ON THE FUNCTIONAL EQUATION

\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]

Tom M. (Mike) Apostol and Herbert S. Zuckerman
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TOM M. APOSTOL AND HERBERT S. ZUCKERMAN

1. Introduction. Let \( f \) be a completely multiplicative arithmetical function. That is, \( f \) is a complex-valued function defined on the positive integers such that

\[ f(mn) = f(m)f(n) \]

for all \( m \) and \( n \). We allow the possibility that \( f(n) = 0 \) for all \( n \). (If \( f \) is not identically zero then we must have \( f(1) = 1 \).) Given such an \( f \) we wish to study the problem of characterizing all numerical functions \( F \) which satisfy the functional equation

(1)

\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]

where \((m, n)\) denotes the greatest common divisor of \( m \) and \( n \). When \( f(n) = n \) for all \( n \), Equation (1) is satisfied by the Euler \( \Phi \) function since we have

\[ \phi(mn)\phi((m, n)) = \phi(m)\phi(n)(m, n) \]

More generally, it is known (see [1], [2]) that an infinite class of solutions of (1) is given by the formula

\[ F(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right) \]

where \( \mu \) is the Möbius function and \( g \) is any multiplicative function, that is,

\[ g(mn) = g(m)g(n) \quad \text{whenever} \ (m, n) = 1 \]

Some work on a special case of this problem has been done by P. Comment [2]. In the case \( f(1) = 1 \) he has investigated those solutions \( F \) of (1) which have \( F(1) \neq 0 \) and which satisfy an additional condition which he calls “property \( O \)” : If there exists a prime \( p_0 \) such that \( F(p_0^\alpha) = 0 \) then \( F(p_0^\beta) = 0 \) for all \( \alpha > 1 \). Comment’s principal theorem states that \( F \) is a solution of (1) with property \( O \) and with \( F(1) \neq 0 \) if, and only if, \( F \) satisfies the two equations

\[ F(mn)F(1) = F(m)F(n) \quad \text{whenever} \ (m, n) = 1 \]

and

\[ \text{PROPERTY OF} \]

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\[ F(p^\alpha) = F(p)f(p)^{\alpha-1} \] for all primes \( p \) and all \( \alpha \geq 1 \).

In this paper we study the problem in its fullest generality. In the case of greatest interest, \( F(1) \neq 0 \), we obtain a complete classification of all solutions of (1).

2. The solutions of (1) with \( f(1) = 0 \). If the given \( f \) has \( f(1) = 0 \) then \( f \) is identically zero and Equation (1) reduces to

\[ F(mn)F((m, n)) = 0 \]

for all \( m, n \). To characterize the solutions of (2) we introduce the following concept.

**Definition 1.** A (finite or infinite) set \( A = \{a_1, a_2, a_3, \ldots \} \) of positive integers is said to have property \( P \) if no \( a_i \) is divisible by any \( a_j^2 \).

Two simple examples of sets with property \( P \) are the set of primes and the set of products of distinct primes. The solutions of (2) may now be characterized as follows:

**Theorem 1.** A numerical function \( F \) satisfies (2) if, and only if, there exists a set \( A \) with property \( P \) such that \( F(n) = 0 \) whenever \( n \notin A \).

**Proof.** Let \( A = \{a_1, a_2, a_3, \ldots \} \) be a set with property \( P \). Define \( F(a_1), F(a_2), F(a_3), \ldots \), in an arbitrary fashion and define \( F(n) = 0 \) if \( n \notin A \). We shall prove that \( F \) satisfies (2).

Choose two integers \( m \) and \( n \) and let \( d = (m, n) \). If \( d \notin A \) then \( F(d) = 0 \) and (2) holds. If \( d \in A \) then \( mn \notin A \) since \( d^2 | mn \). In this case we have \( F(mn) = 0 \) and again (2) holds. Therefore \( F \) satisfies (2) in all cases.

To prove the converse, assume \( F \) satisfies (2) and let \( A \) be the set of integers \( n \) such that \( F(n) \neq 0 \). We shall prove that \( A \) has property \( P \). Choose any element \( b \) in \( A \). If \( b \) were divisible by \( k^2 \) for some \( k \) in \( A \), say \( b = qk^2 \), then we could take \( m = qk, n = k \) in (2) to obtain

\[ F(b)F(k) = 0 \]

which is impossible since both \( b \) and \( k \) are in \( A \). Therefore \( A \) has property \( P \) and the proof of Theorem 1 is complete.

3. The solutions of (1) with \( f(1) = F(1) = 1 \). Since we have characterized all solutions of (1) when \( f(1) = 0 \) we assume from now on that \( f(1) \neq 0 \) which means \( f(1) = 1 \). We divide the discussion in
two parts according as \( F(1) \neq 0 \) or \( F(1) = 0 \). In the first case we introduce \( G(n) = F(n)/F(1) \) and we see that (1) is equivalent to

\[
G(mn)G((m, n)) = G(m)G(n)f((m, n))
\]

with \( G(1) = 1 \). This means that the case with \( F(1) \neq 0 \) reduces to the case \( F(1) = 1 \). In this case we make a preliminary reduction of the problem as follows.

**Theorem 2.** Assume \( f(1) = 1 \). A numerical function \( F \) satisfies (1) with \( F(1) = 1 \) if, and only if, \( F \) is multiplicative and satisfies the equation

\[
F(p^{a+b})F(p^b) = F(p^a)F(p^b)f(p^b)
\]

for all primes \( p \) and all integers \( a \geq b \geq 1 \).

**Proof.** Assume \( F \) satisfies (1). Taking coprime \( m \) and \( n \) in (1) we find \( F(mn) = F(m)F(n) \), so \( F \) is multiplicative. Taking \( m = p^a \), \( n = p^b \) in (1) we obtain (3).

To prove the converse, assume \( F \) is a multiplicative function satisfying (3) for primes \( p \) and \( a \geq b \geq 1 \). Choose two positive integers \( m \) and \( n \). If \( (m, n) = 1 \), Equation (1) is satisfied because it simply states that \( F \) is multiplicative. Therefore, assume \( (m, n) = d > 1 \) and use the prime-power factorizations

\[
m = \prod_{i=1}^{\infty} p_i^{a_i}, \quad n = \prod_{i=1}^{\infty} p_i^{b_i}, \quad d = \prod_{i=1}^{\infty} p_i^{c_i}
\]

where \( a_i \geq 0 \), \( b_i \geq 0 \), \( c_i = \min(a_i, b_i) \), the products being extended over all primes. Since \( F \) is multiplicative we have

\[
F(mn)F(d) = \prod_{i=1}^{\infty} F(p_i^{a_i+b_i})F(p_i^{c_i})
= \prod_{0 \leq i \leq a_i} F(p_i^{a_i})F(p_i^{b_i}) \cdot \prod_{0 \leq i < b_i} F(p_i^{a_i})F(p_i^{c_i})f(p_i^{c_i})
\]

The factors corresponding to \( b_i = 0 \) or \( a_i = 0 \) are

\[
\prod_{0 \leq i \leq a_i} F(p_i^{b_i}) \cdot \prod_{0 \leq i < b_i} F(p_i^{a_i}) = \prod_{a_i = b_i = 0} F(p_i^c)F(p_i^{c_i})f(p_i^{c_i})
\]

since \( F(1) = f(1) = 1 \). For the remaining factors we apply (3) to each product and we obtain

\[
F(mn)F(d) = \prod_{0 \leq i \leq a_i} F(p_i^{a_i})F(p_i^{b_i})f(p_i^{b_i}) \cdot \prod_{0 \leq i < b_i} F(p_i^{a_i})F(p_i^{c_i})f(p_i^{c_i})
= \prod_{i=1}^{\infty} F(p_i^{a_i})F(p_i^{b_i})f(p_i^{b_i}) = F(m)F(n)f(d)
\]

This completes the proof of Theorem 2.
We turn now to the problem of finding all solutions of (3). If \( p \) is a prime for which \( f(p) = 0 \), then for this prime (3) becomes

\[
F(p^{a+b})F(p^b) = 0 \quad \text{whenever } a \geq b \geq 1.
\]

For a fixed \( p \) the solutions of (4) may be characterized as follows:

**Theorem 3.** An arithmetical function \( F \) satisfies (4) for a given prime \( p \) if, and only if, there exists an integer \( c \geq 1 \) such that

\[
F(p^i) = 0 \quad \text{for } 1 \leq i \leq c - 1 \quad \text{and for } i \geq 2c.
\]

**Proof.** Assume \( F \) satisfies (5) for some \( c \geq 1 \). Choose two integers \( a \) and \( b \) with \( a \geq b \geq 1 \). If \( b \leq c - 1 \) then (5) implies \( F(p^b) = 0 \) so (4) is satisfied. If \( b \geq c \) then \( a + b \geq 2b \geq 2c \) so \( F(p^{a+b}) = 0 \) and (4) is again satisfied.

To prove the converse, assume \( F \) is an arithmetical function satisfying (4) for some prime \( p \). If \( F(p^i) = 0 \) for all integers \( t \geq 1 \) then (5) holds with \( c = 1 \). Otherwise, we let \( c \) be the smallest \( t \geq 1 \) for which \( F(p^t) \neq 0 \). Then \( F(p^i) = 0 \) for all \( i \leq c - 1 \). Now take any \( i \geq 2c \) and write \( i = a + c \) where \( a \geq c \). Taking \( b = c \) in (4) we find \( F(p^i) = 0 \) for \( i \geq 2c \). Therefore (5) is satisfied for this choice of \( c \) and the proof of Theorem 3 is complete.

We consider next those primes \( p \) for which \( f(p) \neq 0 \). For such \( p \) the problem of solving (3) may be reduced as follows:

**Theorem 4.** Let \( p \) be a prime for which \( f(p) \neq 0 \). An arithmetical function \( F \) satisfies (3) if, and only if, there exists an arithmetical function \( g \) (which may depend on \( p \)) such that

\[
F(p^a) = g(a)f(p)^a \quad \text{for all } a \geq 1,
\]

where \( g \) satisfies the functional equation

\[
g(a + b)g(b) = g(a)g(b) \quad \text{for all } a \geq b \geq 1.
\]

**Proof.** Assume there exists a function \( g \) satisfying (7) and let \( F(p^a) = g(a)f(p)^a \). Then if \( a \geq b \geq 1 \) we have

\[
F(p^{a+b})F(p^b) = g(a + b)f(p)^{a+b}g(b)f(p)^b
\]

and

\[
F(p^a)F(p^b)f(p^a)f(p^b) = g(a)f(p)^ag(b)f(p)^bf(p)^b.
\]

\(^1\) If \( c = 1 \) the inequality \( 1 \leq i \leq c - 1 \) is vacuous; in this case it is understood that (5) is to hold for all \( i \geq 2 \).
Using (7) we see that $F$ satisfies (3).

To prove the converse, assume $F$ satisfies (3) and let

$$g(a) = \frac{F(p^a)}{f(p)^a}$$

for $a \geq 1$. From (3) we see at once that $g$ satisfies (7), so the proof of Theorem 4 is complete.

Next we determine all the solutions of the functional equation (7).

**Theorem 5.** Assume $g$ is an arithmetical function satisfying (7). Then there exists an integer $k \geq 1$, a divisor $d$ of $k$, and a complex number $C$ such that

(8) $g(n) = 0$ for $1 \leq n \leq k - 1$, and for $n \geq k, n \equiv 0 \pmod{d}$,

(9) $g(n) = C$ for $n \geq k, n \equiv 0 \pmod{d}$.

Conversely, choose any integer $k \geq 1$, any divisor $d$ of $k$, and any complex number $C$. For those $n$ satisfying $n \geq k$ and $n \equiv 0 \pmod{d}$ let $g(n) = C$, and let $g(n) = 0$ for all other $n$. Then this $g$ satisfies (7).

**Proof.** Assume $g$ satisfies (7). If $g$ is identically zero then (8) and (9) hold with any choice of $k$ and $d$ and with $C = 0$. If $g$ is not identically zero, let $k$ be the smallest positive integer $n$ for which $g(n) \neq 0$ and let $C = g(k)$. Then $g(n) = 0$ for $1 \leq n \leq k - 1$. If $n \geq 2k$ we may write $n = k + r, r \geq k$, and use (7) with $a = r, b = k$ to obtain the periodicity relation

(10) $g(k + r) = g(r)$ for $r \geq k$.

In particular, $g(2k) = g(k)$. Therefore, to completely determine $g$ we need only consider $g(n)$ for $n$ in the interval $k + 1 \leq n \leq 2k - 1$. If $g(n) = 0$ for all $n$ in this interval then $g(n) = 0$ for all $n \equiv 0 \pmod{k}$ and (8) and (9) hold with $d = k, C = g(k)$. Suppose, then, that $g(n) \neq 0$ for some $n$ in the interval $k + 1 \leq n \leq 2k - 1$ and let $k + d$ be the smallest such $n$. Then $1 \leq d \leq k - 1$. We prove next that $d \mid k$, that $g(n) = 0$ if $n \equiv 0 \pmod{d}$, and that $g(n) = C$ if $n \equiv 0 \pmod{d}$.

For this purpose we define a new function $h$ by the equation

$$h(n) = \frac{g(n + k)}{g(k)}$$

for $n \geq 0$.

Then the periodicity property (10) implies

(11) $h(n + k) = h(n)$ if $n \geq 0$.

We also have
(12) \[ h(0) = h(k) = 1, \quad h(n) = 0 \quad \text{if} \quad 1 \leq n < d, \quad h(d) \neq 0. \]

Now for \( n \geq 0 \) we have

\[ h(n + d) = h(n + d + 2k) = \frac{g(n + d + 3k)}{g(k)} \quad \text{and} \quad h(d) = \frac{g(d + k)}{g(k)}, \]

Since \( n + 2k > d + k > 1 \) we may use (7) with \( a = n + 2k, b = d + k \), to obtain

\[ h(n + d)h(d) = \frac{g(n + d + 3k)g(d + k)}{g(k)^2} = \frac{g(n + 2k)g(d + k)}{g(k)^2} = h(n + k)h(d) = h(n)h(d). \]

Since \( h(d) \neq 0 \) this implies

(13) \[ h(n + d) = h(n) \quad \text{if} \quad n \geq 0. \]

Using (13) along with (12) we find

\[ h(n) = 0 \quad \text{if} \quad n \not\equiv 0 \pmod{d}, \quad h(n) = 1 \quad \text{if} \quad n \equiv 0 \pmod{d}. \]

Also, \( d \mid k \) since \( h(k) = 1 \). This implies that \( g(n) = 0 \) if \( n \not\equiv 0 \pmod{d} \), and that \( g(n) = g(k) = C \) if \( n \equiv 0 \pmod{d} \).

Now we prove the converse. Given \( k \geq 1 \), a divisor \( d \) of \( k \), and a complex number \( C \), define \( g \) as indicated in (8) and (9). We must prove that this \( g \) satisfies (7). Choose integers \( a \) and \( b \) with \( a \geq b \geq 1 \). If \( a \leq k - 1 \) then \( b \leq k - 1 \) and \( g(a) = g(b) = 0 \) so (7) is satisfied. Suppose, then, that \( a \geq k \). We consider two cases: (i) \( a \not\equiv 0 \pmod{d} \), and (ii) \( a \equiv 0 \pmod{d} \).

If \( a \not\equiv 0 \pmod{d} \) we have \( g(a) = 0 \) and the right member of (7) vanishes. If \( a + b \not\equiv 0 \pmod{d} \) then \( g(a + b) = 0 \). If \( a + b \equiv 0 \pmod{d} \) then \( b \equiv 0 \pmod{d} \) and \( g(b) = 0 \). Therefore we always have \( g(a + b)g(b) = 0 \) so the left member of (7) also vanishes. This settles case (i).

In case (ii), \( a \equiv 0 \pmod{d} \), we again consider the two alternatives \( a + b \not\equiv 0 \pmod{d}, a + b \equiv 0 \pmod{d} \). If \( a + b \not\equiv 0 \pmod{d} \) then \( b \not\equiv 0 \pmod{d} \) and both sides of (7) vanish. If \( a + b \equiv 0 \pmod{d} \) then \( b \equiv 0 \pmod{d} \) so \( g(a) = g(b) = g(a + b) = C \) and Equation (7) is satisfied. This completes the proof of Theorem 5.

Theorems 2 through 5 give us a complete classification of all solutions of (1) in the case \( f(1) = F(1) = 1 \).

4. **The case** \( f(1) = 1, \quad F(1) = 0 \). **In this case any** \( F \) **which satisfies** (1) **must also satisfy**

(14) \[ F(m)F(n) = 0 \quad \text{whenever} \quad (m, n) = 1. \]
These functions may be characterized by means of sets of integers with the following property.

**Definition 2.** A (finite or infinite) set \( S = \{k_1, k_2, k_3, \ldots\} \) of positive integers will be said to have property \( Q \) if \( 1 < k_i < k_{i+1} \) and \( (k_i, k_j) > 1 \) for all \( i \) and \( j \).

For example, the set of all multiples of a given integer \( k > 1 \) has property \( Q \), but there are more complicated sets with this property.

**Theorem 6.** A numerical function \( F \) satisfies (14) if, and only if, there exists a set \( S \) with property \( Q \) such that \( F(n) = 0 \) whenever \( n \in S \), and \( F(n) \neq 0 \) whenever \( n \in S \).

**Proof.** Assume \( F \) satisfies (14). Then \( F(1) = 0 \). If \( F \) is identically zero the theorem holds with \( S \) the empty set. If \( F \) is not identically zero there is a smallest integer \( k_i > 1 \) with \( F(k_i) \neq 0 \). The set \( \{k_i\} \) has property \( Q \). If \( F(n) = 0 \) for all \( n > k_i \) we may take \( S = \{k_i\} \). Otherwise there exists a smallest integer \( k_3 > k_i \) with \( F(k_3) \neq 0 \). The set \( \{k_i, k_3\} \) has property \( Q \) because (14) implies \( (k_i, k_3) > 1 \). If \( F(n) = 0 \) for all \( n > k_3 \) we may take \( S = \{k_i, k_3\} \). If \( F(n) \neq 0 \) for some \( n > k_3 \) we let \( k_3 \) be the smallest such \( n \). Then (14) implies \( (k_i, k_3) > 1 \) and \( (k_2, k_3) > 1 \) so the set \( \{k_i, k_2, k_3\} \) has property \( Q \). Continuing in this way we obtain a set \( S = \{k_i, k_2, k_3, \ldots\} \) (finite or infinite) with the properties indicated in the theorem.

To prove the converse, choose any set \( S \) with property \( Q \), assign arbitrary nonzero values to the elements of \( S \) and let \( F(n) = 0 \) if \( n \in S \). To show that \( F \) satisfies (14), choose integers \( m \) and \( n \) with \( (m, n) = 1 \). Both \( m \) and \( n \) cannot be in \( S \) since \( S \) has property \( Q \). Therefore at least one of \( m \) or \( n \) is not in \( S \) so at least one of \( F(m) \) or \( F(n) \) is zero. This completes the proof of Theorem 6.

Since Theorem 6 characterizes all solutions of (14), all solutions of the more general equation (1) with \( F(1) = 0 \) must be found among those described in Theorem 6. For those solutions \( F' \) of (14) which also satisfy (1) more can be asserted about the set \( S \) on which \( F \) does not vanish. We shall treat only the case in which \( f \) is never zero. In this case, if we write \( G(n) = F(n)/f(n) \), Equation (1) is equivalent to

\[
G(mn)G((m, n)) = G(m)G(n) .
\]

In other words, if \( f \) never vanishes the problem reduces to the case in which \( f \) is identically 1. Moreover, \( G(n) = 0 \) if, and only if, \( F(n) = 0 \) so the set \( S \) on which \( G \) does not vanish is the same as that on which \( F \) does not vanish. For those \( G \) satisfying (15) with \( G(1) = 0 \) we shall prove:
THEOREM 7. Let G be a solution of (15) with G(1) = 0 and let
S = \{k_1, k_2, \ldots\} be a set with property Q such that G(n) \neq 0 if, and
only if, n \in S. Then S contains mn and (m, n) whenever it contains
m and n. Moreover, every element in S is a multiple of k. If
\(tk_1^a \in S\) for some \(t \geq 1, a \geq 1\), then G is constant on the subset
\(\{tk_1^a, tk_1^{a+1}, tk_1^{a+2}, \ldots\}\).

Proof. If \(m, n \in S\), then \(G(m) \neq 0\) and \(G(n) \neq 0\). Therefore
Equation (15) implies \(G(mn) \neq 0\) and \(G((m, n)) \neq 0\), so S contains mn
and \((m, n)\). Let \(d = (k_i, k_i)\). Then \(d \in S\) so \(d = k_i\) since \(k_i\) is the
smallest member of S. Therefore each \(k_i\) in S is a multiple of \(k_i\), as
asserted.

If \(tk_1^a \in S\), let \(S(t) = \{tk_1^a, tk_1^{a+1}, tk_1^{a+2}, \ldots\}\). This is a subset of S.
Taking \(m = k_i\) and \(n = tk_i^{a+r}\) in Equation (15) we find
\(G(tk_i^{a+r+1}) = G(tk_i^{a+r})\) so G is constant on \(S(t)\).

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