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ON AN INEQUALITY OF P. R. BESSACK

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In a recent paper [1], P. R. Beesack derived the inequality

$$(1) \quad |g(x, s)| \leq \frac{\prod_{\nu=1}^n |x - a_\nu|}{(a_n - a_1)(n-1)!}$$

for the Green's function $g(x, s)$ of the differential system

$$(2) \quad \begin{aligned} y^{(n)} &= 0, & y(a_\nu) &= 0, & \nu &= 1, 2, \dots, n, \\ -\infty &< a_1 < a_2 < \dots < a_n < \infty. \end{aligned}$$

In addition to being interesting in its own right, this inequality is a useful tool in the study of the oscillatory behavior of n th order differential equations. It would therefore appear to be worth while to give a short proof of (1). The derivation of this inequality in [1] is rather complicated.

We denote by $[x_0, x_1, \dots, x_k]$ the k th difference quotient of the function $g(x) = g(x, s)$, i.e., we set

$$\begin{aligned} [x_0, x_1] &= \frac{g(x_0) - g(x_1)}{x_0 - x_1}, \\ [x_0, x_1, \dots, x_\nu] &= \frac{[x_0, x_1, \dots, x_{\nu-1}] - [x_1, x_2, \dots, x_\nu]}{x_0 - x_\nu}, \quad \nu = 2, \dots. \end{aligned}$$

This difference quotient can also be represented in the form

$$(3) \quad [x_0, \dots, x_k] = \int \dots \int g^{(k)}(t_0 x_0 + t_1 x_1 + \dots + t_k x_k) dt_0 dt_1 \dots dt_{k-1},$$

where the integration is to be extended over all the positive values of the t_ν for which

$$(4) \quad t_0 + t_1 + \dots + t_k = 1.$$

This formula, which goes back to Hermite, is easily verified by induction (cf., e.g., [2]). It holds if $g(x)$ has continuous derivatives up to the order $k-1$, and if $g^{(k)}$ is piecewise continuous.

Since, by its definition, $g(x, s)$ has continuous derivatives up to the order $n-2$, while $g^{(n-1)}$ has the jump

$$(5) \quad g_+^{(n-1)}(s) - g_-^{(n-1)}(s) = -1$$

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at $x = s$, we may apply (3) with $k = n - 1$. We shall do so twice, identifying the points x_0, \dots, x_{n-1} with x, a_1, \dots, a_{n-1} and x, a_2, \dots, a_n , respectively. Since, because of $g(a_\nu, s) = 0$, $\nu = 1, \dots, n$, we have

$$[x, a_1, \dots, a_{n-1}] = \frac{g(x, s)}{\prod_{\nu=1}^{n-1} (x - a_\nu)}$$

and

$$[x, a_2, \dots, a_n] = \frac{g(x, s)}{\prod_{\nu=2}^n (x - a_\nu)},$$

we obtain, upon subtracting these expressions from each other,

$$(6) \quad \frac{(a_n - a_1)g(x, s)}{\prod_{\nu=1}^n (x - a_\nu)} = \int_D g^{(n-1)}(v)dt - \int_D g^{(n-1)}(u)dt,$$

where, for brevity, $dt = dt_0 dt_1 \dots dt_{n-2}$, D denotes the region defined by (4) (with $k = n - 1$ and $t_\nu > 0$, $\nu = 0, \dots, n - 1$), and

$$(7) \quad u = t_0 x + t_1 a_1 + \dots + t_{n-1} a_{n-1}, \quad v = t_0 x + t_1 a_2 + \dots + t_{n-1} a_n.$$

Both for $a_1 \leq x < s$ and $s < x \leq a_n$, $g(x, s)$ is a polynomial of degree $n - 1$. Accordingly, the function $g^{(n-1)}(x, s)$ is capable only of two constant values, say α and β , which according to (5) are related by $\alpha = \beta + 1$. If we denote by D_1 the subset of D in which $a_1 \leq u < s$ (where u is defined in (7)), we have

$$\begin{aligned} \int_D g^{(n-1)}(u)dt &= \alpha \int_{D_1} dt + \beta \int_{D-D_1} dt = \alpha \int_{D_1} dt + (\alpha - 1) \int_{D-D_1} dt \\ &= \alpha \int_D dt - \int_{D-D_1} dt. \end{aligned}$$

Similarly,

$$\int_D g^{(n-1)}(v)dt = \alpha \int_D dt - \int_{D-D_2} dt,$$

where D_2 is the subset of D in which $a_1 \leq v < s$. Substituting these expressions in (6), we obtain

$$(8) \quad \frac{(a_n - a_1)g(x, s)}{\prod_{\nu=1}^n (x - a_\nu)} = \int_{D-D_2} dt - \int_{D-D_1} dt.$$

The differential dt is positive, and we thus have

$$-\int_D dt \leq -\int_{D-D_1} dt \leq \int_{D-D_2} dt - \int_{D-D_1} dt \leq \int_{D-D_2} dt \leq \int_D dt.$$

Since

$$\int_D dt = \frac{1}{(n-1)!}$$

(as can be seen by applying (3) to the function x^{n-1} and setting $k = n - 1$), this shows that

$$\left| \int_{D-D_2} dt - \int_{D-D_1} dt \right| \leq \frac{1}{(n-1)!}$$

In view of (8), this establishes the inequality (1).

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