

Pacific Journal of Mathematics

**THE RELATIONSHIP BETWEEN THE RADICAL OF A
LATTICE-ORDERED GROUP AND COMPLETE
DISTRIBUTIVITY**

PAUL F. CONRAD

THE RELATIONSHIP BETWEEN THE RADICAL OF A LATTICE-ORDERED GROUP AND COMPLETE DISTRIBUTIVITY

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1. Introduction. Throughout this note let G be a lattice-ordered group (notation 1-group). G is said to be *representable* if there exists an 1-isomorphism of G onto a subdirect sum of a cardinal sum of totally ordered groups (notation 0-groups). In particular, every abelian 1-group is representable. G is said to be *completely distributive* if for $g_{ij} \in G$

$$\bigwedge_{i \in I} \bigvee_{j \in J} g_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} g_{if(i)}$$

provided the indicated joins and intersections exist.

For each $0 \neq g$ in G let R_g be the subgroup of G that is generated by the set of all 1-ideals of G not containing g . Then R_g is an 1-ideal of G and the radical of G is defined to be

$$R(G) = \bigcap R_g \quad (0 \neq g \in G).$$

In [2] it is shown that if G is a divisible abelian 1-group, then there exists a minimal Hahn-type embedding of G into an 1-group of real valued functions if and only if $R(G) = 0$. Thus it would be useful to identify the class of abelian 1-groups with zero radicals, and to examine the properties of non-abelian 1-groups with zero radicals. In our main theorem we show that a representable 1-group G is completely distributive if and only if $R(G) = 0$. We also show $R(G) = 0$ if and only if G has a regular representation. This settles a question raised by Weinberg [6].

With no restrictions on G we show that $R(G)$ is completely determined by the lattice \mathcal{L} of all 1-ideals of G . In particular, if G is a representable 1-group, then whether or not G is completely distributive depends only on \mathcal{L} .

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2. Regular and essential L -ideals. If $g \in G$ and M is an 1-ideal of G that is maximal with respect to $g \notin M$, then M is called a *regular 1-ideal* of G . Let M^* be the intersection of all 1-ideals of G that

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properly contain M . Then since $g \in M^*$, it follows that M^* is the unique 1-ideal of G that covers M . Let Γ be an index set for the set of all pairs (G^γ, G_γ) of 1-ideals of G such that G_γ is regular and G^γ covers G_γ . Define $\alpha < \beta$ if $G^\alpha \subseteq G_\beta$. Then Γ is a *po*-set, and we say that $\gamma \in \Gamma$ is a *value* of g if $g \in G^\gamma \setminus G_\gamma$. In particular, the set of all values of g is a trivially ordered subset of Γ . An element $\gamma \in \Gamma$ is called *essential* if there exists an $0 \neq h$ in G such that all the values of h are $\leq \gamma$. In this case G_γ is called an *essential 1-ideal* of G , and if $g \in G^\gamma \setminus G_\gamma$, then we say that γ is an *essential value* of g .

Clearly the set E of all essential elements in Γ is a dual ideal of Γ ($\alpha < \beta \in \Gamma, \alpha \in E \rightarrow \beta \in E$). The following lemma shows that the radical $R(G)$ of G is completely determined by the essential ideals of G .

LEMMA 1. *The radical of G is the intersection of essential 1-ideals of G : $R(G) = \bigcap G_\gamma (\gamma \in E)$.*

Proof. If $g \notin R(G)$, then $g \notin R_h$ for some h in G and by Zorn's lemma there exists an 1-ideal M of G that is maximal with respect to $g \notin M \supseteq R_h$. Thus $M = G_\gamma$ for some $\gamma \in E, g \in G^\gamma \setminus G_\gamma$ and hence g has an essential value. If $x \in \bigcap G_\gamma$, then x has no essential value and hence $x \in R(G)$. Therefore $\bigcap G_\gamma \subseteq R(G)$. If E is the null set, then $G = \bigcap G_\gamma \supseteq R(G)$ and if $\gamma \in E$, then there exists $0 \neq h_\gamma \in G$ such that if δ is a value of h_γ , then $\delta \leq \gamma$ and hence $G_\delta \subseteq G_\gamma$. Thus $R_{h_\gamma} \subseteq G_\gamma$ and so

$$\bigcap_{\gamma \in E} G_\gamma \supseteq \bigcap_{\gamma \in E} R_{h_\gamma} \supseteq \bigcap_{0 \neq g \in G} R_g = R(G).$$

COROLLARY. *$R(G) = 0$ if and only if each nonzero element in G has at least one essential value.*

We next show that $R(G)$ depends only on the lattice \mathcal{L} of all 1-ideals of G . Note that a regular 1-ideal M of G is characterized by the fact that it is meet irreducible in \mathcal{L} . That is, if M^* is the intersection of all 1-ideals of G that properly contain M , then M is properly contained in M^* .

LEMMA 2. *$\beta \in \Gamma$ is essential if and only if $\bigcap \{G_\gamma : \gamma \in \Gamma \text{ and } \gamma \not\leq \beta\} \neq 0$.*

Proof. Suppose that $0 < h \in \bigcap \{G_\gamma : \gamma \in \Gamma \text{ and } \gamma \not\leq \beta\}$ and let α be a value of h . Then $h \notin G_\alpha$ and so $\alpha \leq \beta$. Thus all the values of h are $\leq \beta$, and hence β is essential. Conversely assume that G_β is essential and pick $0 < h \in G$ such that all the values of h are $\leq \beta$. Then

$h \in \cap \{G_\gamma : \gamma \in \Gamma \text{ and } \gamma \not\leq \beta\}$. For if $h \notin G_\gamma$, where $\gamma \not\leq \beta$, then h must have a value $\alpha \geq \gamma$ which is impossible.

COROLLARY. $R(G)$ is an invariant of the lattice \mathcal{L} of all 1-ideals of G .

LEMMA 3. For an 1-group G the following are equivalent.

- (1) G/M is an 0-group for each regular 1-ideal M of G .
- (2) G is representable.

Proof. For each $0 \neq g$ in G pick an l -ideal M_g of G that is maximal with respect to not containing g . Then $\cap M_g = 0$, and if (1) is satisfied, then each G/M_g is an 0-group and the mapping of $x \in G$ upon $(\dots, M_g + x, \dots)$ is a representation of G . Conversely suppose that G has a representation, then clearly

(3) if $a, b \in G^+$ and $a \wedge b = 0$, then $a \wedge (-x + b + x) = 0$ for all $x \in G$. In fact, Sik [5] established that (2) and (3) are equivalent, but we only need that (2) implies (3). Let M be an 1-ideal of G that is maximal with respect to not containing $0 < a \in G$, and let $A = M + a$. Suppose (by way of contradiction) that G/M is not an 0-group. Then there exist strictly positive elements X and Z in G/M such that $X \wedge Z = M$.

Case I. $X \wedge A = M$. Then $P(A) = \{Y \in G/M : |Y| \wedge A = M\}$ is a convex 1-subgroup of G/M that contains X but not A . If $M < Y \in P(A)$, then $Y = M + y$, where $0 < y \in G$, and $a = a \wedge y + a'$, $y = a \wedge y + y'$, $a' \wedge y' = 0$. Moreover

$$M = A \wedge Y = M + a \wedge M + y = M + a \wedge y.$$

Thus $a \wedge y \in M$ and so $Y = M + y'$ and $A = M + a'$. But by (3), $a' \wedge (-g + y' + g) = 0$ for all g in G and hence $A \wedge -(M + g) + Y + (M + g) = M$. Thus $P(A)$ is a nonzero 1-ideal of G/M that does not contain A , and hence there exists an 1-ideal of G that properly contains M but not a , but this contradicts the maximality of M .

Case II. $X \wedge A \neq M$. Then $P(X)$ is an 1-ideal of G/M that contains Z but not A , and once again we contradict the maximality of M . Therefore G/M is an 0-group, and hence (2) implies (1).

COROLLARY. If G is representable and $R(G) = 0$, then an element g is positive in G if and only if $G_\gamma + g$ is positive for all essential values γ of g .

Proof. If g is positive in G , then $G_\gamma + g$ is positive for all values γ of g , essential or otherwise. If g is not positive, then $g = g \vee 0 +$

$g \wedge 0 = g^+ + g^-$, where $g^- \neq 0$ and $g^+ \wedge -g^- = 0$. By the Corollary to Lemma 1 there exists an essential value γ of g^- and by Lemma 3, G/G_γ is an 0-group, and so $g^+ \in G_\gamma$. Thus γ is also an essential value of g and $G_\gamma + g = G_\gamma + g^-$ is negative.

LEMMA 4. *If $0 < g \in \vee A_\lambda$, where the A_λ are 1-ideals of G , then $g = g_1 \vee \cdots \vee g_n$, where $0 \leq g_i \in \cup A_\lambda$ for $i = 1, \dots, n$.*

Proof. This proof is due to T. Lloyd. Clearly $g = a_1 + \cdots + a_n$, where the $a_i \in A_{\lambda_i}$ for $i = 1, \dots, n$. Thus it suffices to show that $g \leq a'_1 \vee \cdots \vee a'_n$, where $a'_i \in A_{\lambda_i}$ for $i = 1, \dots, n$. For then

$$\begin{aligned} g &= ((a'_1 \vee 0) \wedge g) \vee \cdots \vee ((a'_n \vee 0) \wedge g) \\ &= g_1 \vee \cdots \vee g_n \end{aligned}$$

where $0 \leq g_i \in A_{\lambda_i}$ for $i = 1, \dots, n$. If $n = 2$, then

$$a_1 + a_2 \leq 2a_1 \vee (a_1 + a_2 - a_1 + a_2) = a'_1 \vee a'_2$$

because

$$\begin{aligned} 0 \leq |a_1 - a_2| &= (a_1 - a_2) \vee (a_2 - a_1) \\ &= -a_1 + (2a_1 \vee (a_1 + a_2 - a_1 + a_2)) - a_2. \end{aligned}$$

Thus $a_1 + \cdots + a_n \leq (a_1 + \cdots + a_{n-1})' \vee a'_n$, and since $(a_1 + \cdots + a_{n-1})' \in \vee A_{\lambda_i}$ ($i = 1, \dots, n-1$), $(a_1 + \cdots + a_{n-1})' = b_1 + \cdots + b_{n-1}$, where $b_i \in A_{\lambda_i}$ for $i = 1, \dots, n-1$. Thus by induction $b_1 + \cdots + b_{n-1} \leq a'_1 \vee \cdots \vee a'_{n-1}$ and hence $g \leq a'_1 \vee \cdots \vee a'_n$.

3. Completely distributive L -groups. Let A be a sublattice and subdirect sum of a cardinal sum B of 0-groups B_λ ($\lambda \in A$). If for each λ in A , the projection ρ_λ of A onto B_λ preserves infinite joins, then A is called a *regular* subgroup of B . An 1-group G is said to have a *regular representation* if it is 1-isomorphic to a regular subgroup of a cardinal sum of 0-groups. It is easy to prove that an 1-group G with a regular representation is completely distributive [6]. Weinberg has also shown ([6] Proposition 1.3) that the natural homomorphism of an 1-group G onto G/J , where J is an 1-ideal of G , preserves infinite joins if and only if J is closed ($\vee j_\lambda \in G, \{j_\lambda : \lambda \in A\} \subseteq J \rightarrow \vee j_\lambda \in J$). Thus it follows that G has a regular representation if and only if there exists a family of closed 1-ideals J_λ of G such that $\cap J_\lambda = 0$ and each G/J_λ is an 0-group.

LEMMA 5. (Weinberg) *An 1-group G is completely distributive if and only if for each $0 < g$ in G there exists $0 < g^*$ in G such that*

$$g = \vee g_\lambda, g_\lambda \in G^+ \rightarrow g^* \leq g_\lambda \text{ for some } \lambda.$$

THEOREM. *For a representable 1-group G the following are equivalent.*

- (1) $R(G) = 0$.
- (2) *Each essential 1-ideal of G is closed and $\cap G_\gamma = 0$ ($\gamma \in E$).*
- (3) *G has a regular representation.*
- (4) *G is completely distributive.*

Proof. By Lemma 3, for each γ in E , G/G_γ is an 0-group, and hence by the preceding discussion (2) implies (3) and (3) implies (4). Suppose that G is completely distributive, and assume (by way of contradiction) that $0 < g \in R(G)$. Then by Lemma 5 there exists $0 < g^* \in G$ such that if $g = \vee g_\alpha$ ($g_\alpha \in G^+$), then $g^* \leq g_\alpha$ for some α . Since $g \in R(G)$ it follows that $g \in R_{g^*} = \vee A_\lambda$, where the A_λ are the 1-ideals of G not containing g^* . Thus by Lemma 4, $g = g_1 \vee \cdots \vee g_n$, where $0 \leq g_i \in \cup A_\lambda$. But then $g^* \leq g_i$ for some i , and hence $g^* \in \cup A_\lambda$ a contradiction. Therefore (4) implies (1).

To complete the proof we must show that (1) implies (2). If (1) is satisfied, then by Lemma 1, $\cap G_\gamma = 0$ ($\gamma \in E$). Let G_δ be an essential 1-ideal of G and assume (by way of contradiction) that G_δ is not closed. Then there exists $g \in G^+ \setminus G_\delta$ such that $g = \vee g_j$ ($g_j \in G_\delta^+$). Since G_δ is essential there exists $0 < h \in G$ such that all the values of h are $\leq \delta$. We shall show that for some such h , $g - h \geq g_j$ for all j , and hence $\vee g_j > \vee g_j - h = g - h \geq \vee g_j$.

Case I. There exists $0 < h \in G$ such that all the values of h are $\leq \delta$ and $G_\delta + h < G_\delta + g$. Since $g - h \notin G_\delta$ and $g_j \in G_\delta$, $g - h - g_j \neq 0$. By the Corollary to Lemma 3 it suffices to show that $G_\beta + g - h - g_j$ is positive for all values β of $g - h - g_j$ in E . If $h \in G_\beta$, then $G_\beta + g - h - g_j = G_\beta + g - g_j$ is positive. If $h \notin G_\beta$, then there exists a value γ of h such that $\gamma \geq \beta$. But then $\beta \leq \gamma \leq \delta$, and since $g - h - g_j \in G^\beta \setminus G_\delta$, $\beta = \delta$. Therefore $G_\beta + g - h - g_j = G_\delta + g - h$ is positive.

Case II. For each $0 < h \in G$ such that all of the values of h are $\leq \delta$, $G_\delta + h \geq G_\delta + g$. If $\delta > \gamma \in E$, then we may choose $0 < k \in G$ such that all of the values of k are $\leq \gamma < \delta$. But then $G_\delta + g > G_\delta = G_\delta + k$. Therefore δ is minimal in E . If all values of $0 < h$ are $\leq \delta$, then $G_\delta + h \geq G_\delta + g$ and so $G_\delta + g \wedge h = G_\delta + g$. If β is a value of $g \wedge h$ in E , then $g \wedge h \in G^\beta \setminus G_\beta$ and hence $h \notin G_\beta$. Thus there exists a value γ of h such that $\beta \leq \gamma \leq \delta$ and since δ is minimal in E , $\beta = \delta$. Thus without loss of generality, $0 < h \in G$, δ is the only value of h in E and $G_\delta + h = G_\delta + g$. If $g - h - g_j \neq 0$ and β is a value of $g - h - g_j$ in E then $h \in G_\beta$. Otherwise $\beta = \delta$, but $g - h - g_j \in G_\delta$. Therefore $G_\beta + g - h - g_j = G_\beta + g - g_j$ is positive for all values β

of $g - h - g_j$ in E . This completes the proof of our theorem. In proving that (4) implies (1) we did not use the hypothesis that G is representable. Thus we have

COROLLARY I. *If G is a completely distributive 1-group, then $R(G) = 0$.*

From the Corollary to Lemma 2 we have

COROLLARY II. *If G is a representable 1-group, then whether or not G is completely distributive depends only on the lattice \mathcal{L} of all 1-ideals of G .*

4. Remarks and examples. Let P be the 1-group of all order preserving permutations of the real line (with $fg(x) = f(g(x))$ and f positive if $f(x) \geq x$ for all x). Let

$A = \{f \in P : f \text{ induces the identity on } (-\infty, a] \text{ for some } a\}$, and

$B = \{f \in P : f \text{ induces the identity on } [a, \infty) \text{ for some } a\}$.

Let $C = A \cap B$. Then Holland [4] has shown that A , B and C are the only proper 1-ideals of G , and Higman [3] has shown that C is algebraically simple. Therefore 0 is the only essential 1-ideal of C and since $C/0$ is not an 0-group it follows from Lemma 3 that C is not representable. Therefore C satisfies property (2) of the theorem, but not property (3).

(G, B) is the only value of each element in $A \setminus B$ and $(C, 0)$ is the only value of each nonzero element in C . Thus B and 0 are essential 1-ideals of P , and in particular, P satisfies (1). For each $n = 1, 2, \dots$ let

$$f_n(x) = \begin{cases} 2x & \text{if } x \leq n \\ \frac{x + 3n}{2} & \text{if } n \leq x \leq 3n \\ x & \text{if } 3n \leq x. \end{cases}$$

Then $(\bigvee f_n)(x) = 2x$, and hence the f_n belong to B , but $\bigvee f_n \notin B$. Therefore P satisfies (1) but not (2).

A simple application of Lemma 5 shows that P is completely distributive (or see [6] Example 3.3). Therefore (4) does not imply (2) or (3). On the other hand for arbitrary 1-groups, (3) \rightarrow (2) \rightarrow (1). The remaining question is whether or not (1) or (2) implies (4) for non-representable 1-groups? Note that if $R(G) = 0$ implies complete distributivity, then every 1-group with no proper 1-ideals is completely distributive, and in particular, every 1-group that is algebraically simple is completely distributive.

If the radical used in this note is replaced by one constructed in

exactly the same way, but with 1-ideals replaced by convex 1-subgroups, then if this new radical is zero, the group is completely distributive. Also the new radical is an invariant of the lattice of all convex 1-subgroups of G . The proofs of these statements are analogous to those in this paper using the fact that if C is a regular convex 1-subgroup, then the set of right cosets of C in G is totally ordered by

$$C + x \leq C + y \text{ if } x \leq y + c \text{ for some } c \in C.$$

Unfortunately the converse to the above is false. For example, the new radical for P is P itself and yet P is completely distributive.

Let G be an Archimedean 1-group. By Theorem 5.7 in [2], $R(G) = 0$ if and only if G has a basis, and by Theorem 7.3 in [1], G has a basis if and only if G is (isomorphic to) a subdirect sum of a cardinal sum of subgroups R_γ of the reals which contains the finite cardinal sum of the R_γ . Thus we have a new proof for one of the main results in [6].

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