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2-SIGNALIZERS OF FINITE GROUPS

JOHN GRIGGS THOMPSON

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DEFINITION. Let π be a set of primes and \mathfrak{G} a group. The subgroup \mathfrak{A} of \mathfrak{G} is a π -signalizer of \mathfrak{G} if and only if $|\mathfrak{A}|$ and $|\mathfrak{G} : N_{\mathfrak{G}}(\mathfrak{A})|$ are π' -numbers.¹

Let $s_x(\mathfrak{G}) = \max |\mathfrak{A}|$, \mathfrak{A} ranging over the π -signalizers of \mathfrak{G} .

THEOREM 1. For each pair of integers m, n , there are only finitely many (isomorphism classes of) finite groups \mathfrak{G} such that

$$|\mathfrak{G}|_2 \leq m \quad \text{and} \quad s_2(\mathfrak{G}) \leq n.$$

Proof. Let $|\mathfrak{G}|_2 = 2^r$ and proceed by induction on r , as the theorem holds for $r = 0$. We must bound $|\mathfrak{G}|$ by a function of r and $s_2(\mathfrak{G})$. If \mathfrak{H} is a normal $2'$ -subgroup of \mathfrak{G} , then $s_2(\mathfrak{G}/\mathfrak{H}) = s_2(\mathfrak{G})/|\mathfrak{H}|$, so we assume without loss of generality that 1 is the only normal $2'$ -subgroup of \mathfrak{G} . Suppose \mathfrak{G} contains a normal 2-subgroup $\mathfrak{R} \neq 1$. It suffices to bound $|\mathfrak{G} : \mathfrak{R}|$, so it suffices to bound $s_2(\mathfrak{G}/\mathfrak{R})$, by our induction hypothesis. Let $\mathfrak{A}/\mathfrak{R}$ be a 2-signalizer of $\mathfrak{G}/\mathfrak{R}$ and let $\mathfrak{X}/\mathfrak{R}$ be a S_2 -subgroup of $\mathfrak{G}/\mathfrak{R}$ normalizing $\mathfrak{A}/\mathfrak{R}$. Let $\mathfrak{B} = C_{\mathfrak{A}}(\mathfrak{R})$, so that $\mathfrak{B} = Z(\mathfrak{R}) \times \mathfrak{C}$, where $\mathfrak{C} \text{ char } \mathfrak{B} \triangleleft \mathfrak{A}\mathfrak{X}$. Hence, $|\mathfrak{C}| \leq s_2(\mathfrak{G})$, \mathfrak{C} being a 2-signalizer of \mathfrak{G} . Hence, $|\mathfrak{A}| \leq |\mathfrak{R}| |\mathfrak{C}| |\text{Aut } \mathfrak{R}|_2 \leq s_2(\mathfrak{G}) 2^{r^2}$. This gives a bound for $|\mathfrak{A} : \mathfrak{R}|$, so also for $|\mathfrak{G} : \mathfrak{R}|$.

We therefore assume without loss of generality that 1 is the only normal $2'$ -subgroup of \mathfrak{G} and 1 is the only normal 2-subgroup of \mathfrak{G} . Hence, 1 is the only normal abelian subgroup of \mathfrak{G} , so that \mathfrak{R} , the join of all minimal normal subgroups of \mathfrak{G} , is the direct product of subgroups \mathfrak{R}_i , each of which is a non abelian simple group, $1 \leq i \leq t$. Since $1 = C_{\mathfrak{G}}(\mathfrak{R})$, it suffices to bound $|\mathfrak{R}|$. Clearly, each \mathfrak{R}_i is of even order, since 1 is the only normal $2'$ -subgroup of \mathfrak{G} . This yields $t \leq r$, so it suffices to bound $|\mathfrak{R}_1|$. Let \mathfrak{X} be a S_2 -subgroup of \mathfrak{G} and let J be an involution in $Z(\mathfrak{X}) \cap \mathfrak{R}$, $J = J_1 \cdots J_t$, J_i in \mathfrak{R}_i , $1 \leq i \leq t$. Let $\mathfrak{A}/\langle J \rangle$ be a 2-signalizer of $C_{\mathfrak{G}}(J)/\langle J \rangle$, so that $\mathfrak{A} = \langle J \rangle \times \mathfrak{B}$, where \mathfrak{B} is a 2-signalizer of \mathfrak{G} . Hence, $s_2(C_{\mathfrak{G}}(J)/\langle J \rangle) \leq s_2(\mathfrak{G})$. By our induction hypothesis, $|C_{\mathfrak{G}}(J)|$ is bounded by a function of r and $s_2(\mathfrak{G})$. Since $C_{\mathfrak{R}}(J) = C_{\mathfrak{R}_1}(J_1) \times \cdots \times C_{\mathfrak{R}_t}(J_t)$, $|C_{\mathfrak{R}_1}(J_1)|$ is bounded. If $J_1 = 1$, we have a bound for $|\mathfrak{R}_1|$; otherwise, $|\mathfrak{R}_1| \leq \{|C_{\mathfrak{R}_1}(J_1)|^2\}!$, by a well known theorem of Brauer and Fowler [1]. The proof is complete.

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¹ The notation in this paper conforms with *Solvability of Groups of Odd Order*, W. Feit and J. Thompson, this Journal, 1963, and is for the most part self-explanatory.

REMARK. Defining $f(0, n) = n$, $n = 1, 2, \dots$, and $f(m + 1, n) = z^z$, where $z = f(m, 2^m n)^2$, $m = 0, 1, \dots$, it follows readily that if $|\mathfrak{G}|_2 = 2^m$ and $s_2(\mathfrak{G}) = n$, then $|\mathfrak{G}| \leq f(m, n)$. While the alternating groups show that $|\mathfrak{G}|$ is not bounded by a polynomial function of $|\mathfrak{G}|_2$ and $s_2(\mathfrak{G})$, the given f is hardly to the point.

THEOREM 2. Suppose the S_2 -subgroup \mathfrak{X} of \mathfrak{G} is abelian and $s_2(\mathfrak{G}) = 1$. Then $\mathcal{O}^1(\mathfrak{X}) \triangleleft \mathfrak{G}$.

Proof. We proceed by induction on $|\mathfrak{G}|$. Let \mathfrak{H} be the largest normal 2-subgroup of \mathfrak{G} , and let $\mathfrak{C} = C_{\mathfrak{G}}(\mathfrak{H})$. Suppose $\mathfrak{C} \subset \mathfrak{G}$. Then by induction, $\mathcal{O}^1(\mathfrak{T}) \triangleleft \mathfrak{C}$, so $\mathcal{O}^1(\mathfrak{X}) \text{ char } \mathfrak{C}$, as $\mathcal{O}^1(\mathfrak{X})$ is the set of squares of 2-elements of \mathfrak{C} . Hence, $\mathcal{O}^1(\mathfrak{X}) \triangleleft \mathfrak{G}$ and we are done. Suppose $\mathfrak{C} = \mathfrak{G}$, but $\mathfrak{H} \neq 1$. Let J be an involution of \mathfrak{H} . If $\langle J \rangle$ is a direct factor of \mathfrak{X} , then $\langle J \rangle$ is a direct factor of \mathfrak{G} and we are done by induction. If $\langle J \rangle$ is not a direct factor of \mathfrak{G} , then $J \in \mathcal{O}^1(\mathfrak{X})$, so $\mathcal{O}^1(\mathfrak{X}/\langle J \rangle) = \mathcal{O}^1(\mathfrak{X})/\langle J \rangle$, and we are again done by induction. Hence, we may assume $\mathfrak{H} = 1$.

Since 1 is the only normal 2'-subgroup of \mathfrak{G} , the join \mathfrak{N} of all minimal normal subgroups of \mathfrak{G} is the direct product of subgroups \mathfrak{N}_i , each being a non abelian simple group, $1 \leq i \leq n$. Furthermore, $1 = C_{\mathfrak{G}}(\mathfrak{N})$.

Let J be an involution of \mathfrak{X} . Since $C_{\mathfrak{G}}(J) \subset \mathfrak{G}$, we have $\mathcal{O}^1(\mathfrak{X}) \triangleleft C_{\mathfrak{G}}(J)$. Hence, $N_{\mathfrak{G}}(\mathcal{O}^1(\mathfrak{X})) = \mathfrak{N}$ contains the centralizer of each of its involutions. Suppose J, K are involutions of \mathfrak{N} which are not conjugate in \mathfrak{G} . If K_1 is any conjugate of K , there is an involution L centralizing J and K_1 . Hence, $L \in C_{\mathfrak{G}}(J) \subseteq \mathfrak{N}$, and $K_1 \in C_{\mathfrak{G}}(L) \subseteq \mathfrak{N}$, and so \mathfrak{N} contains the normal closure of K in \mathfrak{G} . Hence, \mathfrak{N} contains the normal closure of each of its involutions, which implies that $\mathfrak{N} \subseteq \mathfrak{N}$. Hence, $[\mathfrak{N}, \mathcal{O}^1(\mathfrak{X})] \subseteq \mathcal{O}^1(\mathfrak{X}) \cap \mathfrak{N}$. As every c.f. of \mathfrak{N} is nonsolvable, $\mathcal{O}^1(\mathfrak{X}) \cap \mathfrak{N} = 1$. Since $1 = C_{\mathfrak{G}}(\mathfrak{N})$, so also $1 = \mathcal{O}^1(\mathfrak{X})$, and we are done. We may therefore assume that any two involutions of \mathfrak{N} are conjugate in \mathfrak{G} . This implies that \mathfrak{X} is homocyclic. If \mathfrak{X} is elementary, we are done. Otherwise, $C_{\mathfrak{G}}(\mathcal{O}^1(\mathfrak{X}))$ has a normal 2-complement, which must be 1, since $s_2(\mathfrak{G}) = 1$. This in turn implies that $\mathfrak{X} = C_{\mathfrak{G}}(\mathcal{O}^1(\mathfrak{X})) \triangleleft \mathfrak{N}$. As \mathfrak{N} contains the centralizer of each of its involutions, \mathfrak{X} is a T.I. set in \mathfrak{G} . By a fundamental result of Suzuki [2], \mathfrak{X} is elementary. The proof is complete.

Conjecture. If \mathfrak{G} is a simple group, then every 2-signalizer of \mathfrak{G} is abelian.

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2. M. Suzuki, *Finite groups whose Sylow 2-subgroups are independent*, (to appear).

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NAVAL ORDNANCE TEST STATION

Richard Arens, <i>Normal form for a Pfaffian</i>	1
Charles Vernon Coffman, <i>Non-linear differential equations on cones in Banach spaces</i>	9
Ralph DeMarr, <i>Order convergence in linear topological spaces</i>	17
Peter Larkin Duren, <i>On the spectrum of a Toeplitz operator</i>	21
Robert E. Edwards, <i>Endomorphisms of function-spaces which leave stable all translation-invariant manifolds</i>	31
Erik Maurice Ellentuck, <i>Infinite products of isolos</i>	49
William James Firey, <i>Some applications of means of convex bodies</i>	53
Haim Gaifman, <i>Concerning measures on Boolean algebras</i>	61
Richard Carl Gilbert, <i>Extremal spectral functions of a symmetric operator</i>	75
Ronald Lewis Graham, <i>On finite sums of reciprocals of distinct nth powers</i>	85
Hwa Suk Hahn, <i>On the relative growth of differences of partition functions</i>	93
Isidore Isaac Hirschman, Jr., <i>Extreme eigen values of Toeplitz forms associated with Jacobi polynomials</i>	107
Chen-jung Hsu, <i>Remarks on certain almost product spaces</i>	163
George Seth Innis, Jr., <i>Some reproducing kernels for the unit disk</i>	177
Ronald Jacobowitz, <i>Multiplicativity of the local Hilbert symbol</i>	187
Paul Joseph Kelly, <i>On some mappings related to graphs</i>	191
William A. Kirk, <i>On curvature of a metric space at a point</i>	195
G. J. Kurowski, <i>On the convergence of semi-discrete analytic functions</i>	199
Richard George Laatsch, <i>Extensions of subadditive functions</i>	209
V. Marić, <i>On some properties of solutions of $\Delta\psi + A(r^2)X\nabla\psi + C(r^2)\psi = 0$</i> ...	217
William H. Mills, <i>Polynomials with minimal value sets</i>	225
George James Minty, Jr., <i>On the monotonicity of the gradient of a convex function</i>	243
George James Minty, Jr., <i>On the solvability of nonlinear functional equations of 'monotonic' type</i>	249
J. B. Muskat, <i>On the solvability of $x^e \equiv e \pmod{p}$</i>	257
Zeev Nehari, <i>On an inequality of P. R. Bessack</i>	261
Raymond Moos Redheffer and Ernst Gabor Straus, <i>Degenerate elliptic equations</i>	265
Abraham Robinson, <i>On generalized limits and linear functionals</i>	269
Bernard W. Roos, <i>On a class of singular second order differential equations with a non linear parameter</i>	285
Tôru Saitô, <i>Ordered completely regular semigroups</i>	295
Edward Silverman, <i>A problem of least area</i>	309
Robert C. Sine, <i>Spectral decomposition of a class of operators</i>	333
Jonathan Dean Swift, <i>Chains and graphs of Ostrom planes</i>	353
John Griggs Thompson, <i>2-signalizers of finite groups</i>	363
Harold Widom, <i>On the spectrum of a Toeplitz operator</i>	365