Pacific Journal of Mathematics

A SUFFICIENT CONDITION THAT AN ARC IN S^n BE CELLULAR

P. H. Doyle, III

Vol. 14, No. 2 June 1964

A SUFFICIENT CONDITION THAT AN ARC IN S^n BE CELLULAR

P. H. DOYLE

An arc A in S^n, the n-sphere, is cellular if $S^n - A$ is topologically E^n, euclidean n-space. A sufficient condition for the cellularity of an arc in E^3 is given in [4] in terms of the property local peripheral unknottedness (L.P.U) [5]. We consider a weaker property and show that an arc in S^n with this property is cellular.

If A is an arc in S^n we say that A is p-shrinkable if A has an end point q and in each open set U containing q in S^n, there is a closed n-cell $C \subset U$ such that q lies in Int C (the interior of C), while BdC (the boundary of C) meets A in exactly one point. We note that A is p-shrinkable is precisely the condition that A be L.P.U. at an endpoint [5]. There is, however, a good geometric reason for using the p-shrinkable terminology here; the letter p denotes pseudo-isotopy.

LEMMA 0. *Let C^n be a closed n-cell and D^n a closed n-cell which lies in int C^n except for a single point q which lies on the boundary of each n-cell. If there is a homeomorphism h of C^n onto a geometric n-simplex such that $h(D^n)$ is also an n-simplex, then there is a pseudo-isotopy ρ_t of C^n onto C^n which is the identity on BdC^n, while $\rho_1(D^n)$, the terminal image of D^n, is the point q.*

The proof of this is omitted since it depends only on the same result when C^n and D^n are simplices.

LEMMA 1. *Let C^n be a closed n-cell and B an arc which lies in int C^n except for an endpoint b of B on BdC^n. Then there is a pseudo-isotopy of C^n onto C^n which is fixed on BdC^n and which carries B to b.*

Proof. Since $B \cap BdC^n = b$ we note that there is in C^n an n-cell D^n which contains B in its interior except for the point b, $D^n - b \subset \text{Int}\, C^n$, and D^n is embedded in C^n as in Lemma 0. Thus Lemma 0 can be applied to shrink B in the manner required by the Lemma.

THEOREM 1. *Let A be an arc in S^n such that for each subarc B of A, B is p-shrinkable. Then every arc in A is cellular.*

Proof. The proof is by contradiction. If A contains a non-cellular

Received January 30, 1963.

subarc there is no loss of generality in assuming this arc is A. Then $S^n - A \neq E^n$. By the characterization theorem of E^n in [1], there is a compact set C in $S^n - A$ and C lies in no open n-cell in $S^n - A$. By the Generalized Schoenflies Theorem [2], this is equivalent to the condition that no bicollared $(n-1)$-sphere in S^n separates C and A.

Let G be the set of all subarcs of A which cannot be separated from C by a bicollared sphere in S^n. We partially order G by set inclusion and select a maximal chain in G. Let B be the intersection of all arcs in this maximal chain. Evidently B cannot be separated from C by a bicollared sphere in S^n. Thus B is an arc and each proper subarc of B can be so separated from C in S^n.

By the hypothesis of the theorem, B is p-shrinkable. So let B be L.P.U. at an endpoint q. Let U be an open set containing q and $U \cap C = \square$. Then there is an n-cell $C^n \subset U$, $C^n \cap B = B^1$, an arc, while $B^1 \cap BdC^n = p$, a point. So by Lemma 1 there is a pseudo-isotopy ρ_t of S^n onto S^n, ρ_t is the identity in $S^n - C^n$, and $\rho_1(B^1) = p$. But $\rho_1(B)$ is a proper subarc of B which cannot be separated from C in S^n by a bicollared sphere. But this is a contradiction. Thus A is cellular as well as each subarc of A.

COROLLARY 1. *Let A be an arc in S^n which is the union of two p-shrinkable arcs, $A_1 \cup A_2$, which meet in a common endpoint p. Then A is cellular if A_1 is* L.P.U.

Proof. Each subarc of A is p-shrinkable.

COROLLARY 2. *Each non-cellular arc A in S^n contains a subarc which is not* L.P.U. *at either of its endpoints.*

Even in S^3 there is a difference between an arc being L.P.U. at each point and having the p-shrinkable property for each subarc. The simplest example is perhaps a mildly wild arc which is not a Wilder arc. [3].

REFERENCES

1. M. Brown, *The monotone union of open n-cells is an open n-cell,* Proc. Amer. Math. Soc., **12** (1961), 812–814.
2. ———, *A proof of the generalized Schoenflies theorem,* Bull. Amer. Math. Soc., **66** (1960), 74–76.
3. R. H. Fox, O. G. Harrold, *The Wilder arcs, Topology of* 3-*manifolds and related topics,* Prentice-Hall, (1962).

4. O. G. Harrold, *The enclosing of simple arcs and curves by polyhedra in* 3-*space,* Duke Math. J., **21** (1959), 615–622.
5. ———, *Combinatorial structures, local unknottedness, and local peripheral unknottedness. Topology of 3-Manifolds and related topics,* Prentice-Hall, (1962).

VIRGINIA POLYTECHNIC INSTITUTE

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 14, No. 2 June, 1964

Tom M. (Mike) Apostol and Herbert S. Zuckerman, *On the functional equation $F(mn)F((m, n)) = F(m)F(n)f((m, n))$* . . . 377
Reinhold Baer, *Irreducible groups of automorphisms of abelian groups* . . . 385
Herbert Stanley Bear, Jr., *An abstract potential theory with continuous kernel* . . . 407
E. F. Beckenbach, *Superadditivity inequalities* . . . 421
R. H. Bing, *The simple connectivity of the sum of two disks* . . . 439
Herbert Busemann, *Length-preserving maps* . . . 457
Heron S. Collins, *Characterizations of convolution semigroups of measures* . . . 479
Paul F. Conrad, *The relationship between the radical of a lattice-ordered group and complete distributivity* . . . 493
P. H. Doyle, III, *A sufficient condition that an arc in S^n be cellular* . . . 501
Carl Clifton Faith and Yuzo Utumi, *Intrinsic extensions of rings* . . . 505
Watson Bryan Fulks, *An approximate Gauss mean value theorem* . . . 513
Arshag Berge Hajian, *Strongly recurrent transformations* . . . 517
Morisuke Hasumi and T. P. Srinivasan, *Doubly invariant subspaces. II* . . . 525
Lowell A. Hinrichs, Ivan Niven and Charles L. Vanden Eynden, *Fields defined by polynomials* . . . 537
Walter Ball Laffer, I and Henry B. Mann, *Decomposition of sets of group elements* . . . 547
John Albert Lindberg, Jr., *Algebraic extensions of commutative Banach algebras* . . . 559
W. Ljunggren, *On the Diophantine equation $Cx^2 + D = y^n$* . . . 585
M. Donald MacLaren, *Atomic orthocomplemented lattices* . . . 597
Moshe Marcus, *Transformations of domains in the plane and applications in the theory of functions* . . . 613
Philip Miles, *B^* algebra unit ball extremal points* . . . 627
W. F. Newns, *On the difference and sum of a basic set of polynomials* . . . 639
Barbara Osofsky, *Rings all of whose finitely generated modules are injective* . . . 645
Calvin R. Putnam, *Toeplitz matrices and invertibility of Hankel matrices* . . . 651
Shoichiro Sakai, *Weakly compact operators on operator algebras* . . . 659
James E. Simpson, *Nilpotency and spectral operators* . . . 665
Walter Laws Smith, *On the elementary renewal theorem for non-identically distributed variables* . . . 673
T. P. Srinivasan, *Doubly invariant subspaces* . . . 701
J. Roger Teller, *On the extensions of lattice-ordered groups* . . . 709
Robert Charles Thompson, *Unimodular group matrices with rational integers as elements* . . . 719
J. L. Walsh and Ambikeshwar Sharma, *Least squares and interpolation in roots of unity* . . . 727
Charles Edward Watts, *A Jordan-Hölder theorem* . . . 731
Kung-Wei Yang, *On some finite groups and their cohomology* . . . 735
Adil Mohamed Yaqub, *On the ring-logic character of certain rings* . . . 741
Paul Ruel Young, *A note on pseudo-creative sets and cylinders* . . . 749