is the product of distinct linear factors we have \( L \mid A \), \( p^{d+1} | (1 + va) \), and \( A^a \) is a \( p^d \)th power. Hence \( p \mid a \), and \( A \) itself is a \( p^d \)th power. It follows that \( H \) is a \( p^d \)th power. Suppose there is a \( b \) such that \( \rho_b \neq 0 \), \( 1 + vb \) is not a power of \( p^{k} \), and \( p^{d+1} \mid (1 + vb) \). Without loss of generality suppose that \( b \) is the smallest integer with these properties. By (17) we have \( 1 + vb > p^d \), and by (18) we have

\[ L^{1+vb} A^{vb} \| \{\zeta x^q L' + B - \Sigma'' \rho_j L^{1+vj} A^{jd}\}, \]

where \( \Sigma'' \) is over those \( j \) such that \( j < b \), \( p^{d+1} | (1 + vj) \). The right side of (21) is a \( p^{d+1} \)st power. Hence the left side of (21) is also a \( p^{d+1} \)st power. Therefore \( p^{d+1} | (1 + vb) \), a contradiction. It follows that for every \( j \) such that \( \rho_j \neq 0 \), either \( 1 + vj \) is a power of \( p^{k} \) or \( p^{d+1} | (1 + vj) \). This establishes the desired result for \( d + 1 \), and
LEMMA 5. Suppose there exists an integer $d$ such that $L'$ is a $p^d$th power but not a $p^{d+1}$st power, and $1 + vr > p^d$. Then $v = 1$ and $p^{d+1} \nmid (1 + r)$.

Proof. Since $L'$ is a $p$th power by (14), we have $d \geq 1$. By Lemma 4 we have

$$R(x) = \sum_{i=0}^{c} \omega_i x^{(p^k i - 1)/v} + \Sigma^* \rho_j x^j + x^r,$$

where the $\omega_i$ are elements of $\mathcal{H}$, $c = [d/k]$, and the summation $\Sigma^*$ is over all $j$ such that $p^d \mid (1 + vj)$, $p^d < 1 + vj$, $j < r$. Moreover since $1 + vr > p^d$ and $\rho_r \neq 0$, we have $p^d \mid (1 + vr)$. Furthermore $H$ is a $p^{d-1}$st power. Since $\zeta \in \mathcal{H}$, it follows that $\zeta L'$ is a $p^d$th power but not a $p^{d+1}$st power. Thus we can set

$$H = A^{p^{d-1}} \text{ and } \zeta L' = C^{p^d},$$

where $C$ is not a $p$th power. Substitution in (15) gives us

$$(22) \quad L^{1+vr} A^{p^d} = x^q C^{p^d} + B,$$

where

$$B = -\zeta xL' - LR(F) + LF^r$$
$$= -\zeta xL' - \sum_{i=0}^{c} \omega_i L^{sk} A^{p^d(\frac{p^k i - 1}{v})} - \Sigma^* \rho_j L^{1+vj} A^{ip^d}.$$

Now the left side of (22) is a $p^d$th power. Moreover

$$q \geq 1 + fr \geq 1 + vr > p^d,$$

so that $p^{d+1} \mid q$. Therefore $B$ is a $p^d$th power, say $B = D^{p^d}$. Extracting the $p^d$th roots of both sides of (22) we obtain

$$(23) \quad L^{(1+vr)p^{-d}} A^r = x^{q p^{-d}} C + D.$$

Differentiation now yields

$$(24) \quad L^{-1+ \frac{1+vr}{p^{-d}}} A^{r-1} (1 + vr)p^{-d} L'A + r L'A' = x^{q p^{-d}} C' + D'.$$

since $p^{d+1} \mid q$. Multiplying (24) by $C$, (23) by $C'$, and subtracting, we get

$$(25) \quad L^{-1+ \frac{1+vr}{p^{-d}}} A^{r-1} E = CD' - C'D,$$

where

$$E = (1 + vr)p^{-d} L'AC + r L'A' C - LAC'.$$
Now $A | H$ and therefore $LA | F$. Moreover
\[ C | L' = G/((\zeta U) = \zeta^{-1}U_1U_2\cdots U_r | F_1F_2\cdots F_r. \]
Hence $C$ is relatively prime to $LA$. Since $C$ is not a $p$th power we have $C' \neq 0$. Hence $C \nmid LAC'$. It follows that $E \neq 0$. From (25) we obtain $CD' \neq C'D$ and
\[ (26) \quad L^{-d+(1+vr)p^{-d}} A^{-1} | (CD' - C'D), \]
where
\[ e = \begin{cases} 
0 & \text{if } p^{d+1} \mid (1 + vr), \\
1 & \text{if } p^{d+1} \nmid (1 + vr).
\end{cases} \]
Comparing degrees in (26) we obtain
\[ (27) \quad (1 + vr - ep^d)l + p^d(r - 1) \deg A < p^d \deg (CD) = \deg (L'B). \]
Now the leading term of $R(x)$ is $x^r$ and $R(x) \neq x^r$. Set $b = \deg \{R(x) - x^r\}$. Then we have $0 \leq b < r$ and
\[ \deg B \leq \deg (LF^b) = (1 + vb)l + bp^d \deg A \leq (1 + vb)l + (r - 1)p^d \deg A. \]
Substitution in (27) gives us, after simplification,
\[ v(r - b)l < ep^d l + \deg L' < (ep^d + 1)l. \]
Hence $v(r - b) \leq ep^d$. Therefore $e \neq 0$. Hence $e = 1$ and
\[ v(r - b) \leq p^d. \]
Since $p^d \mid (1 + vr)$ and $1 + vr > p^d$, we have $1 + vr \geq 2p^d$ and
\[ 1 + vb = 1 + vr - v(r - b) \geq p^d. \]
Since $p^d \neq 0$, this gives us $p^d \mid (1 + vb)$. Since $p^d \mid (1 + vr)$, it follows that $p^d \mid v(r - b)$ and $p \nmid v$. Hence $v(r - b) = p^d$ and $v = 1$. Finally since $e = 1$ we have
\[ p^{d+1} \mid (1 + vr) = 1 + r, \]
which completes this proof.

**Lemma 6.** If $d$ is an integer such that $p^d < 1 + vr$, then $L'$ is a $p^{d+1}$st power.

**Proof.** Suppose the result is false. Then $L'$ is not a $p^{d+1}$st power and $p^d < 1 + vr$. Without loss of generality we suppose that $L'$ is a $p^d$th power. By Lemma 5 we have $v = 1$ and $p^{d+1} \mid (1 + r)$. 

\[ p^{d+1} \mid (1 + vr) = 1 + r, \]

which completes this proof.
Therefore \( k = 1 \) and \( p^a < 1 + r \). It follows from Lemma 4 that \( R(x) \) is of the form

\[
R(x) = \sum_{i=0}^{d-1} \omega_i x^{\alpha_i} + \sum \rho_j x^j,
\]

where the summation \( \sum \) is over all \( j \) such that \( p^d \mid (1 + j), j \leq r \). Moreover \( H \) is a \( p^{d-1} \)st power and \( p^d \mid (1 + r) \). Now

\[
FR(F') = \prod_{i=0}^{r} (F - \gamma_i) = \prod_{i=0}^{r} F_i = (x^a - x)G
\]

by (4), so that

\[
(28) \quad \sum \rho_j F^{j+1} = x^a G + B,
\]

where \( \deg B \leq p^{d-1}f \). The left side of (28) is a \( p^d \)th power. Moreover \( q \geq 1 + fr \geq 1 + r > p^d \), so that \( x^a \) is a \( p^{d+1} \)st power. Since \( G = \zeta L'U \) and \( U = L^{-1}H^p = H^q \), it follows that \( G \) is a \( p^d \)th power. Hence \( B \) is also a \( p^d \)th Power. We set

\[
G = C^{p^d} \quad \text{and} \quad B = D^{p^d}.
\]

Then, extracting the \( p^d \)th roots of both sides of (28), we get

\[
(29) \quad \sum \xi^a F^{j} = x^{a-p^d}C + D,
\]

where \( a = (r + 1)p^d \geq 2 \), the \( \xi \) are in \( \mathcal{A} \), \( \xi_a = 1 \), and \( \deg D \leq f/p \). Now \( p \nmid a \) since \( p^{d+1} \mid (r + 1) \). We set \( \bar{F} = F + \xi_{a-1}a \). Then (29) becomes

\[
(30) \quad \sum \eta^a \bar{F}^{j} = x^{a-p^d}C + D,
\]

where the \( \eta \) are in \( \mathcal{A} \), \( \eta_a = 1 \), and \( \eta_{a-1} = 0 \). Differentiating (30) we obtain

\[
(31) \quad \sum j \eta^a \bar{F}^{j-1}F' = x^{a-p^d}C' + D'.
\]

Eliminating \( x^{a-p^d} \) from (30) and (31) we get

\[
\eta_0 C' + \sum \eta^a \bar{F}^{j-1}(C'F' - jC \bar{F}') = C'D - CD'.
\]

Since \( \eta_{a-1} = 0 \), it follows that

\[
\bar{F}^{a-1}(C'F' - aC \bar{F}') = C'D - CD' - E,
\]

where

\[
\deg E < (a - 2)f + \deg C.
\]
Now
\[
\deg C = p^{-a} \deg G < p^{-a} f \leq f/p
\]
by (5). Hence \(\deg E < (a - 1)f\), and
\[
\deg (C'D - CD') < \deg (CD) < 2f/p \leq (a - 1)f.
\]
Therefore
\[
\deg (C'D - CD' - E) < (a - 1)f = \deg \bar{F}^{a-1},
\]
and (32) yields
\[
C'\bar{F} = aC\bar{F}'.
\]
Now \(\bar{F}' = F' = \theta^{-1}G \neq 0\) by Lemma 1. Therefore \(aC\bar{F}' \neq 0\). Hence \(C' \neq 0\) and thus \(C' \nmid C\). It follows that \((\bar{F}, C) \neq 1\). Since
\[
C'\hat{F} = G = \prod_{i=0}^{r} U_i
\]
we have \((\bar{F}, U_b) \neq 1\) for some \(b, 0 \leq b \leq r\). Hence \((\bar{F}, F_b) \neq 1\). Since \(\bar{F} - F_b \in \mathcal{X}\), we must have \(\bar{F} = F_b\). Therefore
\[
C'F_b = aCF_b'.
\]
Since \(v = 1\), we have \(F_b = L_bH_b\), whether or not \(b = 0\). Hence
\[
C'L_bH_b = aCL_b'H_b',
\]
and \(C'L_b = aCL_b'\). Now \(L_b\) is relatively prime to \(L'_b\). Therefore \(L_b \mid C\). Since \(v = 1\) we have
\[
C'\hat{F} = G = \prod_{i=0}^{r} U_i = \prod_{i=0}^{r} H_i.
\]
It follows that \(L_b \mid H_bH_1 \cdots H_r\). On the other hand \(L_b \nmid H_b\), while for \(i \neq b\) we have \((L_b, H_i) = 1\). Therefore \(L_b \nmid H_bH_1 \cdots H_r\), a contradiction. This completes the proof of this lemma.

We are now in a position to prove the most general theorem of this paper. We drop the assumption \(\gamma_0 = 0\).

**Theorem 1.** Let \(\mathcal{X}\) be a finite field of characteristic \(p\) that contains exactly \(q\) elements. Let \(F(x)\) be a polynomial over \(\mathcal{X}\) of degree \(f, f > 0\). Let \(\gamma_0, \gamma_1, \ldots, \gamma_r\) be the distinct values \(F(\tau)\) as \(\tau\) ranges over \(\mathcal{X}\), and let \(l_i\) denote the number of distinct roots in \(\mathcal{X}\) of the polynomial \(F(x) - \gamma_i\). Let the \(\gamma_i\) be arranged in such a way that \(l_0 \leq l_i, 1 \leq i \leq r\). Set \(L = \Pi(x - \pi)\), where the product is over the distinct roots \(\pi\) of \(F(x) - \gamma_0\) that lie in \(\mathcal{X}\). Suppose that
Then there exist positive integers \( v, k, m \); a polynomial \( N \) over \( \mathcal{K} \); and \( \omega_0, \omega_1, \ldots, \omega_m \) in \( \mathcal{K} \) such that \( L \nmid N, v \mid (p^k - 1) \), \( 1 + vr = p^{mk} \), \( L' \) is a \( p^{mk} \)-th power, \( \omega_0 \neq 0 \), \( \omega_m = 1 \),

\[
F(x) = L^n N^{p^{mk}} + \gamma_0,
\]

(33) \[
\prod_{i=1}^{r} (x - \gamma_i + \gamma_0) = \sum_{i=0}^{m} \omega_i \varphi^{(p^{ki}-1)/v},
\]

and

(34) \[
\sum_{i=0}^{m} \omega_i L^{v p^{ki}} N^{p^{km} (p^{ki}-1)/v} = -\omega_0 (x^v - x) L'.
\]

Proof. Without loss of generality we can suppose that \( \gamma_0 = 0 \), so that our previous discussion applies. Let \( d \) be the integer such that

\[
p^d \geq 1 + vr > p^{d-1}.
\]

It follows from Lemma 6 that \( L' \) is a \( p^d \)-th power. We now apply Lemma 4 to conclude that either \( 1 + vr \) is a power of \( p^k \) or \( p^d \mid (1 + vr) \). In either case we must have \( p^d = 1 + vr \). Since \( k \) is the smallest positive integer such that \( v \mid (p^k - 1) \), it follows that \( k \mid d \). We put \( m = d/k \). Then \( L' \) is a \( p^{mk} \)-th power and \( 1 + vr = p^{mk} \). Applying Lemma 4 again we find that \( R(x) \) is of the form

\[
R(x) = \sum_{i=0}^{m} \omega_i \varphi^{(p^{ki}-1)/v},
\]

so that (33) holds. Moreover \( H \) is a \( p^{d-1} \)-st power by Lemma 4, and therefore \( H^p \) is a \( p^{mk} \)-th power. Thus there is a polynomial \( N \) over \( \mathcal{K} \) such that

\[
F = L^n H^p = L^n N^{p^{mk}}.
\]

Furthermore since \( L \nmid H \), it follows that \( L \nmid N \). Using (16) we obtain \( \omega_0 = \rho_0 = -\zeta \neq 0 \). It follows at once from (33) that \( \omega_m = 1 \). Finally we substitute in (15) to obtain (34). This completes the proof of the theorem.

In the next two sections we apply Theorem 1 to a number of special cases.

3. A special case. There are two general types of polynomials known for which (1) holds \([1, \S \,5]\). For every polynomial of the first type both \( L' \) and \( N \) are constants. Thus this case is of special interest. Here we have the following result:
LEMMA 7. Suppose that $U$ and $N$ are both constants. Then $q$ is a power of $p^k$, and $F$ is of the form

$$F = \alpha L^v + \gamma, \quad L = \beta + \sum_{j=0}^{d} \varphi_j x^{p^kj},$$

where $L$ factors into distinct linear factors over $\mathbb{K}$ and $v \mid (p^k - 1)$.

Proof. Since $N$ is a constant it follows from Theorem 1 that $F = \alpha L^v + \gamma$, where $\alpha \in \mathbb{K}$ and $\gamma = \gamma_0 \in \mathbb{K}$. Suppose that $L$ is not of the form given in (35). Then, since $L'$ is a constant, we can write

$$L = \beta + \sum_{j=0}^{c} \varphi_j x^{p^kj} + \sum_{j=a}^{l/p} \delta_j x^{p^j}$$

where $a$ and $c$ are integers such that $p^{k(c+1)} > pa > p^k$, $l \geq pa$, and $\delta_a \neq 0$. Moreover $L' = \varphi_0 \neq 0$. Now (34) becomes

$$\sum_{i=0}^{m} \chi_i L^{p^kt} = -\omega_0 \varphi_0 (x^q - x),$$

where the $\chi_i$ are in $\mathbb{K}$, $\chi_0 = \omega_0 \neq 0$, and $\chi_m \neq 0$. Substituting (36) in (37) we get

$$\varphi + \sum_{j=0}^{c} \psi_j x^{p^kj} + \chi_0 \delta_a x^{pa} + \sum_{j=pa+1}^{l/pm} \sigma_j x^j = -\omega_0 \varphi_0 (x^q - x),$$

for suitable $\varphi$, $\psi_j$, $\sigma_j$ in $\mathbb{K}$. Since $\chi_0 \delta_a \neq 0$, this implies that either $pa = 1$ or $pa = q$. Comparing degrees we obtain

$$q = lp^{km} > l \geq pa.$$

Clearly $pa \neq 1$. This contradiction implies that $L$ is of the desired form, which completes this proof.

The converse of Lemma 7 is already known [1]: If $q$ is a power of $p^k$, and if $F$ is of the form (35), then the polynomial $F$ satisfies the equality $r = [(q - 1)/f]$. This was proved in [1] as follows: Let $\pi$ be a root of $L$. Replacing $x$ by $x + \pi$ we can assume that $\beta = 0$. Let $l = \deg L$ as before, and set $L(x) = L$. Because of the form of $L$ the values assumed by $L(\tau)$ as $\tau$ ranges over $\mathbb{K}$ form a vector space over the subfield $GF(p^k)$. Since we have assumed that $L$ factors into distinct linear factors over $\mathbb{K}$, it follows that $L$ has exactly $l$ distinct roots in $\mathbb{K}$. Therefore this vector space contains exactly $q/l$ distinct elements. Then since $F = \alpha L^v + \gamma$, where $v \mid (p^k - 1)$, it follows that the number of values assumed by $F(\tau)$ as
\(\tau\) ranges over \(\mathcal{H}\) is exactly
\[
1 + (-1 + q/\ell)/v = 1 + (q - \ell)/f = 1 + [(q - 1)/f].
\]
Hence \(r = [(q - 1)/f]\).

Thus we have a complete characterization of those polynomials for which \(r = [(q - 1)/f] \geq 2\), subject to the condition that \(L'\) and \(N\) are both constants. One significance of this result can be seen from the following lemma:

**Lemma 8.** If \(f \leq \sqrt{q}\), and \(r = [(q - 1)/f] \geq 2\), then \(L'\) and \(N\) are both constants.

**Proof.** Theorem 1 applies so that we have \(1 + rv = p^{mk}\), and \(f = vl + p^{mk} \deg N\). Moreover \(f^2 \leq q\) and \(r = [(q - 1)/f]\) so that
\[
f \leq q/f \leq r + 1 = 1 + (p^{mk} - 1)/v \leq p^{mk}.
\]
Thus \(p^{mk} \deg N < f \leq p^{mk}\), \(\deg N = 0\), and \(N\) is a constant. Furthermore \(L'\) is a \(p^{mk}\)th power by Theorem 1 and \(\deg L' < l \leq f \leq p^{mk}\).

Hence \(L'\) is also a constant, and the proof of this lemma is complete.

The above results give us a complete characterization of those polynomials \(F\) for which \(r = [(q - 1)/f] \geq 2\) and \(0 < f \leq \sqrt{q}\). Now suppose that \(r = [(q - 1)/f] < 2\) and \(0 < f \leq \sqrt{q}\). Then
\[
2 > (q - 1)/f \geq (f^2 - 1)/f,
\]
f^2 - 2f - 1 < 0, and thus \(f = 1\) or \(f = 2\). Now \(q\) is a prime power and \(f^2 \leq q < 2f + 1\). Hence we have either (i) \(f = 1\) and \(q = 2\), or (ii) \(f = 2\) and \(q = 4\). If \(f = 1\), then \(F\) is clearly of the form (35) with \(v = k = 1\) and \(d = 0\). If \(f = 2\) and \(q = 4\), then \(r = 1\), and since \(F_0\) and \(F_1\) together have 4 distinct roots in \(\mathcal{H}\), it follows that \(F_0\) has two distinct roots in \(\mathcal{H}\); so that \(F\) is still of the form (35), this time with \(p = 2\) and \(v = k = d = 1\). Thus we see that the condition \(r \geq 2\) can be dropped here. Combining all these results we obtain one of our major results:

**Theorem 2.** Let \(F(x)\) be a polynomial over the finite field \(\mathcal{H}\) of characteristic \(p\) and let \(q\) denote the number of elements of \(\mathcal{H}\). Let \(r + 1\) denote the number of distinct values assumed by \(F(\tau)\) as \(\tau\) ranges over \(\mathcal{H}\), and let \(f\) be the degree of \(F(x)\). Suppose that \(0 < f \leq \sqrt{q}\). Then
\[
r = [(q - 1)/f]
\]
if and only if \(F\) is of the form
where \( L \) is a polynomial that factors into distinct linear factors over \( \mathcal{K} \) and that has the form

\[
L = \beta + \sum_{i=0}^{d} \varphi_i x^{p^k i} .
\]

and where \( v \) and \( k \) are integers such that \( v \mid (p^k - 1) \), \( q \) is a power of \( p^k \), and \( \alpha, \beta, \gamma, \) and the \( \varphi_i \) are elements of \( \mathcal{K} \).

4. The cases \( q = p \) and \( q = p^2 \). The results of § 1 enable us to treat the case \( q = p \) quickly.

Suppose \( q = p \) and \( r = [(q - 1)/f] \geq 2 \). If \( \gamma_0 = 0 \), then the results of § 1 apply, so that

\[
F = L^v H^p, \quad L = xS^p + T^p
\]

by (12) and (14). Since

\[
\deg F = f \leq \frac{1}{2}(q - 1) = \frac{1}{2}(p - 1) < p,
\]

the polynomials \( H, S, \) and \( T \) are all constants. Thus \( F \) is of the form \( \alpha(x + \beta)^p \) and \( v = f \). It is easily shown that \( v \mid (q - 1) \) here. Dropping the assumption \( \gamma_0 = 0 \), we see that if \( q = p \) and \( r = [(q - 1)/f] \geq 2 \), then \( f \mid (q - 1) \) and \( F \) is of the form

\[
F = \alpha(x + \beta)^r + \gamma.
\]

We note that in this case \( L' \) and \( N \) must both be constants, so that we could have obtained this result from Lemma 7.

Let us now consider the case \( q = p^2 \). Comparing the degrees of the two sides of (34) we obtain

\[
p^{mk} + rp^{mk} \deg N = q + \deg L' \leq q + l - 1 = p^2 + l - 1.
\]

Therefore

\[
(38) \quad pl + p \deg N \leq p^2 + l - 1 .
\]

Thus \( pl \leq p^2 + l - 1 \) or \( l \leq p + 1 \). Since \( L' \) is a \( p \)-th power, it follows that \( l \equiv 0 \) or \( 1 \) (mod \( p \)). Therefore \( l = 1, p, \) or \( p + 1 \). If \( l = p \) or \( p + 1 \), the inequality (38) gives us

\[
p \deg N \leq p^2 - l(p - 1) - 1 \leq p - 1 ,
\]

\( \deg N = 0 \) and \( N \) is a constant. If \( l = 1 \), then \( L \) is of the form \( x + \beta, L' = 1, \) and (34) gives us
\[ N \mid (-\omega_0 x^q + \omega_0 x - \omega_0 L) = -\omega_0 (x^q + \beta) = -\omega_0 L^r. \]

Thus in case \( l = 1 \), we see that \( N \) is a constant times a power of \( L \). Since \( L \not| N \), this implies that \( N \) is a constant. Thus \( N \) is a constant in all three cases.

If \( L' \) is also a constant then Lemma 7 applies, and \( F \) is of the form (35) with either (i) \( l = 1, \ d = 0, \) and \( v \mid (p^3 - 1), \) or (ii) \( l = p, \ k = d = 1, \) and \( v \mid (p - 1) \).

Now suppose that \( L' \) is not a constant. Since \( L' \) is a \( p^{m_k} \)th power by Theorem 1, we must have \( l = p + 1 \) and \( m = k = 1 \). Since \( N \) is a constant we have \( F = \alpha L^r + \gamma \), where \( \alpha \in \mathcal{F} \) and \( \gamma = \gamma_0 \in \mathcal{F} \). Moreover \( L \) is of the form \( L = xS^p + T^p \) by (14). Since \( L \) has leading coefficient 1, \( S \) is of the form \( S = x + \varphi \). Moreover \( T \) is of the form \( T = \mu x + \nu \). Now (34) becomes

\[
\omega_0 L + \chi L^p = -\omega_0 (x^q - x)S^p,
\]

where \( \chi \in \mathcal{F} \). Comparing leading coefficients we see that \( \chi = -\omega_0 \). Therefore

\[
L^p = (x^q - x)S^p + L = x^p S^p + T^p.
\]

Extracting \( p \)th roots we obtain \( L = x^S + T \). Thus

\[
xS^p + T^p = x^p S + T,
\]

or

\[
(39) \quad x^{p+1} + \mu x^p + \varphi x + \nu^p = x^{p+1} + \varphi x^p + \mu x + \nu.
\]

Comparing the coefficients of \( x \) in (39) we obtain \( \mu = \varphi^p \). Therefore

\[
L = x^S + T = x^{p+1} + \varphi x^p + \varphi x + \nu = (x + \varphi)^{p+1} + \beta,
\]

where \( \beta = \nu - \varphi^{p+1} \). Comparing the constant terms of (39) we get \( \nu^p = \nu \). Therefore \( \nu \in GF(p) \), the prime field of \( \mathcal{F} \). Now \( \varphi^{p+1} \in GF(p) \). Hence \( \beta \in GF(p) \). Since \( L \) has distinct roots we have \( \beta \neq 0 \). Now if \( \nu = 1 \), then \( F = \alpha L^r + \gamma \), and \( F - \gamma - \alpha \beta \) has exactly one distinct root in \( \mathcal{F} \); contradicting (11). Thus \( \nu \geq 2 \). We have shown that if \( q = p^r \), \( r = [(q - 1)/f] \geq 2 \) and \( L' \) is not constant, then \( F \) is of the form \( \alpha L^r + \gamma \), where \( L \) is of the form

\[
L = (x + \varphi)^{p+1} + \beta,
\]

where \( \beta \in GF(p), \ \beta \neq 0, \ v \mid (p - 1), \ v \geq 2 \).

Conversely if \( q = p^r \) and \( F \) has this form, then \( L(\tau) \in GF(p) \) for all \( \tau \in \mathcal{F} \), and thus \( F \) assumes at most

\[
1 + (p - 1)/v = 1 + (q - 1)/f = 1 + [(q - 1)/f]
\]
distinct values. Since we always have \( r \geq [(q - 1)/f] \), this implies that \( r = [(q - 1)/f] \).

We have completed the discussion of the case \( q = p^2 \). We sum up our results for this case in our final theorem:

**Theorem 3.** Let \( \mathcal{K} \) be a field of characteristic \( p \) that contains exactly \( p^2 \) elements. Let \( F(x) \) be a polynomial over \( \mathcal{K} \) of degree \( f \), \( f > 0 \). Let \( F(\tau) \) assume exactly \( r + 1 \) distinct values as \( \tau \) ranges over \( \mathcal{K} \). If \( r = [(p^2 - 1)/f] \geq 2 \), then \( F(x) \) has one of the following three forms:

(i) \( F(x) = \alpha(x + \beta)^r + \gamma \), where \( v \mid (p^2 - 1) \), \( \alpha \neq 0 \),

(ii) \( F(x) = \alpha(x^p + px + \beta)^r + \gamma \), where \( x^p + px + \beta \) has \( p \) distinct roots in \( \mathcal{K} \), \( v \mid (p - 1) \), \( \alpha \neq 0 \),

(iii) \( F(x) = \alpha((x + \varphi)^{p+1} + \beta)^r + \gamma \), where \( \beta \in GF(p) \), \( \beta \neq 0 \), \( v \geq 2 \), \( v \mid (p - 1) \), and \( \alpha \neq 0 \).

Conversely if \( F(x) \) has one of these three forms, then \( r = [(q - 1)/f] \).

For \( q > p^2 \), the question of the characterization of all polynomials \( F \) for which (1) holds, remains open. The most general types of polynomials known for which (1) holds are described in [1, §5]. At present it seems unlikely that there are any more.

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