ON AN INEQUALITY OF P. R. BESSACK

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In a recent paper [1], P. R. Beesack derived the inequality

\[ |g(x, s)| \leq \frac{\prod_{\nu=1}^{n} |x - a_{\nu}|}{(a_{n} - a_{\nu})(n - 1)!} \tag{1} \]

for the Green's function \( g(x, s) \) of the differential system

\[ y^{(\nu)} = 0, \quad y(a_{\nu}) = 0, \quad \nu = 1, 2, \ldots, n, \]
\[ -\infty < a_{1} < a_{2} < \cdots < a_{n} < \infty. \tag{2} \]

In addition to being interesting in its own right, this inequality is a useful tool in the study of the oscillatory behavior of \( n \)th order differential equations. It would therefore appear to be worth while to give a short proof of (1). The derivation of this inequality in [1] is rather complicated.

We denote by \( [x_{0}, x_{1}, \ldots, x_{k}] \) the \( k \)th difference quotient of the function \( g(x) = g(x, s) \), i.e., we set

\[
[x_{0}, x_{1}] = \frac{g(x_{0}) - g(x_{1})}{x_{0} - x_{1}}, \\
[x_{0}, x_{1}, \ldots, x_{\nu}] = \frac{[x_{0}, x_{1}, \ldots, x_{\nu-1}] - [x_{1}, x_{2}, \ldots, x_{\nu}]}{x_{0} - x_{\nu}}, \quad \nu = 2, \ldots.
\]

This difference quotient can also be represented in the form

\[ [x_{0}, \ldots, x_{k}] = \int \cdots \int g^{(k)}(t_{0}x_{0} + t_{1}x_{1} + \cdots + t_{k}x_{k})dt_{0}dt_{1}\cdots dt_{k-1}, \tag{3} \]

where the integration is to be extended over all the positive values of the \( t_{\nu} \) for which

\[ t_{0} + t_{1} + \cdots + t_{k} = 1. \tag{4} \]

This formula, which goes back to Hermite, is easily verified by induction (cf., e.g., [2]). It holds if \( g(x) \) has continuous derivatives up to the order \( k - 1 \), and if \( g^{(k)} \) is piecewise continuous.

Since, by its definition, \( g(x, s) \) has continuous derivatives up to the order \( n - 2 \), while \( g^{(n-1)} \) has the jump

\[ g^{(n-1)}_{+}(s) - g^{(n-1)}_{-}(s) = -1 \tag{5} \]

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261
at $x = s$, we may apply (3) with $k = n - 1$. We shall do so twice, identifying the points $x_0, \ldots, x_{n-1}$ with $x, a_1, \ldots, a_{n-1}$ and $x, a_2, \ldots, a_n$, respectively. Since, because of $g(a, s) = 0$, $\nu = 1, \ldots, n$, we have

$$[x, a_1, \ldots, a_{n-1}] = \frac{g(x, s)}{\prod_{\nu=1}^{n-1} (x - a_\nu)}$$

and

$$[x, a_2, \ldots, a_n] = \frac{g(x, s)}{\prod_{\nu=2}^{n} (x - a_\nu)}$$

we obtain, upon subtracting these expressions from each other,

$$\frac{(a_n - a_1)g(x, s)}{\prod_{\nu=1}^{n} (x - a_\nu)} = \int_D g^{(n-1)}(v)dt - \int_D g^{(n-1)}(u)dt,$$

where, for brevity, $dt = dt_0dt_1 \cdots dt_{n-2}$, $D$ denotes the region defined by (4) (with $k = n - 1$ and $t_v > 0$, $\nu = 0, \ldots, n - 1$), and

$$u = t_0x + t_1a_1 + \cdots + t_{n-1}a_{n-1}, \quad v = t_0x + t_1a_2 + \cdots + t_{n-1}a_n.$$

Both for $a_1 \leq x < s$ and $s < x \leq a_n$, $g(x, s)$ is a polynomial of degree $n - 1$. Accordingly, the function $g^{(n-1)}(x, s)$ is capable only of two constant values, say $\alpha$ and $\beta$, which according to (5) are related by $\alpha = \beta + 1$. If we denote by $D_1$ the subset of $D$ in which $a_1 \leq u < s$ (where $u$ is defined in (7)), we have

$$\int_D g^{(n-1)}(u)dt = \alpha \int_{D_1} dt + \beta \int_{D-D_1} dt = \alpha \int_{D_1} dt + (\alpha - 1) \int_{D-D_1} dt = \alpha \int_D dt - \int_{D-D_1} dt.$$  

Similarly,

$$\int_D g^{(n-1)}(v)dt = \alpha \int_D dt - \int_{D-D_2} dt,$$

where $D_2$ is the subset of $D$ in which $a_1 \leq v < s$. Substituting these expressions in (6), we obtain

$$\frac{(a_n - a_1)g(x, s)}{\prod_{\nu=1}^{n} (x - a_\nu)} = \int_{D-D_1} dt - \int_{D-D_1} dt.$$  

The differential $dt$ is positive, and we thus have

$$-\int_D dt \leq -\int_{D-D_1} dt \leq \int_{D-D_1} dt - \int_{D-D_1} dt \leq \int_{D-D_2} dt \leq \int_D dt.$$
Since

\[ \int_{D} dt = \frac{1}{(n - 1)!} \]

(as can be seen by applying (3) to the function \(x^{n-1}\) and setting \(k = n - 1\)), this shows that

\[ \left| \int_{D_2} dt - \int_{D_1} dt \right| \leq \frac{1}{(n - 1)!} \]

In view of (8), this establishes the inequality (1).

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George James Minty, Jr., *On the monotonicity of the gradient of a convex function* ................................................................. 243
George James Minty, Jr., *On the solvability of nonlinear functional equations of ‘monotonic’ type* ............................................... 249
J. B. Muskat, *On the solvability of \( x^e \equiv e \pmod{p} \) ................................ 257
Zeev Nehari, *On an inequality of P. R. Bessack* .................................... 261
Raymond Moos Redheffer and Ernst Gabor Straus, *Degenerate elliptic equations* ................................................................. 265
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Bernard W. Roos, *On a class of singular second order differential equations with a non linear parameter* ........................................ 285
Tôru Saitô, *Ordered completely regular semigroups* ............................... 295
Edward Silverman, *A problem of least area* .......................................... 309
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Harold Widom, *On the spectrum of a Toeplitz operator* ....................... 365