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Given a function $\phi \in L_\infty(-\pi, \pi)$, the Toeplitz operator T_ϕ is the operator on H_2 (the set of $f \in L_2$ with Fourier series of the form $\sum_0^\infty c_n e^{in\theta}$) which consists of multiplication by ϕ followed by P, the natural projection of L_2 onto H_2: if $f \sim \sum_{-\infty}^\infty c_n e^{in\theta}$ then $Pf \sim \sum_0^\infty c_n e^{in\theta}$. Succinctly,

$$T_\phi f = P(\phi f) \qquad f \in H_2.$$

In [5] a necessary and sufficient condition on ϕ was given for the invertibility of T_ϕ. This will be stated below. (The paper [5] is needlessly complicated. In a recent paper of Devinatz [1], however, all results of [5] and more are proved without undue complication in a general Dirichlet algebra setting.) Halmos [2] has posed the following as a test question for any theory of invertibility of Toeplitz operators: *Is the spectrum of a Toeplitz operator necessarily connected?* We shall shown here that the answer is *Yes.*

The proof consists mainly of applications of Theorem I of [5], which says the following.

A necessary and sufficient condition for the invertibility of T_ϕ is the existence of function ϕ_+ and ϕ_- belonging respectively to H_2 and $\bar{H}_2$ (the set of complex conjugates of H_2 functions) such that

(a) $\phi = \phi_+\phi_-$,

(b) $\phi_+^{-1} \in H_2$ *and* $\phi_-^{-1} \in \bar{H}_2$,

(c) *for* $f \in L_\infty$, $Sf = \phi_+^{-1}P\phi_-^{-1}f \in L_2$, *and* $f \to Sf$ *extends to a bounded operator on* L_2.

We don't want to prove the theorem here but we do have to say where the functions $\phi_\pm$ come from under the assumption that T_ϕ is ivertible. If we set

$$\psi_+ = T_\phi^{-1}1, \ \bar{\psi}_- = T_\phi^{*-1}1$$

then it can be shown that $\phi\psi_+\psi_- = c$, a constant. We must have $c \neq 0$ since $\psi_\pm$ can vanish only on sets of measure zero and ϕ is not identically zero. One then defines

$$\phi_+ = 1/\psi_+, \quad \phi_- = c/\psi_-$$

and (a) and (b) hold.

As for the relevance of condition (c), it turns out that the ex-

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tension of S, restricted to H_2, is exactly T_ϕ^{-1}. It follows that

$$(1) \qquad \| (Pf)\phi_- \|_2 \leqq \| \phi \|_\infty \| T_\phi^{-1} \| \| f\phi_- \|_2 \qquad f \in L_\infty .$$

Conversely, suppose there exists an M such that

$$\| (Pf)\phi_- \|_2 \leqq M \| f\phi_- \|_2 \qquad f \in L_\infty .$$

Then we can deduce

$$\| \phi_+^{-1} P \phi_-^{-1} f \|_2 \leqq M \| \phi^{-1} \|_\infty \| f \|_2 \qquad f \in L_\infty .$$

It is a simple consequence of (c) that $\| \phi^{-1} \|_\infty < \infty$. (See [5], Theorem I, corollary, or [1], Lemma 2.) Thus (c) may be replaced by

(c′) *$\phi^{-1} \in L_\infty$ and the map $f \to Pf$ is bounded in the space $L_2(|\phi_-|^2 d\theta)$.*

We shall need this fact.

To begin the proof of the connectedness of $\sigma(T_\phi)$, the spectrum of T_ϕ, let Λ be a compact set disjoint from $\sigma(T_\phi)$. (Think of Λ as being a simple closed curve surrounding a portion of $\sigma(T_\phi)$.) For each $\lambda \in \Lambda$ the operator $T_\phi - \lambda = T_{\phi-\lambda}$ is invertible, so we have the corresponding functions

$$\psi_+(\lambda) = (T_\phi - \lambda)^{-1} 1, \quad \bar{\psi}_-(\lambda) = (T_\phi - \lambda)^{*-1} 1$$

and the constant $c(\lambda)$ as described above, and

$$(2) \qquad \phi - \lambda = \phi_+(\lambda)\phi_-(\lambda)$$

where

$$\phi_+(\lambda) = 1/\psi_+(\lambda) , \quad \phi_-(\lambda) = c(\lambda)/\psi_-(\lambda) .$$

Let us consider the continuity of these various function of λ. It follows from the definition of $\psi_\pm(\lambda)$ and the continuity, in the uniform operator topology, of the mappings $\lambda \to (T_\phi - \lambda)^{-1}$ and $\lambda \to (T_\phi - \lambda)^{*-1}$, that $\lambda \to \psi_\pm(\lambda)$ are continuous functions from Λ to L_2. This implies that $\lambda \to c(\lambda)/(\phi - \lambda)$ is continuous from Λ to L_1. Since $\lambda \to \phi - \lambda$ is continuous from Λ to L_∞ we conclude that $\lambda \to c(\lambda)$ is continuous from Λ to L_1, so $c(\lambda)$ is a continuous complex valued function. Since $c(\lambda) \neq 0$ it follows also that $\lambda \to \phi_+(\lambda) = (\phi - \lambda)\psi_-(\lambda)/c(\lambda)$ and $\lambda \to \phi_-(\lambda) = (\phi - \lambda)\psi_+(\lambda)$ are continuous from Λ to L_2. To recapitulate, the four functions $\phi_\pm(\lambda)^{\pm 1}$ are L_2 continuous.

The next step is to take logarithms. Since both $\phi_+(\lambda)$ and $1/\phi_+(\lambda)$ belong to H_2, $\phi_+(\lambda)$ is an outer function. Recall that this means it has the representation

$$\phi_+(\lambda) = \alpha_+(\lambda) e^{\log|\phi_+(\lambda)| + i[\log|\phi_+(\lambda)|]^\sim}$$

where the tilde denotes conjugate function and

$$\alpha_+(\lambda) = \operatorname{sgn} \int \phi_+(\lambda) d\theta$$

is a constant of absolute value 1. Since $\phi_+(\lambda)^{\pm 1}$ are L_2 continuous so is $\log |\phi_+(\lambda)|$, and therefore also $[\log |\phi_+(\lambda)|]^{\sim}$ (since $u \to \tilde{u}$ is L_2 continuous). The continuity of the complex valued function $\alpha_+(\lambda)$ follows from the fact that $\int \phi_+(\lambda) d\theta$ is continuous and nonzero.

Similarly we can write

$$\phi_-(\lambda) = \alpha_-(\lambda) e^{\log|\phi_-(\lambda)| - i[\log|\phi_-(\lambda)|]^{\sim}}$$

with $\alpha_-(\lambda)$ continuous and nonzero. Putting our representations together and using (2) we have

$$(3) \qquad \phi - \lambda = \alpha(\lambda) e^{\log|\phi_+(\lambda)| + i[\log|\phi_+(\lambda)|]^{\sim}} e^{\log|\phi_-(\lambda)| - i[\log|\phi_-(\lambda)|]^{\sim}}$$

where $\alpha(\lambda) = \alpha_+(\lambda)\alpha_-(\lambda)$ is a continuous nowhere vanishing complex valued function.

The sum of the two exponents in (3), which we shall call $l(\lambda, \theta)$, is for each λ an element of L_2, and the map $\lambda \to l(\lambda, \cdot)$ is L_2 continuous. It is important that we be able to say that for each θ (or almost every θ), $l(\lambda, \theta)$ is a continuous function of λ. This is false for general L_2 valued functions but in our situation something as good is true.

LEMMA 1. *There is a null set* $N \subset (-\pi, \pi)$ *and a function* $L(\lambda, \theta)$ *defined on* $\Lambda \times N'$ *such that for each* λ

$$L(\lambda, \theta) = l(\lambda, \theta) \ a.e.,$$

for each $\theta \in N'$

$$L(\lambda, \theta) \text{ is continuous in } \lambda \, .$$

and for all $\lambda \in \Lambda$, $\theta \in N'$

$$\phi(\theta) - \lambda = \alpha(\lambda) e^{L(\lambda, \theta)} \, .$$

Proof. First we make sure that ϕ is defined everywhere and that its range has positive distance from Λ. This we can do since Λ is a compact set disjoint from $R(\phi)$, the essential range of ϕ. (Recall that $T_{\phi-\lambda}$ invertible implies $(\phi - \lambda)^{-1} \in L_\infty$.)

Take $\lambda_0 \in \Lambda$ and let $L_0(\lambda_0, \theta)$ be a function of θ which equals $l(\lambda_0, \theta)$ a.e. and for which

$$\phi(\theta) - \lambda_0 = \alpha(\lambda_0) e^{L_0(\lambda_0, \theta)}$$

everywhere. Let $U = \{\lambda \in \Lambda : |\lambda - \lambda_0| < \delta\}$ be a neighborhood of λ_0 so small that $\lambda \in U$ implies

$$\left|\frac{\alpha(\lambda)}{\alpha(\lambda_0)} - 1\right| < 1 ,$$

$$\left|\frac{\phi(\theta) - \lambda}{\phi(\theta) - \lambda_0} - 1\right| < 1 , \qquad \text{all } \theta.$$

We extend $L_0(\lambda_0\, \theta)$ to a function defined on $U \times (-\pi, \pi)$ by

$$(4) \qquad L_0(\lambda, \theta) = L_0(\lambda_0, \theta) + \log \frac{\phi(\theta) - \lambda}{\phi(\theta) - \lambda_0} - \log \frac{\alpha(\lambda)}{\alpha(\lambda_0)}$$

where the logarithms are defined by the usual power series. Clearly $L_0(\lambda, \theta)$ is continuous on U for each θ and $\phi(\theta) - \lambda = \alpha(\lambda)e^{L_0(\lambda,\theta)}$ everywhere on $U \times (-\pi, \pi)$. We shall show $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$, at least if δ is small enough. Let us set

$$u_+(\lambda) = \frac{\phi_+(\lambda)}{\alpha_+(\lambda)} = e^{\log|\phi_+(\lambda)| + i[\log|\phi_+(\lambda)|]^\sim}$$

$$u_-(\lambda) = \frac{\phi_-(\lambda)}{\alpha_-(\lambda)} = e^{\log|\phi_-(\lambda)| - i[\log|\phi_-(\lambda)|]^\sim}$$

and

$$v_\pm(\lambda) = e^{1/2L_0(\lambda\ \theta) \pm i/2\tilde{L}_0(\lambda,\theta)} .$$

We know $u_+(\lambda)^{\pm 1} \in L_2$. Actually for each λ, $u_+(\lambda)^{\pm 1} \in L_p$ for some $p > 2$ (the p depending on λ). The reason is the following. Condition (c′) in the criterion given above for invertibility implies that the map $f \to Pf$ is bounded in the space $L_2(|u_-(\lambda)|^2 d\theta)$. Helson and Szegö have determined ([3], Theorem 1) all measures $d\mu$ such that $f \to Pf$ is bounded in $L_2(d\mu)$. They are measures of the form

$$d\mu = e^{\rho + \tilde{\sigma}} d\theta$$

with $\rho \in L_\infty$ and $\|\sigma\|_\infty < \pi/2$. However

$$\|\sigma\|_\infty < \frac{\pi}{2} \text{ implies } e^{\tilde{\sigma}} \in L_1 .$$

This is a theorem of Zygmund. (See [6], p. 257.) A statement which is only at first glance stronger is

$$\|\sigma\|_\infty < \frac{\pi}{2} \text{ implies } e^{\pm\tilde{\sigma}} \in L_{1+\varepsilon} \text{ for some } \varepsilon > 0 .$$

Putting these things together we can conclude that $u_-(\lambda)^{\pm 1} \in L_p$ for some $p > 2$, and so also $u_+(\lambda)^{\pm 1} \in L_p$.

Since $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e., a routine check shows $|v_+(\lambda_0)| = c\,|u_+(\lambda_0)|$ a.e., where c is a nonzero constant, so we have $v_+(\lambda_0)^{\pm 1} \in L_{p_0}$. We shall show from this that $v_+(\lambda)^{\pm 1} \in L_2$ for all $\lambda \in U$ is δ is sufficiently small. We have

$$\frac{v_+(\lambda)}{v_+(\lambda_0)} = e^{1/2[L_0(\lambda,\theta) - L_0(\lambda_0,\theta)]} e^{i/2[\widetilde{L}_0(\lambda,\theta) - \widetilde{L}_0(\lambda_0,\theta)]} .$$

It follows from (4) that

$$\lim_{\lambda \to \lambda_0} \| L_0(\lambda, \theta) - L_0(\lambda_0, \theta) \|_\infty = 0 .$$

Therefore, from Zygmund's theorem again, we can say this: given any $q_0 < \infty$ there exists a δ so that $v_+(\lambda)/v_+(\lambda_0) \in L_{q_0}$ whenever $|\lambda - \lambda_0| < \delta$. If we choose q_0 so that $p_0^{-1} + q_0^{-1} = 1/2$ then we shall have $v_+(\lambda) \in L_2$. In fact me shall have $v_+(\lambda) \in H_2$. (Any function of the form $\exp(\sigma + i\tilde{\sigma})$, $\sigma \in L_2$, which belongs to L_2 also belongs to H_2; see [6], pp. 282–3.) Similarly

$$v_+(\lambda)^{-1} \in H_2 \text{ and } v_-(\lambda)^{\pm 1} \in \bar{H}_2 .$$

Now almost everywhere

$$u_+(\lambda) u_-(\lambda) = v_+(\lambda) v_-(\lambda) \left(= \frac{\phi - \lambda}{\alpha(\lambda)} \right)$$

so

$$\frac{u_+(\lambda)}{v_+(\lambda)} = \frac{v_-(\lambda)}{u_-(\lambda)} .$$

The left side belongs to H_1 and the right to $\bar{H}_1$ so both sides must be a constant $\beta = \beta(\lambda)$, and

$$\frac{v_-(\lambda)}{v_+(\lambda)} = \beta(\lambda)^2 \frac{u_-(\lambda)}{u_+(\lambda)} .$$

If we take the logarithm of the absolute value of both sides we obtain

$$[\mathscr{I} L_0(\lambda, \theta)]^\sim = 2 \log |\beta(\lambda)| + \log |\phi_-(\lambda)| - \log |\phi_+(\lambda)|$$

and so

$$\mathscr{I} L_0(\lambda, \theta) = [\log |\phi_+(\lambda)|]^\sim - [\log |\phi_-(\lambda)|]^\sim + \gamma(\lambda)$$

where $\gamma(\lambda)$ is, for each λ, a constant. Since

$$\mathscr{R} L_0(\lambda, \theta) = \log \left| \frac{\phi(\theta) - \lambda}{\alpha(\lambda)} \right| = \log |\phi_+(\lambda)| + \log |\phi_-(\lambda)|$$

we have upon adding,

$$L_0(\lambda, \theta) = l(\lambda, \theta) + i\gamma(\lambda) \qquad \text{a.e.}$$

Given a sequence $\lambda_n \to \lambda(\lambda_n, \lambda \in U)$ there is a subsequence $\lambda_{n'}$ for which $l(\lambda_{n'}, \theta) \to l(\lambda, \theta)$ a.e. (This follows from the L_2 continuity of l.) Since $L_0(\lambda_{n'}, \theta) \to L_0(\lambda, \theta)$ everywhere we have $\gamma(\lambda_{n'}) \to \gamma(\lambda)$. This shows that γ is a continuous function of λ. Since $\gamma(\lambda_0) = 0$ (recall that by definition, $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e.) and γ is for each λ an integral multiple of 2π, we must have $\gamma(\lambda) = 0$. Thus $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$.

Because of what we have done and the compactness of Λ we can find a finite open covering $\{U_k\}$ of Λ and for each k a function $L_k(\lambda, \theta)$ defined on $U_k \times (-\pi, \pi)$ so that $L_k(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U_k$, $L_k(\lambda, \theta)$ is continuous on U_k for each θ, and $\phi(\theta) - \lambda = \alpha(\lambda)e^{L_k(\lambda,\theta)}$ on $U_k \times (-\pi, \pi)$. Consider a pair of these open sets U_j and U_k, and let $\lambda_1, \lambda_2, \cdots$ be dense in $U_j \cap U_k$. For each λ_n there is a θ-set E_n of measure zero outside of which both $L_j(\lambda_n, \theta)$ and $L_k(\lambda_n, \theta)$ equal $l(\lambda_n, \theta)$. Thus if θ does not belong to $\bigcup E_n$ we have $L_j(\lambda_n, \theta) = L_k(\lambda_n, \theta)$ for all n. By the continuity of L_j and L_k in λ and the density of $\{\lambda_n\}$ we conclude that $L_j(\lambda, \theta) = L_k(\lambda, \theta)$ for all $\lambda \in U_j \cap U_k$ as long as θ does not belong to the set $F_{j,k} = \bigcup E_n$. Thus as long as θ does not belong to the set $N = \bigcup_{j,k} F_{j,k}$ any two of the functions $L_k(\lambda, \theta)$ agree where they are both defined. We can therefore combine all the L_k to define a single function $L(\lambda, \theta)$ on $\Lambda \times N'$ which has all the required properties.

LEMMA 2. *If Λ is a simple closed curve disjoint from $\sigma(T_\phi)$ then $R(\phi)$, the essential range of ϕ, lies entirely inside or entirely outside Λ.*

Proof. Lemma 1 says that $\phi(\theta) - \lambda = \alpha(\lambda)e^{L(\lambda,\theta)}$ where $L(\lambda, \theta)$ is continuous in λ for each $\theta \in N'$. For each θ the index (winding number) of Λ with respect to $\phi(\theta)$ is the index of $-\alpha(\lambda)$ with respect to the origin, and so is independent of θ. But the index is 1 if $\phi(\theta)$ is inside Λ and 0 if $\phi(\theta)$ is outside Λ, and this establishes the lemma.

LEMMA 3. *If Λ is a simple closed curve disjoint from $\sigma(T_\phi)$ and such that $R(\phi)$ lies entirely outside Λ, then $\sigma(T_\phi)$ lies entirely outside Λ.*

Proof. Write

$$\phi(\theta) - \lambda = e^{L(\lambda,\theta) + \log \alpha(\lambda)} \qquad \lambda \in \Lambda,\ \theta \in N'$$

where $\log \alpha(\lambda)$ denotes a continuous logarithm of $\alpha(\lambda)$. This exists since $\alpha(\lambda)$ has index zero. Let $d\mu_z$ be the Borel measure on Λ which solves the interior Dirichlet problem, i.e., if f is a continuous function on Λ then $\int f(\lambda)d\mu_z(\lambda)$ is the value at the point z inside Λ of the function harmonic inside Λ, continuous on the union of Λ and its inside, and equal to f on Λ. Now $L(\lambda, \theta) + \log \alpha(\lambda)$ is (for fixed $\theta \in N'$) a continuous logarithm of $\phi(\theta) - \lambda$. Since $\phi(\theta)$ is outside Λ this can be extended to a continuous logarithm of $\phi(\theta) - z$ for z inside Λ. The extension is a harmonic function, so

$$\int [L(\lambda, \theta) + \log \alpha(\lambda)] d\mu_z(\lambda)$$

is the value of the extension at z. Consequently

$$\phi(\theta) - z = e^{\int [L(\lambda,\theta) + \log\alpha(\lambda)] d\mu_z(\lambda)} \,. \tag{5}$$

The integral $I(\theta) = \int L(\lambda, \theta) d\mu_z(\lambda)$ is a pointwise integral, i.e., for each θ, $L(\lambda, \theta)$ is a Borel measurable function of λ and $I(\theta)$ is its integral. We prefer to think of it as a weak integral, i.e., I is the unique L_2 function which satisfies, for all $u \in L_2$,

$$(I, u) = \int (L(\lambda, \theta), u(\theta)) d\mu_z(\lambda) \,.$$

This identity follows from Fubini's theorem. If we use the fact that $L(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each λ, we can write (5) as

$$\phi(\theta) - z = e^{\int \log\alpha(\lambda) d\mu_z(\lambda)} e^{\int \log|\phi_+(\lambda)| d\mu_z(\lambda) + i\int [\log|\phi_+(\lambda)|]^\sim d\mu_z(\lambda)} \cdot e^{\int \log|\phi_-(\lambda)| d\mu_z(\lambda) - i\int [\log|\phi_-(\lambda)|]^\sim d\mu_z(\lambda)}$$

where all integrals are weak integrals. Now $\sim$ commutes with integration respect to $d\mu_z(\lambda)$; this follows from the definition of $\sim$ in terms of Fourier coefficients. Thus if we set

$$A = e^{\int \log\alpha(\lambda) d\mu_z(\lambda)}$$
$$t_+ = \int \log|\phi_+(\lambda)| \, d\mu_z(\lambda)$$
$$t_- = \int \log|\phi_-(\lambda)| \, d\mu_z(\lambda)$$

we have

$$\phi - z = Ae^{t_+ + i\tilde{t}_+} e^{t_- - i\tilde{t}_-} \,.$$

We shall show that this factorization exhibits the invertibility of $T_\phi - z$. Set

$$\phi_+ = Ae^{t_+ + i\tilde{t}_+}, \quad \phi_- = e^{t_- - i\tilde{t}_-}.$$

We must verify that $\phi_+^{\pm 1} \in H_2$, that $\phi_-^{\pm 1} \in \bar{H}_2$, and that the map $f \to Pf$ is bounded in $L_2(|\phi_-|^2 d\theta)$.

The following fact is crucial. *If* $w_1, w_2 \geqq 0$ *satisfy*

$$\int |Pf|^2 w_i d\theta \leqq M \int |f|^2 w_i d\theta \qquad (i = 1, 2)$$

for all $f \in L_\infty$, *and* $w = w_1^\alpha w_2^{1-\alpha} (0 \leqq \alpha \leqq 1)$, *then also*

$$\int |Pf|^2 w \, d\theta \leqq M \int |f|^2 w \, d\theta .$$

This follows from an interpolation theorem first proved for general operators and weight functions by Stein ([4], Theorem 2). We shall need an extension of this theorem to families of weight functions, and for convenience we state this extension together with another little fact as,

SUBLEMMA. *Assume* $\lambda \to r(\lambda, \theta)$ *is continuous from the compact set* Λ *to real* L_2 *and such that for all* λ

$$\int e^{r(\lambda,\theta)} d\theta \leqq K .$$

Let μ *be a nonnegative Borel measure on* Λ *with* $\mu(\Lambda) = 1$. *Then*

$$\int e^{\int r(\lambda,\theta) d\mu(\lambda)} d\theta \leqq K .$$

If in addition

$$\int |Pf|^2 e^{r(\lambda,\theta)} d\theta \leqq M \int |f|^2 e^{r(\lambda,\theta)} d\theta$$

for all $f \in L_\infty$, *then also*

$$\int |Pf|^2 e^{\int r(\lambda,\theta) d\mu(\lambda)} d\theta \leqq M \int |f|^2 e^{\int r(\lambda,\theta) d\mu(\lambda)} d\theta .$$

Suppose for the moment that this has been established. If we apply the first part of the sublemma to the four functions $\pm \log |\phi_\pm(\lambda)|^2$ and recall that by continuity the norms $\| \phi_\pm(\lambda)^{\pm 1} \|_2$ are uniformly bounded on Λ, we conclude that

$$e^{\pm t_\pm} = e^{\int \log |\phi_\pm(\lambda)|^{\pm 1} d\mu_z(\lambda)}$$

belong to L_2, and so $\phi_+^{\pm 1} \in H_2$ and $\phi_-^{\pm 1} \in \bar{H}_2$. Next it follows from (c′)

of the criterion for invertibility and the fact that $T_\varphi - \lambda$ is invertible for each $\lambda \in \Lambda$ that

$$\int | Pf |^2 | \phi_-(\lambda) |^2 d\theta \leqq M \int | f |^2 | \phi_-(\lambda) |^2 d\theta$$

for all $f \in L_\infty$; M can be chosen independently of λ since Λ is bounded away from $\sigma(T_\phi)$. (See (1).) Therefore, by the sublemma again,

$$\int | Pf |^2 e^{2t_-} d\theta \leqq M \int | f |^2 e^{2t_-} d\theta ,$$

i.e., $f \to Pf$ is bounded in $L_2(| \phi_- |^2 d\theta)$. This concludes the proof of invertibility of $T_\phi - z$. Since $T_\phi - z$ is invertible for any z inside Λ we conclude that $\sigma(T_\phi)$ lies entirely outside Λ.

It remains to prove the sublemma. For each integer n let $E_{n,i}$ $(i = 1, 2, \cdots)$ be a finite partition of Λ into Borel sets so that

$$(6) \qquad \| r(\lambda, \theta) - r(\lambda', \theta) \|_2 < \frac{1}{n}$$

if λ, λ' belong to the same $E_{n,i}$. Choose points $\lambda_{n,i} \in E_{n,i}$ and set

$$w_n = \exp \{ \sum_i r(\lambda_{n,i}, \theta) \mu(E_{n,i}) \}$$
$$w = \exp \left\{ \int r(\lambda, \theta) d\mu(\lambda) \right\} .$$

It follows from (6) that $\log w_n \to \log w$ in L_2 and our problem is to justify various passages to the limit under the integral sign. It follows from Hölder's inequality that for each n we have $\| w_n \|_1 \leqq K$. There is a sequence n' so that $w_{n'} \to w$ a.e., so Fatou's lemma gives $\| w \|_1 \leqq K$. This is the first part of the sublemma.

The unextended interpolation theorem has a trivial generalization to arbitrary finite logarithmically convex combinations of weight functions. Since $0 \leqq \mu(E_{n,i}) \leqq 1$ and $\sum_i \mu(E_{n,i}) = \mu(\Lambda) = 1$ we can conclude that for each n

$$\int | Pf |^2 w_n d\theta \leqq M \int | f |^2 w_n d\theta .$$

A slight modification of this which also follows from the unextended interpolation theorem is

$$(7) \qquad \int | Pf |^2 w_n^{1-\varepsilon} w_1^{\varepsilon} d\theta \leqq M \int | f |^2 w_n^{1-\varepsilon} w_1^{\varepsilon} d\theta$$

for all $\varepsilon (0 < \varepsilon < 1/2), n, f$. (Here w_1 is just w_n with $n = 1$.) By Hölder's inequality $\| w_n^{1-\varepsilon} w_1^{\varepsilon} \|_1 \leqq K$. This implies that $w_n^{1-2\varepsilon}$ have uniformly bounded norm in $L_p(w_1^{\varepsilon} d\theta)$, where $p = (1 - \varepsilon)/(1 - 2\varepsilon)$.

Since $f \in L_\infty$ the functions $|f|^2 w_n^{1-2\varepsilon}$ also have uniformly bounded norm. Since $p > 1$ we can find a sequence n' so that $|f|^2 w_{n'}^{1-2\varepsilon}$ converge weakly to a function in $L_p(w_1^\varepsilon d\theta)$. But n' has a subsequence n'' so that $|f|^2 w_{n''}^{1-2\varepsilon}$ converges a.e. to $|f|^2 w^{1-2\varepsilon}$. It follows that

$$|f|^2 w_{n'}^{1-2\varepsilon} \longrightarrow |f|^2 w^{1-2\varepsilon}$$

weakly. The conjugate space of $L_p(w_1^\varepsilon d\theta)$ is $L_q(w_1^\varepsilon d\theta)$ where $q=(1-\varepsilon)/\varepsilon$. Since $w_1^\varepsilon \in L_q(w_1^\varepsilon d\theta)$ it follows from the weak convergence that

$$\int |f|^2 w_{n'}^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \longrightarrow \int |f|^2 w^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \; . \tag{8}$$

This holds of course if n' is replaced by any subsequence, in particular one such that $w_{n''} \to w$ a.e. Then (7) with ε replaced by 2ε, (8), and Fatou's lemma give

$$\int | Pf |^2 w^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \leqq \int |f|^2 w^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \; .$$

Since $w^{1-2\varepsilon} w_1^{2\varepsilon} \leqq \max(w, w_1) \in L_1$ we can take the limit as $\varepsilon \to 0$ under the integral on the right, and apply Fatou's lemma to the integral on the left, to obtain the final conclusion of the sublemma.

Now we are in a position to prove, without much more difficulty, that $\sigma(T_\phi)$ is connected. Suppose not. Then we can find a simple closed curve Λ, disjoint from $\sigma(T_\phi)$, so that a non-empty portion of $\sigma(T_\phi)$ lies inside Λ and a non-empty portion of $\sigma(T_\phi)$ lies outside Λ. Call these portions σ_1 and σ_2 respectively. By Lemmas 2 and 3, $R(\phi)$ lies entirely inside Λ. Let Γ_ε be a simple closed curve surrounding a non-empty portion σ_3 of σ_2 and such that each point of Γ_ε is within ε of σ. Since σ_2 is contained in the convex hull of $R(\phi)$ (in fact all of $\sigma(T_\phi)$ is; this will be explained in a moment) Γ_ε will be contained in the convex hull of Λ if ε is sufficiently small. Thus of the three possibilities for disjoint simple closed curves (Λ and Γ_ε will be disjoint is ε is small enough),

Λ inside Γ_ε
Γ_ε inside Λ
Γ_ε, Λ have disjoint insides,

the first is eliminated since Γ_ε is contained in the convex hull of Λ, the second is eliminated since σ_3 lies entirely outside Λ, and the third is eliminated by Lemma 3: since $R(\phi)$ lies outside Γ_3 so does $\sigma(T_\phi)$. The assumption that $\sigma(T_\phi)$ is disconnected has led to a contradiction.

It remains to see why $\sigma(T_\phi)$ is contained in the convex hull of $R(\phi)$. It suffices to show that T_ϕ is invertible if $R(\phi)$ is contained in an open angle of opening less than π with vertex 0, and since invertibility of T_ϕ is not destroyed by multiplying ϕ by a nonzero constant we may assume that this angle has the positive real axis as bisector. But then for sufficiently small ε we shall have $\|1 - \varepsilon\phi\|_\infty < 1$, i.e. $\|I - \varepsilon T_\phi\| < 1$, and this implies T_ϕ is invertible.

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