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1. Introduction. Let m be the space of bounded sequences of real numbers, $\sigma = \{s_n\}$, $n = 1, 2, 3, \cdots$ under the norm $\|\sigma\| = \sup |s_n|$. Then m includes the set of all convergent sequences of real numbers. Let $A = (a_{kn})$, k, $n = 1, 2, 3, \cdots$ be a real Toeplitz-matrix, i.e. an infinite matrix of real numbers which satisfies the conditions

$$(1.1) \qquad \lim_{k \to \infty} a_{kn} = 0 , \qquad\qquad n = 1, 2, \cdots ;$$

$$(1.2) \qquad \sum_{n=1}^{\infty} |a_{kn}| < M ,$$

where M is a positive number which is independent of k;

$$(1.3) \qquad \lim_{k \to \infty} \sum_{n=1}^{\infty} a_{kn} = 1 .$$

It is well-known that A defines a regular method of summation. That is to say, if $\{s_n\}$ is a convergent sequence in M, with limit s, then the transform $\{t_k\}$ of $\{s_n\}$ by A exists, where

$$(1.4) \qquad t_k = \sum_{n=1}^{\infty} a_{kn} s_n$$

and $\{t_k\}$ converges to the same limit as $\{s_n\}$,

$$(1.5) \qquad \lim_{k \to \infty} t_k = \lim_{n \to \infty} s_n = s .$$

If $\{s_n\}$ is an arbitrary element of m then $\{t_k\}$ exists anyhow and may (or may not) converge to a limit, which is then called the A-limit of $\{s_n\}$. However, it has been shown by Steinhaus that for every Toeplitz-matrix A there exists a sequence $\{s_n\}$ in m which is not summed by A, i.e. such that $\{t_k\}$ does not converge (compare e.g. Ref. 2). Nevertheless the method of summation by infinite matrices is very useful in various branches of real and complex Function Theory. We shall call the limits obtained in this way *Toeplitz-limits*, to distinguish them from other types of generalized limits such as those discussed below.

A straightforward application of the Hahn-Banach extension theorem shows that there exist continuous linear functionals $F(x)$ defined *on* m such that for every convergent sequence $\sigma = \{s_n\}$ in m, $F(\sigma) = \lim_{n \to \infty} s_n$. Such a functional will be called a *Hahn-Banach-limit*.

In view of Steinhaus' theorem, mentioned above, a Hahn-Banach-limit cannot be a Toeplitz-limit, although every Toeplitz-limit can be extended to a Hahn-Banach-limit, again by applying the Hahn-Banach extension theorem.

If $F(x)$ is a Hahn-Banach-limit such that

1.6.   (positivity) $s_n \geqq 0$ for $n = 1, 2, \cdots$ entails $F(\{s_n\}) \geqq 0$ and such that

1.7.   (shift invariance) $F(\{s_n\}) = F(\{s_{n+1}\})$ for all $\{s_n\} \in m$—
then $F(x)$ will be called a *Mazur-Banach limit*. It is known that there exist Mazur-Banach limits (compare ref. 1 or ref. 10).

In spite of their greater efficiency, the generalized limits which are defined by means of the methods of Functional Analysis (Hahn-Banach limits) cannot in general be used to replace the more concrete matrix methods of summation (Toeplitz-limits). In the present paper, we propose to use the methods of Non-standard Analysis in order to bridge the gap between the Hahn-Banach-limits on one hand and Toeplits-limits on the other hand. Thus, we shall derive generalized limits which, while clearly related to the matrix methods, are efficient for *all* bounded sequences. These generalized limits are given by certain linear forms with coefficients in a nonstandard model of analysis, and some of them satisfy also conditions 1.6 and 1.7. More generally, it will be shown that there exist non-standard models of analysis in which all continuous linear functionals on m can be represented by linear forms in a sense which will be made precise in due course.

The foundations of Non-standard Analysis are sketched in Ref. 7 and developed in greater detail in Refs. 8 and 9. The scope of the theory of Ref. 8 is more comprehensive inasmuch as it is based on a higher order language, L. We shall adopt L as the basis of the present paper, but in order to follow it, it will be sufficient to suppose that L is some formal language which is appropriate to the predicate calculus of order ω.

As mentioned in Ref. 7, particular non-standard models of analysis are provided by the ultrapowers of the real numbers, R_0, which constitute proper extensions of R_0. The ultrapower technique applies both to the lower predicate calculus and to the higher order language which will be considered here. For the notions which can be expressd in the lower predicate calculus, W. A. J. Luxemburg has given a detailed and expert development of Non-standard Analysis in terms of ultrapowers (Ref. 5, compare also Ref. 6). When specific ultrapowers are used, the explicit use of a formal language can be avoided

and the fact that a non-standard model of analysis has in a definite formal sense the *same* properties as the standard model remains in the background, as a heuristic principle. This has the advantage of making the subject comprehensible to analysts who are not familiar with the formal languages of Mathematical Logic and also reveals certain aspects of the procedure which are not apparent if the structure of the non-standard model remains unspecified. On the other hand, by using a formal language we may establish the truth of a vast number of useful assertions about the non-standard model simply but rigorously by transfer from the standard case. If no formal language is used we have to prove all these results ab initio.

2. Q-sequences. Let R_0 be the field of real numbers, and let K_0 be the set of all sentences which are formulated in a higher order language L, as in Ref. 8 (compare § 1, above) and which hold in R_0. Thus, L contains (symbols for) all real numbers, for all n-ary relations on real numbers, including unary relations, or *sets*, and also for concepts of higher type such as relations between sets, relations between relations, etc. In such a language, an infinite sequence $\{s_n\}$, $n = 1, 2, 3, \cdots$ is represented by a binary relation $\sigma(x, y)$ in which the domain of the first variable is the set of positive integers, such that for every positive integer n there is a unique real number a for which $\sigma(n, a)$ holds in R_0 (and hence, is a sentence of K_0). The set of all these relations then determines a unary relation in L, $\theta(x)$, say, i.e. $\theta(\sigma)$ holds in R_0 if and only if σ represents an infinite sequence as indicated.

Now let $^*R$ be a non-standard model of K_0, i.e. a model of K_0 which is different from R_0 and hence, is a proper extension of R_0. Then the unary relation of L which determines the set of natural numbers, N, in R_0 determines a non-standard model of the natural numbers, $^*N$, in $^*R$, and the predicate $\theta(x)$ determines a set of non-standard infinite sequences in $^*R$, i.e. a set of functions whose domain is the set of positive integers in $^*R$, $^*N$-$\{0\}$, and which take values in $^*R$. The sequences which are determined in this way are called Q-sequences (Q for quasi-standard). In particular, if $\{s_n\}$ is an infinite sequence in R_0, which corresponds to a binary relation $\sigma(x, y)$ as above, then $\sigma(x, y)$ determines in $^*R$ a Q-sequence which coincides with $\{s_n\}$ for all finite values of the subscript n, and which will be denoted by $^*\{s_n\}$. A Q-sequence which is obtained in this way by the extension of a sequence in R_0 is called an S-sequence (S for standard). For example, the sequence which is given by $s_n = 1/n$ for all positive integers, finite or infinite, is an S-sequence. There exist Q-sequences which are not S-sequences. For example, if ω is a fixed infinite natural number then the sequence which is given by

$$\begin{aligned} a_n &= 1/\omega \quad \text{for} \quad 1 \leqq n \leqq \omega \\ a_n &= 0 \quad \text{for} \quad n > \omega \end{aligned}$$

(2.1)

is not an S-sequence, since every S-sequence takes standard real values for standard (finite) subscripts, but it is a Q-sequence, for the assertion,

"For every positive integer ω there exists an infinite sequence which is given by 2.1,"

holds in R_0, and therefore, can be formulated as a sentence of K_0 and holds also in $*R$.

More generally, if (a_{kn}) is a Toeplits-matrix, and ω is a fixed infinite natural number then the sequence

2.2. $a_n = a_{\omega n}$ for all positive integers n, finite or infinite,

is a Q-sequence. It cannot be an S-sequence, for 1.1 implies that for every finite positive integer n, $a_{\omega n}$ is infinitesimal. If the sequence were an S-sequence then $a_{\omega n} = a_n$ would at the same time have to be a standard number and so $a_n = 0$ for all finite n. It is easy to see that the only S-sequence which has this property is the zero sequence, $a_n = 0$ for *all* n, and this contradicts 1.3, which implies that $\sum_{n=1}^{\infty} a_{\omega n}$ is infinitely close to 1. In this connection, $\sum_{n=1}^{\infty} a_{\omega n}$ is to be understood in the sense of the classical (Weierstrass) definition.

The equation 2.1 is a special case of 2.2. It is obtained by taking for (a_{kn}) the matrix of arithmetic means, $a_{kn} = 1/k$ for $n \leqq k$, $a_{kn} = 0$ for $n > k$.

On the other hand it is not difficult to define sequence which are not even Q-sequences. For example, the sequence which is given by

$$\begin{aligned} a_n &= 0 \quad \text{for all finite positive integers } n \text{ ,} \\ a_n &= 1 \quad \text{for all infinite positive } n \text{ ,} \end{aligned}$$

(2.3)

cannot be a Q-sequence, for if it were, then the set of all infinite natural numbers would be definable in $*R$ as the set of all natural numbers for which $a_n = 1$. Every nonempty set of natural numbers which is definable in $*R$ must have a first element, for this is a property of subsets of natural numbers which can be formulated as a sentence of K_0. The set of infinite natural numbers in $*R$ is not empty but does not possess a first element. This shows that 2.3 cannot be a Q-sequence.

A more general property of Q-sequences, which will be made use of in the sequel is as follows.

2.3. THEOREM. *Let $\{A_n\}$ be a Q-sequence such that A_n is infinitesimal for all finite n. Then there exists an infinite natural number ω such that A_n is infinitesimal for all $n < \omega$.*

Proof. A *Q-set* is a set which exists in $*R$ as a model of K_0 (just as a *Q-sequence* is a sequence which exists in $*R$). Supposing that $\{A_n\}$ satisfies the assumptions of the theorem, let B be the set of all positive integers in $*R$ such that $n|A_n| \geqq 1$. Then B is a Q-set (since it is defined in terms of a Q-sequence). If B is empty then $n|A_n| < 1$ and so $|A_n| < 1/n$ for all positive integers n. This shows in particular that A_n is infinitesimal for all infinite n since $1/n$ is then infinitesimal. We conclude that in this case the conclusion of the theorem is satisfied by all infinite positive integers ω. If B is not empty then it includes a smallest element, ω, for it is a property of every nonempty subset of the natural numbers in R_0, and hence also in $*R$, to possess a smallest element. Moreover, ω must be infinite, otherwise $\omega|A_\omega|$ would be infinitesimal and hence, smaller than 1. For $n < \omega$ we have again $n|A_n| < 1$, $|A_n| < 1/n$, A_n is infinitesimal also for all infinite n less than ω. This completes the proof of 2.3.

Let $\{a_n\}$ be a Q-sequence. Then the infinite sum $\sum_{n=1}^\infty |a_n|$ may or may not exist in the sense of the classical (Weierstrass) definition, as applied to $*R$. Thus, $\sum_{n=1}^\infty |a_n|$ exists if and only if the partial sums $\sum_{n=1}^k |a_n|$ is uniformly bounded in $*R$ as k varies over all finite or infinite positive integers. In particular, if $\sum_{n=1}^\infty |a_n|$ exists and equals a *finite* number in $*R$ (i.e. a number of $*R$ which is smaller than some number of R_0) then $\{a_n\}$ will be called an *S-bounded form*. The reason for this terminology will become apparent presently. A Q-sequence $\{a_n\}$ is an S-bounded form if and only if there exists a standard real number A and that $\sum_{n=1}^k |a_n| \leqq A$ for all k, finite or infinite.

Let $*m$ be the extension of the space m to $*R$. Thus, $*m$ consists of all Q-sequences which are bounded in $*R$. In particular, $*m$ contains all the S-sequences $*\sigma$ which are the extensions to $*R$ of bounded sequences $\sigma = \{s_n\}$ in R_0.

Let $\alpha = \{a_n\}$ be an S-bounded from, such that $\sum_{n=1}^\infty |a| = A$, where A is a finite number. We shall associate with α a functional $F_\alpha(x)$ with domain m and range in R_0, in the following way. For every $\sigma = \{s_n\}$ in m, the sum $\sum_{n=1}^\infty a_n s_n$ exists in $*R$ since, for every finite or infinite positive integer k,

$$(2.4) \qquad \left| \sum_{n=1}^k a_n s_n \right| \leqq \sum_{n=1}^k |a_n||s_n| \leqq \|\sigma\| \sum_{n=1}^k |a_n| \leqq \|\sigma\|A,$$

taking into account that $\|\sigma\|$ which is a bound for the absolute values of the element of σ must be a bound also for the absolute values of the elements of $*\sigma$. Moreover, (2.4) shows that $\sum_{n=1}^\infty a_n s_n$ is actually finite, since it cannot exceed the finite number $\|\sigma\|A$, and accordingly

possesses a standard part. Denoting by ^0a the standard part of any finite a, we now put

(2.5) $$F_\alpha(\sigma) = {}^0\!\left(\sum_{n=1}^{\infty} a_n s_n\right)$$

and we claim that $F_\alpha(x)$ is a continuous linear functional on m.

Indeed, if $\sigma = \{s_n\}$ and $\sigma' = \{s'_n\}$ are two sequences in m and λ is a real number (in R_0) then

$$F_\alpha(\sigma + \sigma') = {}^0\!\left(\sum_{n=1}^{\infty} a_n(s_n + s'_n)\right) = {}^0\!\left(\sum_{n=1}^{\infty} a_n s_n + \sum_{n=1}^{\infty} a_n s'_n\right)$$
$$= {}^0\!\left(\sum_{n=1}^{\infty} a_n s_n\right) + {}^0\!\left(\sum_{n=1}^{\infty} a_n s'_n\right) = F_\alpha(\sigma) + F_\alpha(\sigma')$$

and

$$F_\alpha(\lambda\sigma) = {}^0\!\left(\sum_{n=1}^{\infty} a_n \lambda s_n\right) = {}^0\!\left(\lambda \sum_{n=1}^{\infty} a_n s_n\right) = \lambda\,{}^0\!\left(\sum_{n=1}^{\infty} a_n s_n\right) = \lambda F_\alpha(\sigma)\,,$$

showing that the functional is linear. Also, by 2.4, $|F_\alpha(\sigma)| \leqq A\|\sigma\|$ showing that $F_\alpha(x)$ is continuous and, moreover that the norm of $F_\alpha(x)$ does not exceed A, $\|F_\alpha(x)\| \leqq A$.

3. Generalized limits. Let α be an S-bounded form, and let $F_\alpha(x)$ be the linear functional associated with α, as introduced in § 2 above. Then

3.1. THEOREM. *In order that $F_\alpha(x)$ be a Hahn-Banach limit for a given S-bounded form $\alpha = \{a_n\}$ is necessary and sufficient that*

3.2. a_n *is infinitesimal for all finite n, $a_n \simeq 0$, and*

3.3. $\sum_{n=1}^{\infty} a_n$ *is infinitely close to 1, $\sum_{n=1}^{\infty} a_n \simeq 1$.*

Proof. The conditions are necessary. For any finite positive integer n, let σ be the sequence (in m) which is defined by

(3.4) $$s_k = 0 \quad \text{for} \quad k \neq n, \quad s_n = 1\,.$$

Then 3.4 holds also for the extension of σ to $^*R$, and so

$$F_\alpha(\sigma) = {}^0\!\left(\sum_{k=1}^{\infty} a_k s_k\right) = {}^0 a_n$$

where $^0 a_n$ is the standard part of a_n. On the other hand, since $F_\alpha(x)$ is a Hahn-Banach limit, we must have $F_\alpha(\sigma) = 0$, which is the limit

of σ. Hence $^0a_n = 0$, i.e. $a_n \simeq 0$, a_n is infinitesimal.

Again, let σ be the sequence in which is defined by

$$(3.5) \qquad\qquad s_n = 1 \qquad \text{for all } n \; .$$

Then the limit of σ is 1, and $s_n = 1$ also for all infinite values of n. Hence, $F_\alpha(\sigma) = {}^0(\sum_{n=1}^\infty s_n) = 1$, and so $\sum_{n=1}^\infty \simeq 1$, proving 3.3.

The conditions are also sufficient. For any positive integer n, put $A_n = \sum_{k=1}^n |a_k|$. Then 3.2 implies that A_n is infinitesimal for all finite n. Hence, by 2.3, there exists an infinite integer ω such that $A_{\omega-1} = \sum_{k=1}^{\omega-1} |a_k| = \eta$, say, is infinitesimal. On the other hand, since the sum $\sum_{n=1}^\infty |a_n|$ exists, there is an infinite integer $\Omega > \omega$, such that $\sum_{n=\Omega}^\infty |a_n| \leq \eta$. Referring to 3.3 we may then conclude that

$$\sum_{n=\omega}^{\Omega} a_n \simeq 1$$

or, which is the same, that

$$\sum_{n=\omega}^{\Omega} a_n = 1 - \varepsilon$$

where ε is infinitesimal (positive, or negative, or zero).

Now let $\sigma = \{s_n\}$ be any convergent sequence in m (i.e. in R_0), with limit s. Then $s - s_n$ is infinitesimal for all infinite n. Let θ be the lowest upper bound of the elements of the set T of all numbers $|s - s_n|$ for $n \geq \omega$. θ exists since T is a bounded Q-set, and the fact that every bounded set has a lowest upper bound transfers from R_0 to $^*R$ since it can be expressed as a sentence of K_0. But the elements of T are all infinitesimal, and so θ also must be infinitesimal (possibly zero). For if τ is an upper bound for T, and τ is not infinitesimal then $\frac{1}{2}\tau$ also is an upper bound for T and so τ cannot be the lowest upper bound of T.

We have to show that

$$s = F_\alpha(\sigma) = {}^0\!\left(\sum_{n=1}^\infty a_n s_n \right)$$

or, which is the same that $|s - \sum_{n=1}^\infty a_n s_n|$ is infinitesimal.

Now, computing in $^*R$,

$$\left| s - \sum_{n=1}^\infty a_n s_n \right| = \left| \left(\sum_{n=\omega}^{\Omega} a_n + \varepsilon \right) s - \sum_{n=1}^\infty a_n s_n \right|$$

$$= \left| \varepsilon s - \sum_{n=1}^{\omega-1} a_n s_n + \sum_{n=\omega}^{\Omega-1} a_n (s - s_n) - \sum_{n=\Omega}^\infty a_n s_n \right|$$

$$\leq |\varepsilon| |s| + \sum_{n=1}^{\omega-1} |a_n| |s_n| + \sum_{n=\omega}^{\Omega-1} |a_n| |s - s_n| + \sum_{n=\Omega}^\infty |a_n| |s_n|$$

$$\leq |\varepsilon| |s| + \eta \|\sigma\| + A\theta + \eta \|\sigma\| \leq (|\varepsilon| + 2\eta) \|\sigma\| + A\theta$$

where the sum on the right hand side is indeed infinitesimal. This completes the proof of 3.1.

Let (a_{kn}) be any Toeplitz-matrix in the ordinary sense, i.e. in R_0, and let ω be an infinite positive integer. Then the sequence $\alpha = \{a_n\}$ which is given by 2.2 is an S-bounded form since, by 1.2,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |a_{\omega n}| < M.$$

At the same time, α satisfies 3.2 and 3.3 by 1.1 and 1.3 respectively. It follows that $F_\alpha(x)$ is a generalized limit.

If we wish to ensure that an S-bounded form defines a Banach-Mazur limit we require an additional condition.

3.6.   THEOREM.   *Let $\alpha = \{a_n\}$ be an S-bounded form which satis-flies 3.2 and 3.3 such that $a_n \geqq 0$ for all n and such that*

$$(3.7) \qquad\qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

is infinitesimal.   Then $F_\alpha(x)$ is a Banach-Mazur limit.

Proof.   We know from 3.1 that $F_\alpha(x)$ is at any rate a Hahn-Banach limit. Thus, it only remains for us to show that $F_\alpha(x)$ satisfies condition 1.7, for any convergent sequence $\sigma = \{s_n\}$ in m. Computing in $*R$, we have in fact

$$\left| \sum_{n=1}^{\infty} a_n s_n - \sum_{n=1}^{\infty} a_n s_{n+1} \right| = \left| a_1 s_1 + \sum_{n=1}^{\infty} (a_{n+1} - a_n) s_{n+1} \right|$$

$$\leqq |a_1| \, \|\sigma\| + \left(\sum_{n=1}^{\infty} |a_{n+1} - a_n| \right) \|\sigma\|,$$

where the right hand side is infinitesimal, by 3.2 and 3.7.   Hence,

$$\sum_{n=1}^{\infty} a_n s_n \simeq \sum_{n=1}^{\infty} a_n s_{n+1}$$

and so

$$F_\alpha(\{s_n\}) = {}^0\!\left(\sum_{n=1}^{\infty} a_n s_n \right) = {}^0\!\left(\sum_{n=1}^{\infty} a_n s_{n+1} \right) = F_\alpha(\{s_{n+1}\}),$$

showing that 1.7 is satisfied.   The truth of 1.6 is obvious, proving 3.6.

If $a_n = a_{\omega n}$, for infinite ω, where (a_{kn}) is a Toeplitz-matrix as above then 3.7 is satisfied provided

$$(3.8) \qquad\qquad \lim_{k \to \infty} |a_{k,\,n+1} - a_{kn}| = 0$$

in the standard sense, i.e. in R_0, and $a_n \geqq 0$ for all n, provided the

elements of the matrix are nonnegative.

A particular example of a sequence $\{a_n\}$ which provides a Banach-Mazur limit is given by 2.1.

4. Representation of continuous linear functionals on m in non-standard analysis. We have seen that every S-bounded form in a non-standard model of analysis gives rise to a continuous linear functional on m. Conversely, we may ask whether every continuous linear functional on m can be represented in this way. In this direction, we have the following rather strong result, whose proof is the main purpose of this section.

4.1. THEOREM. *There exists a non-standard model of analysis, $^*R$, such that for every continuous linear functional $F(x)$ on m, there is an S-bounded form α in $^*R$ such that $F(x) \equiv F_\alpha(x)$ identically on m.*

In order to prove 4.1, we require some auxiliary considerations which refer to the standard case (i.e. to R_0).

Let $\sigma^1, \cdots, \sigma^\nu$ be any finite sequence of elements of m, where

$$\sigma^i = \{s^i_1, s^i_2, \cdots, s^i_n, \cdots\}, \qquad\qquad i = 1, \cdots, \nu$$

and let ε be positive, otherwise arbitrary. Then (compare Ref. 1, p. 69) there exists a positive integer μ, such that for every set of real numbers $\lambda_1, \cdots, \lambda_\nu$,

$$\| \lambda_1\sigma^1 + \lambda_2\sigma^2 + \cdots + \lambda_\nu\sigma^\nu \| \leqq \max_{1 \leqq i \leqq \mu} | \lambda_1 s^1_i + \lambda_2 s^2_i + \cdots + \lambda_\nu s^\nu_i |(1 + \varepsilon) .$$

It follows that if we define the μ-dimensional vectors τ^i, $i = 1, \cdots, \nu$, by

$$\tau^i = (s^i_1, s^i_2, \cdots, s^i_\mu)$$

and use the sup (lowest upper bound) norm in the space m' spanned by these vectors, then for every set of real numbers $\lambda_1, \cdots, \lambda_\nu$

$$\| \lambda_1\sigma^1 + \lambda_2\sigma^2 + \cdots + \lambda_\nu\sigma^\nu \| \leqq (1 + \varepsilon)\| \lambda_1\tau^1 + \lambda_2\tau^2 + \cdots + \lambda_v\tau^\nu \| .$$

Hence, for any continuous linear functional $F(x)$ on m,

$$| F(\lambda_1\sigma^1 + \cdots + \lambda_\nu\sigma^\nu) | \leqq \| F(x) \| \, \| \lambda_1\sigma^1 + \cdots + \lambda_\nu\sigma^\nu \|$$
$$\leqq (1 + \varepsilon)\| F(x) \| \, \| \lambda_1\tau^1 + \cdots + \lambda_\nu\tau^\nu \|$$

which is equivalent to

$$(4.2) \quad | \lambda_1 F(\sigma^1) + \cdots + \lambda_2 F(\sigma^\nu) | \leqq (1 + \varepsilon)\| F(x) \| \, \| \lambda_1\tau^1 + \cdots + \lambda_\nu\tau^\nu \| .$$

In m', define a functional $G(x)$ by

$$(4.3) \qquad G(\lambda_1 \tau^1 + \cdots + \lambda_\nu \tau^\nu) = \lambda_1 F(\sigma^1) + \cdots + \lambda_\nu F(\sigma^\nu) \,.$$

This definition is unique, for 4.2 shows that if two representations of the form $\lambda_1 \tau^1 + \cdots + \lambda_\nu \tau^\nu$ coincide, then the corresponding expressions on the right hand side of 4.3 coincide. Moreover, by 4.3, $G(x)$ is clearly additive, and by 4.2 it is also continuous, with norm $\|G(x)\| \leqq (1 + \varepsilon)\| F(x) \|$ (where the norm of F and G refer to the spaces m and m' respectively).

Now let m_μ be the full μ-dimensional real space under the sup norm, so that m' is a subspace of m_μ. Extend $G(x)$ to all of m_μ without increasing its norm and, without fear of confusion, call the result again $G(x)$. Let $\xi^i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{i\mu})$, $i = 1, \cdots, \mu$, where δ_{ik} is the Kronecker delta, and put $a_i = G(\xi^i)$. Let

$$\xi^0 = \xi^1 \, sg \, a_1 + \xi^2 \, sg \, a_2 + \cdots + \xi^\mu \, sg \, a_\mu$$

(where $sg \, a$ is -1, 0, or 1 according a is negative, zero, or positive) then $\| \xi^0 \| = 1$ provided at least one of the a_i is different from zero. Also,

$$G(\xi^0) = |\, a_1 \,| + |\, a_2 \,| + \cdots + |\, a_\mu \,| \leqq \| G(x) \| \, \| \xi^0 \|$$
$$\leqq (1 + \varepsilon)\| F(x) \| \, |\, \xi^0 \,|$$

and so, in any case

$$|\, a_1 \,| + |\, a_2 \,| + \cdots + |\, a_3 \,| \leqq (1 + \varepsilon)\| F(x) \| \,.$$

Now

$$\tau^i = s_1^i \xi^1 + s_2^i \xi^2 + \cdots + s_\mu^i \xi^\mu$$

and so

$$F(\sigma^i) = G(\tau^i) = a_1 s_1^i + a_2 s_2^i + \cdots + a_\mu s_\mu^i \,.$$

Summing up, we have established (compare Ref. 1, p. 70).

4.4. THEOREM. *Let $F(x)$ be a continuous linear functional on m, let $\varepsilon > 0$, and let $\sigma^i = \{s_n^i\}$ be arbitrary elements of m, $\varepsilon = 1, \cdots, \nu$. Then there exist real numbers $a_1, \cdots, a_\mu$, such that*

$$(4.5) \qquad a_1 s_1^i + a_2 s_\lambda^i + \cdots + a_\mu s_\mu^i = F(\sigma_i) \,, \qquad\qquad i = 1, \cdots, \nu$$

while

$$(4.6) \qquad a_1 \,| + |\, a_2 \,| + \cdots + |\, a_\mu \,| \leqq (1 + \varepsilon)\| F(x) \| \,.$$

We are now in a position to prove 4.1.

Consider the statement:

"x is an absolutely convergent sequence, $x = \{x_1, x_2, \cdots\}$ such that $\sum_{n=1}^{\infty} |x_n|$ does not exceed y, and z is a bounded sequence, $z = \{z_1, z_2, z_3, \cdots\}$, such that $\sum_{n=1}^{\infty} x_n z_n$ is equal to w."

This may be formulated in L as a predicate $Q(x, y, z, w)$ whose extralogical constants occur in K_0. We now extend our vocabulary by introducing for each bounded linear functional $F(x)$ on m a constant a_F, different a_F being used for different functionals. For any continuous linear functional $F(x)$ on m, and for any σ in M, we define a sequence of formal sentences $Y_n(F, \sigma)$ by

$$(4.7) \qquad Y_n(F, \sigma) = Q\left(a_F, \left(1 + \frac{1}{n}\right)\| F(x) \|, \sigma, F(\sigma)\right) \qquad n = 1, 2, \cdots .$$

Let K_1 be the set of all sentences which are obtained in this way. We claim that the set $H = K_0 \cup K_1$ is consistent. In order to establish this fact it is sufficient to show that $H' = K_0 \cup K_1'$ is consistent where K_1' is an arbitrary finite subset of K_1. We may limit the class of K_1' to be considered somewhat by observing that for any F and σ,

$$Y_n(F, \sigma) \supset Y_m(F, \sigma)$$

is deducible from K_0 provided $n > m$. This implies that we may suppose the subscripts of the sentences which belong to the given K_1' to be all equal. Indeed, if this is not the case from the outset we replace them all by the greatest subscripts which occurs in K_1' to begin with.

Suppose in accordance with this remark that K_1' consists of the sentences

$$(4.8) \qquad \begin{aligned} &Y_n(F_1, \sigma^1), \ Y_n(F_1, \sigma^2), \ \cdots, \ Y_n(F_1, \sigma^{\nu_1}) , \\ &Y_n(F_2, \sigma^{\nu_1+1}), \ Y_n(F_2, \sigma^{\nu_1+2}), \ \cdots, \ Y_n(F_2, \sigma^{\nu_2}) , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &Y_n(F_k, \sigma^{\nu_{k-1}+1}), \ Y_n(F_k, \sigma^{\nu_{k-1}+2}), \ \cdots, \ Y_n(F_k, \sigma^{\nu_k}) . \end{aligned}$$

In order to prove that H' is consistent we shall show that, with a suitable interpretation of the constants $a_{F_1}, \cdots, a_{F_k}$, R_0 becomes a model of H'. It is in fact evident that R_0 is a model of K_0. Interpreting a_{F_1} as the infinite sequence

$$\{a_1, \cdots, a_\mu, 0, 0, 0, \cdots\}$$

for a suitable set of real numbers $(a_1, \cdots, a_\mu)$ such as exists according

to Theorem 4.4, if we put $\varepsilon = 1/n$ and identify F with F_1, and $\sigma^1, \cdots, \sigma^\nu$ with the present $\sigma^1, \cdots, \sigma^{\nu_1}$ respectively, we find that the sentences in the first line of 4.8 also holds in R_0. A similar procedure shows that, with the appropriate interpretation, the remaining sentences of 4.8 also hold in R_0. This shows that H' is consistent and hence, that H is consistent.

Let $^*R$ be a model of H. For any continuous linear functional on m, $F(x)$, the constant a_F denotes an infinite sequence

$$\alpha = \{a_1, a_2, a_3, \cdots, a_k, \cdots\}$$

(where the subscript varies over the positive integers in $^*R$) such that for any $\sigma = \{s_k\}$ in m,

(4.9) $$\sum_{k=1}^{\infty} a_k s_k = F(\sigma) .$$

Moreover, for any finite positive integer n,

(4.10) $$\sum_{k=1}^{\infty} |a_k| \leqq \left(1 + \frac{1}{n}\right)\| F(x) \| .$$

Indeed, the validity of 4.9 and 4.10 is asserted by $Y_n(F, \sigma)$, and this sentence holds in $^*R$. Since n is a positive and finite integer, but otherwise arbitrary, we may conclude from 4.10 that

(4.11) $$^0\!\left(\sum_{k=1}^{\infty} |a_k|\right) \leqq \| F(x) \|$$

so that a is an S-bounded form. Equation 4.9 shows that $F_a(x) \equiv F(x)$ identically on m since it implies the weaker relation $^0(\sum_{k=1}^{\infty} a_k s_k) = F(\sigma)$. $^*R$ is a model of K_0 but it cannot be the standard model, R_0, for it is not true that every continuous linear functional in ^0R can be represented as in 4.9. This completes the proof of 4.1.

Suppose now that we have strict inequality in 4.11, then there exists a standard number M, $0 < M < \| F(x) \|$ such that

$$\sum_{k=1}^{\infty} |a_k| \leqq M .$$

Hence, by 4.9,

$$|F(\sigma)| = \left|\sum_{k=1}^{\infty} a_k s_k\right| \leqq \| \sigma \| \sum_{k=1}^{\infty} |a_k| \leqq \| \sigma \| M$$

for all σ in m. But this contradicts the definition of $\| F(x) \|$ and shows that 4.11 may be replaced by

(4.12) $$^0\!\left(\sum_{k=1}^{\infty} |a_k|\right) = \| F(x) \| .$$

We also observe that the sequences which correspond to the a_{F_i} in the consistency proof for H' are all finite, i.e. they take the value zero for sufficiently large subscripts. We may therefore add this as a requirement for the sequences $\alpha = \{a_1, a_2, \cdots, a_k, \cdots\}$ and we may attain in this way that every α is Q-finite i.e. that a_k is equal to zero for subscripts greater than some ν which depends on α and which may be infinite. Our remarks are summed up in the following corollary.

4.13. COROLLARY TO 4.1. *In 4.1, we may add the three conditions that the sequences α are finite, that $\sum_{k=1}^{\infty} a_k s_k = F(\sigma)$ for every σ in m—and not only $^0(\sum_{k=1}^{\infty} a_k s_k) = F(\sigma)$—and that $^0(\sum_{k=1}^{\infty} |a_k|) = \| F(x) \|$.*

5. Representation of continuous linear functionals on m by ultra-filter limits. In order to be able to follow the contents of this section, the reader should have some familiarity with the theory of ultra-powers. He will then see that the non-standard model of analysis constructed in the preceding section may be obtained as an ultrapower of R_0. Let $J = \{r\}$ be the index set of the ultrapower and let D be the ultrafilter on J which determines $^*R$. An infinite sequence in $^*R$ then is an equivalence class, with respect to D, of sets of infinite sequences $\sigma^\nu = \{s_1^\nu, s_2^\nu, \cdots\}$ which are indexed in J.

A set of real numbers a_ν which is indexed in J tends to the limit a in the ultrafilter D, and we write

$$\lim_J a_\nu = a(D) \, ,$$

if for every positive ε, the set $\{\nu \,|\, |\, a - a_\nu \,| < \varepsilon\}$ belongs to D. With this notation, we have the following "translation" of 4.1, taking into account 4.13.

5.1. THEOREM. *There exists an index set $J = \{\nu\}$ and an ultra-filter D on J such that—*

for every continuous linear functional $F(x)$ on the space m there exists a set of sequences $a^\nu = \{a_1^\nu, a_2^\nu, \cdots\}$ which is indexed by J and which satisfies the following conditions.

5.2. *The sequences a^ν are finite in the sense that $a_n^\nu = 0$ for all ν greater than some positive integer k_ν,*

$$(5.3) \qquad\qquad \lim \sum_{n=1}^{k_\nu} |\, a_n \,| = \| F(x) \| \qquad\qquad (D) \, ,$$

and

5.4. *for any $\sigma = \{s_1, s_2, \cdots\}$ in m,*

$$\lim_J \sum_{n=1}^{k_\nu} a_n s_n = F(\sigma)(D) \ .$$

Moreover, in view of 4.9 we may replace 5.4 by the condition

5.5. *for any σ in m, the set*

$$\left\{\nu \,\middle|\, \sum_{n=1}^{k_\nu} a_n s_n = F(\sigma)\right\}$$

belongs to D—
which is somewhat stronger than 5.4.

Theorem 5.1 is not only an immediate consequence of 4.1 and 4.13, but conversely it implies those results as we may see by taking as the required non-standard model of analysis the ultrapower $(R_0)_D^J$, i.e. the direct product of a set of copies of R_0 which are indexed in J, reduced with respect to the ultrafilter D. Theorem 5.1 has the "advantage" that it does not involve the notion of a non-standard model of analysis but it lacks the intuitive significance of the preceeding theory.

Except for the requirement of positivity in the definition of the Banach-Mazur limit all our concepts and results carry over to the complex case. The proofs also remain applicable with minor modifications. In particular, in § 4.4 *sg a* has to be replaced by $e^{-i \arg a}$ for $a \neq 0$.

The reader may find it interesting to compare 5.1 above with the representation of continuous linear functionals on m by Moore-Smith limits, which is due to Hildebrandt (Ref. 4) and with the matrix representation due to Mazur (compare Ref. 1, p. 72) which, however, applies only to separable subspaces of m.

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